Malliavin calculus for Hawkes processes: a new approach

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Existing Malliavin calculus

[1] C. Hillairet, L. Huang, M. Khabou, and A. Réveillac. *The Malliavin-Stein method for Hawkes functionals.* Latin American Journal of Probability and Mathematical Statistics, 2022.
[2] C. Hillairet, A. Réveillac, and M. Rosenbaum. *An expansion formula for Hawkes processes and application to cyber-insurance derivatives.* Stochastic Processes and their Applications, 2023.

Idea: to perturb the system by adding a jump.

Profits:

- an expansion formula for functionals of the Hawkes process,
- a Stein method,
- to compute some prices of financial or insurance derivatives.

Abstract of our Malliavin calculus

Idea: to perturb the jump instants and to formally differentiate with respect to these jump instants.

Profits:

- to define a local derivative satisfying the chain rule,
- an absolute continuity criterion (in particular for the solution of a SDE),
- computations of Greeks.

Setting and first notations

Notations:

- Ω the set of càdlàg real functions on $[0, +\infty)$,
- $N_t(\omega)$ the number of jumps between 0 and $t \in [0, +\infty)$ of $\omega \in \Omega$,
- $\mathbb{P}$ the probability measure such that N is a Hawkes process with intensity

$$\lambda^*(t) = \lambda + \int_0^t \mu(t-s)dN_s, \ t \geq 0,$$

  with $\lambda \in \mathbb{R}_+^*$ and μ differentiable with bounded derivative and $\|\mu\|_1 < 1$,
- $T \in (0, +\infty)$ a terminal instant,
- $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the filtration generated by N.

Directional derivative: Definition

A reparametrization τ_ε of time depending on $\varepsilon \in (0, +\infty)$ and $m \in \mathcal{H}$ where

$$\mathcal{H} = \left\{ m \in L^2((0,T)) \ \int_0^T m(s)ds = 0 \right\},$$

and such that $\tau_\varepsilon(0) = 0, \tau_\varepsilon(T) = T$ and the number and the order of jump times of $\omega \circ \tau_\varepsilon$ remain unchanged for any $\omega \in \Omega$.

For any $F \in L^2(\Omega)$ such that the following limit exists in $L^2(\Omega)$,

$$D_m F = \lim_{\varepsilon \to 0} \frac{F \circ \mathcal{T}_\varepsilon - F}{\varepsilon} \quad \text{with} \quad \mathcal{T}_\varepsilon(\omega) = \omega \circ \tau_\varepsilon.$$

Example

For any $i \in \mathbb{N}^*$, $\overline{T}_i = T_i \wedge T$ is Malliavin differentiable and $D_m \overline{T}_i = -\widehat{m}(\overline{T}_i)$ where $\widehat{m}(t) = \int_0^t m(s)ds$.

Directional derivative: Properties

On $\mathcal{S}$ the set of

$$F = a1_{\{N_T=0\}} + \sum_{n=1}^{+\infty} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}$$

where f_n is smooth with bounded derivatives of any order, the random variables are Malliavin differentiable and the directional derivative D satisfies on $\mathcal{S}$

$$D_m(FG) = (D_m F)G + F(D_m G)$$

and the chain rule: for any $\Phi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and $F_1, \cdots, F_n \in \mathcal{S}$,

$$D_m \Phi(F_1, \cdots, F_n) = \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j}(F_1, \cdots, F_n) D_m F_j.$$

Directional derivative: Integration by parts

Integration by parts

For any $F \in \mathcal{S}$,

$$\mathbb{E}[D_m F] = \mathbb{E}\left[\left(\int_{(0,T]} (\psi(m,t) + \widehat{m}(t)\mu(T-t) + m(t))dN_t\right) F\right]$$

where $\widehat{m}(t) = \int_0^t m(s)ds$ and

$$\psi(m,t) = \frac{1}{\lambda^*(t)} \int_{(0,s)} (\widehat{m}(t) - \widehat{m}(s))\mu'(t-s)dN_s.$$

Idea of the proof: $\mathbb{P}\mathcal{T}_\varepsilon^{-1} \ll \mathbb{P}$ with density G^ε which satisfies $\mathbb{E}[D_m F] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{G^\varepsilon - 1}{\varepsilon} F\right] = \mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} F\right]$.

