# Desingularization of time-periodic vortex motion in bounded domains via KAM tools

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**Abstract**

We examine the Euler equations within a simply-connected bounded domain. The dynamics of a single point vortex are governed by a Hamiltonian system, with most of its energy levels corresponding to time-periodic motion. We show that for the single point vortex, under certain non-degeneracy conditions, it is possible to desingularize most of these trajectories into time-periodic concentrated vortex patches. We provide concrete examples of these non-degeneracy conditions, which are satisfied by a broad class of domains, including convex ones. The proof uses Nash-Moser scheme and KAM techniques, in the spirit of the recent work on the leapfrogging motion [59], combined with complex geometry tools. Additionally, we employ a vortex duplication mechanism to generate synchronized time-periodic motion of multiple vortices. This approach can be, for instance, applied to desingularize the motion of two symmetric dipoles (with four vortices) in a disc or a rectangle. To our knowledge, this is the first result showing the existence of non-rigid time-periodic motion for Euler equations in generic simply-connected bounded domain. This answers an open problem that has been pointed, for example, in [10].

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# 1 Introduction

In this study, we investigate the vortex dynamics governed by the 2D incompressible Euler equations in a simply-connected domain $\mathbf{D}$. These equations are formulated in the velocity-vorticity form as follows:

$$\partial_t \boldsymbol{\omega} + u \cdot \nabla \boldsymbol{\omega} = 0, \qquad u = \nabla^{\perp} \Delta_{\mathbf{D}}^{-1} \omega, \qquad \nabla^{\perp} = (-\partial_y, \partial_x), \tag{1.1}$$

where $\Delta_{\mathbf{D}}$ is the standard Laplacian supplemented with Dirichlet boundary condition. This system has been studied for a long time and is still the subject of intensive activities touching multiple facets. Here, we don't in any case pretend to make a complete review of the main significant results, which is an ambitious program, but we shall simply focus on two important subjects that fit with the scope of the paper. The first one is related to the existence and uniqueness of global solutions which is proved at the level of integrable and bounded vortices for smooth domains by Yudovich in his seminal work [93]. However, the problem turns out to be more tricky for rough domains, for instance when the boundary contains corners or fractal sets. The existence part for very general domains is done by Gérard-Varet and Lacave [47]. As to the uniqueness part, only partial results have been performed in the last decade, under some angle constraints or positive vorticity condition, see for instance [55, 56, 78]. We also refer the reader to the recent work of Agrawal and Nahmod [2] where the uniqueness is obtained under some constraints on $C^{1,\alpha}$ domains. Another important topic that will be explored in part in this paper is related to the emergence of coherent structures. One particularly significant facet of this field revolves around relative equilibria, which represent equilibrium states in a rigid body frame. They stand as one of the main central field in the study of vortex motion, garnering considerable theoretical and experimental examination. For a comprehensive exploration of bifurcation techniques used to construct rigid time-periodic solutions stemming from various stationary radial states, we refer the reader to an exhaustive list that outlines numerous pertinent questions and findings in this domain [14, 19, 20, 28, 29, 45, 46, 49, 61, 67, 68, 70]. Additional steady solutions have been discovered in [22, 32]. We observe that all these results are concerned with rigid motion where the shape of the vorticity is conserved without any deformation over the time. Very recently, new structures on quasi-periodic vortex patches for Euler equations have been constructed via KAM methods near ellipses [13], Rankine vortices [62], and annuli [60]. Similar studies have been conducted over the past few years for more active scalar equations such as [48, 58, 71, 88]. More investigations can be found in [21, 25, 33, 37, 38, 39]. On the other hand, as we shall see later, the point vortex system offers multiple configurations of non-rigid motion where the dynamics can be tracked explicitly as for the leapfrogging motion. A classic example of this is provided by Love [80], who studied two symmetric dipoles. Recently, Hassainia, Hmidi and Masmoudi [59] have proven the desingularization of this system, using KAM tools.

The primary objective of this paper is to construct non-rigid, time-periodic solutions for the Euler equations in simply connected bounded domains. This will be achieved by desingularizing the motion of point vortices using KAM approach. It is important to note that in bounded domains, the point vortex system, where the vorticity is concentrated at a finite number of points $z_1, ..., z_N$ (with $N \in \mathbb{N}^*$) with circulations $\Gamma_1, .., \Gamma_N$, follows the ODE

$$\forall k \in [\![1, N]\!], \qquad \frac{d\xi_k(t)}{dt} = \sum_{\substack{\ell=1\\ \ell\neq k}}^{N} \frac{\mathrm{i}\Gamma_\ell}{\pi} \partial_{\overline{\xi}} G_{\mathbf{D}}\big(\xi_\ell(t), \xi_k(t)\big) + \frac{\mathrm{i}\Gamma_k}{2\pi} \partial_{\overline{\xi}} \mathcal{R}_{\mathbf{D}}\big(\xi_k(t)\big). \tag{1.2}$$

Here, $G_{\mathbf{D}}$ and $\mathcal{R}_{\mathbf{D}}$ are the Green and Robin functions of the domain $\mathbf{D}$ whose definitions are provided in Section 2.1. Notice that in [64], Helmholtz introduced this asymptotic system in the flat case $\mathbf{D} = \mathbb{R}^2$ with $G_{\mathbb{R}^2}(z, w) = \log|z - w|$ and $\mathcal{R}_{\mathbb{R}^2} \equiv 0$, to describe the behavior of mutual interactions of several concentrated vortices. Later on, Kirchhoff and Routh [76, 89] recast this system in the Hamiltonian form, with the Hamiltonian

$$H(\xi_1, .., \xi_N) = \sum_{1\leqslant k<\ell\leqslant N} \tfrac{\Gamma_k\Gamma_\ell}{2\pi} G_{\mathbf{D}}(\xi_k, \xi_\ell) + \sum_{k=1}^{N} \tfrac{\Gamma_k^2}{4\pi} \mathcal{R}_{\mathbf{D}}(\xi_k).$$

The point vortex system provides a tractable model describing in suitable regimes the fluid motion. Despite its simplicity, it captures the main features of vortex interactions and serves as a foundation for understanding more intricate fluid behaviors. A huge literature has been performed over the past century addressing various aspects of the point vortex system. While it is impossible to cover all the significant contributions in this area, we will concentrate on the most relevant topics related to this paper. Below, we will highlight some key results obtained for the flat case $\mathbf{D} = \mathbb{R}^2$, which can be appropriately adapted for more general domains. The Cauchy-Lipschitz Theorem ensures that the trajectories of the vortices are well-defined and smooth as long as any two point vortices remain distant from each other. Collapse in finite time may occur with signed circulations as documented in [4, 6, 51, 52, 65, 77, 86], although this phenomenon is rare [30, 81]. On the other hand, this Hamiltonian system enjoys three important independent conservation laws, which are in involution: total circulation, linear impulse and angular impulse. This implies that the point vortex system is completely integrable for $N \leqslant 3$ and starts to exhibit chaotic motion from $N = 4$, for instance see [7]. Within the complex structure of point vortex system, certain long-lived or coherent structures emerge. A specific class is given by configurations keeping their geometric form unchanged throughout their evolution. An illustration of this is given by Thomson polygons where the point vortices are located on the vertices of a regular polygon with the same circulation. This configuration rotates uniformly about its center. A general review on this topic can be found in [5]. One can desingularize such configurations into steady vortex patch motion through various methods such as variational techniques [16, 17, 18, 91, 92] or gluing method [27]. The use of the contour dynamics approach, more efficient to track the dynamics, was first developped by Hmidi and Mateu for symmetric vortex pairs [69]. The asymmetric case was solved in [57] and the global bifurcation has been addressed in [44]. For symmetric configurations involving more point vortices, we refer the reader to García's works on Kármán vortex street [42] and Thomson polygons [43]. Finally, the desingularization of general configurations satisfying a natural non-degeneracy condition has been exposed in [63].