Directional derivative: Consequences

Theorem

The quadratic bilinear form

$$\mathcal{E}_m(F,G) = \mathbb{E}[D_m F D_m G], \quad F, G \in \mathcal{S},$$

is closable on $L^2(\Omega)$.

Thus we denote $(\mathbb{D}_m^{1,2}, \mathcal{E}_m)$ its closed extension and $(\mathbb{D}_m^{1,2}, D_m)$ the extension of $(\mathcal{S}, D_m)$.

The previous formulas remain valid for any $F \in \mathbb{D}_m^{1,2}$.

The local Dirichlet form: Definition using a Hilbert basis

For any F, G in

$$\mathbb{D}^{1,2} = \left\{ F \in \bigcap_{i\in\mathbb{N}} \mathbb{D}^{1,2}_{m_i}, \quad \sum_{i=0}^{+\infty} \|D_{m_i}F\|^2_{L^2(\Omega)} < +\infty \right\}$$

where $(m_i)_{i\in\mathbb{N}}$ is a Hilbert basis of $\mathcal{H}$,

$$\begin{aligned} \mathcal{E}(F,G) &= \sum_{i=0}^{+\infty} \mathbb{E}[D_{m_i}F D_{m_i}G] \\ DF &= \sum_{i=0}^{+\infty} D_{m_i}F m_i \in L^2(\Omega;\mathcal{H}). \end{aligned}$$

The local Dirichlet form: Properties

Proposition

The bilinear form $(\mathbb{D}^{1,2}, \mathcal{E})$ is a local Dirichlet form admitting the carré du champ $\Gamma[F, G] = \langle DF, DG \rangle_{\mathcal{H}}$ for $F, G \in \mathbb{D}^{1,2}$, and the gradient D.

Moreover $\mathbb{D}^{1,2}$ is a Hilbert space for the norm

$$\|F\|^2_{\mathbb{D}^{1,2}} = \|F\|_{L^2(\Omega)} + \mathcal{E}(F),$$

the operator D satisfies the chain rule for any Lipschitz function and doesn't depend on the choice of $(m_i)_{i \in \mathbb{N}}$ because, for $F \in \mathcal{S}$,

$$DF = \sum_{n=1}^{d} \sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n) \left(\frac{T_j}{T} - 1_{[0,T_j]} \right) 1_{\{N_T = n\}}.$$

Divergence operator: Definition by duality

$\mathrm{Dom}(\delta)$ is the set of $u \in L^2(\Omega; \mathcal{H})$ such that there exists $c \in \mathbb{R}_+^*$ such that

$$\forall F \in \mathbb{D}^{1,2}, \; \left| \mathbb{E}\left[\int_0^T D_t F u_t dt \right] \right| \leq c \|F\|_{\mathbb{D}^{1,2}}$$

and, for any $u \in \mathrm{Dom}(\delta)$, $\delta(u)$ is the unique element in $L^2(\Omega)$ such that

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF \rangle_{\mathcal{H}}].$$

Divergence operator: Properties

Proposition

For any u predictable process in $L^2(\Omega; \mathcal{H})$,

$$\delta(u) = \int_{(0,T]} (\psi(u,t) + \widehat{u}(t)\mu(T-t) + u(t))dN_t$$

where $\widehat{m}(t) = \int_0^t m(s)ds$ and

$$\psi(m,t) = \frac{1}{\lambda^*(t)} \int_{(0,t)} (\widehat{m}(t) - \widehat{m}(s))\mu'(t-s)dN_s.$$

We do not have the Clark-Ocone formula because $N_T \in \mathbb{D}^{1,2}$ with $DN_T = 0$ but $N_T \neq \mathbb{E}[N_T]$.