In this work, we focus on the motion of a single point vortex within a bounded simply-connected domain. In this special case, only the second term in the right-hand side of (1.2) persists and the equation takes the Hamiltonian form

$$\dot{\xi}(t) = \nabla^\perp H_{\mathbf{D}}\big(\xi(t)\big), \qquad \nabla^\perp \triangleq \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix}, \qquad H_{\mathbf{D}} \triangleq \tfrac{\Gamma}{4\pi} \mathcal{R}_{\mathbf{D}}. \tag{1.3}$$

This 2D system will be explored with more details in Section 2. In particular, we infer from (2.18) that the set of energy levels is an unbounded connected interval, that is,

$$H_{\mathbf{D}}(\mathbf{D}) \triangleq [\lambda_{\min}, \infty). \tag{1.4}$$

By virtue of the Hamiltonian structure, all the particle trajectories are closed curves defined by the energy levels

$$\mathcal{E}_\lambda = \big\{z \in \mathbb{C}, \quad H_{\mathbf{D}}(z) = \lambda\big\}, \qquad \lambda \geqslant \lambda_{\min}.$$

The orbit for $\lambda = \lambda_{\min}$ is a single point given by the lowest energy level of the critical points to the Hamiltonian. It is important to mention that the regular values of the Hamiltonian give rise to time-periodic trajectories and, in view of Sard's theorem, it is the generic situation. However, the study of critical points of the Robin function is a complex and delicate subject and the literature contains only a few known results on this topic. For example, when the domain is convex and bounded, there is only one critical point and the Hamiltonian is strictly convex. This implies that all the orbits are periodic and enclose convex subdomains, see [15, 54]. However, except in radial domains, when the trajectory is closed, the particle does not generally exhibit rigid body motion. In fact, it remains far from equilibrium states (critical points) and its trajectory is significantly influenced and deformed by the geometry of the domain. More periodic configurarions with multiple vortices near the boundary were discovered by Bartsch and Sacchet in [10]. Desingularizing these trajectories using classical solutions to the Euler equations that replicate similar time-periodic dynamics is particularly challenging, especially due to the time-space resonance, which can disrupt the formation of confined structures. Our primary objective is to achieve this construction using KAM theory, which requires that the energy levels belong to a suitable Cantor set, carefully constructed through an extraction procedure. This approach builds upon the recent work on the leapfrogging phenomenon established in [59].
To state our main result, we will introduce some necessary objects and assumptions. Let $\lambda_{\min} < \lambda_* < \lambda^*$ such that for any $\lambda \in [\lambda_*, \lambda^*]$ the orbit $\mathcal{E}_\lambda$ is periodic with minimal period $\mathtt{T}(\lambda)$ and parametrized by $t \in \mathbb{R} \mapsto \xi_\lambda(t)$. This assumption holds provided that the interval $[\lambda_*, \lambda^*]$ does not contain any critical value for the Hamiltonian. According to Sard's theorem, the set of critical values has zero Lebesgue measure. We believe that this set, which is trivially compact, is actually discrete–likely even finite–due to the specific structure of the Hamiltonian. However, we have not yet been able to produce any proof of this formal conjecture. Remark that the map $\lambda \in [\lambda_*, \lambda^*] \mapsto \mathtt{T}(\lambda)$ is real analytic regardless of the boundary regularity of the domain $\mathbf{D}$. Next, for each $\lambda \in [\lambda_*, \lambda^*]$, we introduce the $\mathtt{T}(\lambda)-$periodic matrix:

$$\mathbb{A}_\lambda(t) = \begin{pmatrix} \mathtt{u}_\lambda(t) & \mathtt{v}_\lambda(t) \\ \overline{\mathtt{v}_\lambda(t)} & \overline{\mathtt{u}_\lambda(t)} \end{pmatrix}, \qquad \mathtt{u}_\lambda(t) = -\tfrac{\mathrm{i}}{2} e^{2\mathcal{R}_{\mathbf{D}}(\xi_\lambda(t))}, \qquad \mathtt{v}_\lambda(t) = \tfrac{\mathrm{i}}{4}\big[\partial_z \mathcal{R}_{\mathbf{D}}\big(\xi_\lambda(t)\big)\big]^2.$$

We associate to this matrix its fundamental matrix $\mathscr{M}_\lambda$, which solves the ODE

$$\partial_t \mathscr{M}_\lambda(t) = \mathbb{A}_\lambda(t)\mathscr{M}_\lambda(t), \qquad \mathscr{M}_\lambda(0) = \mathrm{Id}.$$

Then the monodromy matrix is defined by $\mathscr{M}_\lambda\big(\mathtt{T}(\lambda)\big)$ and its spectrum is denoted by $\mathrm{sp}\big(\mathscr{M}_\lambda\big(\mathtt{T}(\lambda)\big)\big)$. We emphasize that due to the zero trace of $\mathbb{A}_\lambda(t)$, the monodromy matrix belongs to the special linear group $\mathrm{SL}(2, \mathbb{C})$. Our first main result reads as follows.

**Theorem 1.1.** *Let $\mathbf{D}$ be a simply connected bounded domain and $\lambda_{\min} < \lambda_* < \lambda^*$. Assume that:*

1. *Non-degeneracy of the period:*
$$\min_{\lambda \in [\lambda_*, \lambda^*]} \big|\mathtt{T}'(\lambda)\big| > 0.$$

2. *Spectral assumption:*
$$\forall \lambda \in [\lambda_*, \lambda^*], \quad 1 \notin \mathrm{sp}\big(\mathscr{M}_\lambda\big(\mathtt{T}(\lambda)\big)\big).$$

*Then, $\exists\, \varepsilon_0 > 0$ such that $\forall\, \varepsilon \in (0, \varepsilon_0)$, there exists a Cantor like set $\mathscr{C}_\varepsilon \subset [\lambda_*, \lambda^*]$, with*

$$\lim_{\varepsilon \to 0} |\mathscr{C}_\varepsilon| = \lambda^* - \lambda_*,$$

*and for any $\lambda \in \mathscr{C}_\varepsilon$, there exists a solution to Euler equation taking the form*

$$\forall t \in \mathbb{R}, \quad \omega(t) = \frac{1}{\varepsilon^2}\mathbf{1}_{D_t^\varepsilon}, \qquad D_t^\varepsilon = \xi_\lambda(t) + \varepsilon O_t^\varepsilon,$$

*with*

$$\forall t \in \mathbb{R}, \quad D_{t+\mathtt{T}(\lambda)}^\varepsilon = D_t^\varepsilon, \qquad \xi_\lambda\big(t + \mathtt{T}(\lambda)\big) = \xi_\lambda(t).$$

**Remark 1.2.** *1. The previous theorem applies to generic simply-connected bounded domains, which are not necessarily a perturbation of explicit domains, such as discs, ellipses or rectangles.*

2. *In contrast to the leapfrogging vortex motion* [59], *the period* $\mathtt{T}(\lambda)$ *is not explicit. This explains the need of adding the first assumption in the previous theorem. We believe that this latter is universally valid, as the energy levels of critical points of the period function are expected to constitute a countable set of isolated points. Indeed, it would be surprising to find domains for which the period remains constant on a non-trivial interval of energy levels.*

3. *As for the spectral assumption, verifying it in its original form is difficult due to the intricate nature of the trajectories of periodic orbits, but we believe it holds for the majority of energy levels. We mention that the spectral assumption corresponds to the classical one appearing in finite dimensional dynamical systems.*

Next, we will present a practical implication of Theorem 1.1, connecting its assumptions to the geometry of the domain $\mathbf{D}$ and applying it to relevant standard examples. This leads us to the following corollary.