Application 1: Absolute continuity criterion

Theorem

For any $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$, the image measure $F_*[\det(\Gamma[F]).\mathbb{P}]$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ where $\Gamma[F] = (\Gamma[F_i, F_j])_{1 \leq i,j \leq d}$:

$$F_*[\det(\Gamma[F]).\mathbb{P}] \ll \lambda_d.$$

Corollary

For any $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$, conditionally to $\Gamma[F] \in GL_d(\mathbb{R})$, the law of F is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$:

$$\mathbb{P}_F(\cdot \mid \Gamma[F] \in GL_d(\mathbb{R})) \ll \lambda_d.$$

Application 1: Absolute continuity criterion

SDE on $[0, T]$

$$dX_t = f(t, X_t)dt + g(t, X_{t-})dN_t, \quad X_0 = x_0,$$

where $f, g : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are measurable functions and satisfy

- For any $t \in [0, T]$, the maps $f(t, \cdot), g(t, \cdot)$ are of class C^1.
- $\sup_{t,x}(|\nabla_x f(t, x)| + |\nabla_x g(t, x)|) < +\infty$.
- For any $x \in \mathbb{R}^d$, the map $g(\cdot, x)$ is differentiable.

Auxiliary function:

$$\varphi(t, x) = f(t, x + g(t, x)) - (I_d + \nabla_x g(t, x))f(t, x) - \frac{\partial g}{\partial t}(t, x).$$

Application 1: Absolute continuity criterion

Proposition

$X_T \in \mathbb{D}^{1,2}$ and we have an explicit expression of DX_T and $\Gamma[X_T]$.

Theorem

If $d = 1$ and $\varphi(t, x) \neq 0$ for any $(t, x) \in [0, T] \times \mathbb{R}$ then, conditionally to $\{N_T \geq 1\}$, the law X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$:

$$\mathbb{P}_{X_T}(\cdot \mid N_T \geq 1) \ll \lambda_1.$$

We also have a theorem for $d \geq 1$ with a spanning condition and conditionally to the fact that the process N admits enough jumps.

Application 2: Greek computation

Dynamics of an asset price

$$dS_t = rS_t dt + \sigma S_{t-} d\widetilde{N}_t = (r - \sigma \lambda^*(t)) S_t dt + \sigma S_{t-} dN_t, \quad S_0 = x_0.$$

We consider a function Φ and we would like to know the variations of $\mathbb{E}[\Phi(S_T)]$ with respect to the different parameters x_0, r and σ.

Our result is true for every classical payoff functions.

Application 2: Greek computation

Theorem

$$\frac{\partial}{\partial x_0}\mathbb{E}[1_{\{N_T\geq 1\}}\Phi(S_T)] = \mathbb{E}\left[\Phi(S_T^{x_0})\delta\left(m1_{\{N_T>0\}}\frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right)\right]$$

$$= -\mathbb{E}\left[\frac{\Phi(S_T^{x_0})\delta(m)1_{\{N_T>0\}}}{\sigma x_0\int_{(0,T]}\mu(T-t)\widehat{m}(t)dN_t}\right]$$

$$-\mathbb{E}\left[\frac{\Phi(S_T^{x_0})\int_{(0,T]}\mu'(T-s)\widehat{m}(s)^2dN_s}{\sigma x_0\left(\int_{(0,T]}\mu(T-s)\widehat{m}(s)dN_s\right)^2}1_{\{N_T>0\}}\right]$$

$$+\mathbb{E}\left[\frac{\Phi(S_T^{x_0})\int_{(0,T]}\mu(T-s)m(s)\widehat{m}(s)dN_s}{\sigma x_0\left(\int_{(0,T]}\mu(T-s)\widehat{m}(s)dN_s\right)^2}1_{\{N_T>0\}}\right].$$

Thank you for your attention.

Do you have some questions ?