**Corollary 1.3.** *Assume that the bounded domain* $\mathbf{D}$ *is convex. Then, the conclusion of Theorem* 1.1 *holds true for any* $\lambda_{\min} < \lambda_* < \lambda^*$ *provided that the conformal mapping* $F : \mathbb{D} \to \mathbf{D}$ *with* $F(0) = \xi_0$ *satisfies*

$$\left|\tfrac{F^{(3)}(0)}{F'(0)}\right| \notin \left\{2\sqrt{1-\tfrac{1}{n^2}},\ n \in \mathbb{N}^* \cup \{\infty\}\right\}.$$

*Here,* $\mathbb{D}$ *stands for the open unit disc of the complex plane and* $\xi_0$ *is the unique critical point of Robin function.*

**Remark 1.4.** *1. The assumption in the previous corollary is an open condition. Therefore, for a given domain* $\mathbf{D}$ *satisfying the assumptions of Corollary 1.3 one might expect that the same conclusion holds for all convex domains* $\mathbf{D}_\varepsilon$ *which are* $\varepsilon$*-perturbation of* $\mathbf{D}$*. This is a consequence of the continuity of the Riemann mapping with respect to the domain, known as Carathéodory's kernel Theorem.*

2. *More refined version, where the domain is not necessarily convex but the Robin function has only one critical point, is given by Corollary* 3.3.

Corollary 1.3 applies to domains such as rectangles and ellipses provided that the aspect ratio avoids a specific discrete set. This issue will be thoroughly examined in Section 5, as outlined in Propositions 5.1 and 5.4. Below, we provide a formal statement.

**Proposition 1.5.** *The following properties hold true.*

1. *Let* $\mathbf{D}$ *be an ellipse with semi-axes* $0 < b \leqslant a$*. Then there exists a countable set* $\mathcal{G}_E \subset (1,\infty)$ *such that Corollary* 1.3 *applies if and only if* $\frac{a}{b} \notin \mathcal{G}_E$.

2. *Let* $\mathbf{D}$ *be a rectangle with sides* $0 < l \leqslant L$*. Then there exists a countable set* $\mathcal{G}_R \subset (0,1)$ *such that Corollary* 1.3 *applies if and only if* $\frac{l}{L} \notin \mathcal{G}_R$.

We shall now present another interesting result related to the duplication method and its application in generating synchronized multiple time periodic vortices. The duplication principle reads as follows. We refer the reader to Section 6.1 for the complete proof.

**Proposition 1.6.** *Let* $\mathbf{D}$ *be a simply-connected bounded domain whose boundary contains a non-trivial segment* $[z,w]$ *with* $z \neq w$ *such that*

$$\partial\mathbf{D} \cap (z,w) = [z,w].$$

*We denote* $\mathtt{S}$ *the reflexion through the axis* $(z,w)$*. Let* $\omega \in L^\infty\big(\mathbb{R}; L^\infty_c(\mathbf{D})\big)$ *be a global weak solution to Euler equations in* $\mathbf{D}$*. Then, there exists* $\omega^\# \in L^\infty\big(\mathbb{R}; L^\infty_c(\mathbf{D}^\star)\big)$ *a global in time weak solution to Euler equations in* $\mathbf{D}^\star \triangleq \mathrm{Int}(\overline{\mathbf{D}} \cup \overline{\mathtt{S}(\mathbf{D})})$ *taking the form*

$$\omega^\#(t,x) \triangleq \begin{cases} \omega(t,x), & \text{if } x \in \mathbf{D},\\ -\omega(t,\mathtt{S}x), & \text{if } x \in \mathtt{S}(\mathbf{D}). \end{cases}$$

*In particular, if $\omega$ is a time periodic vortex patch, then $\omega^{\#}$ is a time periodic counter-rotating pair of patches.*

This construction can be repeated as long as the new domain retains the same properties as the initial one. In this manner, we can generate multiple synchronized vortices, with their dynamics replicating that of a single vortex, up to appropriate reflections. To illustrate, let's apply the previous statement to construct synchronized non-rigid periodic motion of four vortex patches within the unit disc. In the planar case, the desingularization of the four point vortices has been studied by Davila, del Pino, Musso and Parmeshwar in [26] using gluing techniques. It is worth to point out that in their case the problem is dispersive and one only can describe the asymptotic dynamics. However in our setting the presence of the boundary constraints the trajectories to be closed and therefore obtain a periodic motion. In the next proposition we provide our result in this direction. A more detailed discussion on $m$-sectors can be found in Section 6.2.2.

**Proposition 1.7.** *Let $\mathbb{D}_2$ be the first quadrant of the unit disc, namely*

$$\mathbb{D}_2 \triangleq \big\{(x,y)\in\mathbb{D}\quad \text{s.t.}\quad x>0\quad\text{and}\quad y>0\big\}.$$

*Denote by $\mathtt{S}_x$ and $\mathtt{S}_y$ the reflexion with respect to the horizontal and vertical axis, respectively. Then, the following properties hold true.*

1. *The domain $\mathbb{D}_2$ satisfies the assumptions of Corollary 1.3 with critical point*

$$\xi_0 \triangleq \tfrac{1}{\sqrt{2}}\Big(4+\sqrt{17}\Big)^{-\frac{1}{4}}(1+\mathrm{i}).$$

2. *Let $\mathcal{E}_\lambda\subset\mathbb{D}_2$ be a point vortex periodic orbit that can be desingularized into a periodic vortex patch motion $\mathbf{1}_{D_t}\in L^\infty\big(\mathbb{R};L^\infty_c(\mathbb{D}_2)\big)$. Then, the function*

$$\omega^\star \triangleq \mathbf{1}_{D_t}-\mathbf{1}_{\mathtt{S}_xD_t}-\mathbf{1}_{\mathtt{S}_yD_t}+\mathbf{1}_{-D_t}\in L^\infty\big(\mathbb{R};L^\infty_c(\mathbb{D})\big)$$

   *is a time periodic solution to the Euler equations in the unit disc $\mathbb{D}$.*

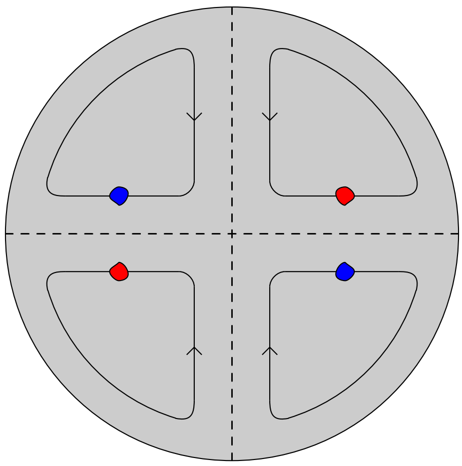

Figure 1: Choreography of four synchronized vortex patches in the unit disc.

Next, we will discuss the key ideas behind the proof of Theorem 1.1. To the best of our knowledge, this result marks the first construction of classical solutions for Euler equations involving non-rigid periodic motion in bounded domains. Our proof is inspired by the approach used in [59], which combines a desingularization procedure with the Nash-Moser scheme and KAM theory to tackle degenerate quasi-linear transport equations driven by time-space periodic coefficients associated with point vortex motion. This method is remarkably robust, making it applicable to various significant ordered structures in geophysical flows. As we will see later, our proof not only addresses the

challenges of small divisor problem caused by time-space resonances but also overcomes several other substantial challenges. These include degeneracy in the time direction, the degeneracy of the mode 1 in the leading term, and the invertibility of an operator with variable coefficients. In addition, a supplementary difficulty in our setting arises from the geometry of the fixed domain $\mathbf{D}$, which is not explicit in the general statement. The key idea to overcome this issue is to employ complex analysis using the Riemann conformal mapping, which encodes the geometrical properties of $\mathbf{D}$. This procedure naturally leads to the appearance of interesting geometrical quantities, such as the Schwarzian derivative and the conformal radius.
Let us now outline the fundamental steps of the proof of Theorem 1.1 and discuss the various technical challenges mentioned before.

➊ *Contour dynamics equations and linearization.* As we shall see in Section 3.1, we will look for a solution to Euler equations (1.1) in the form

$$\boldsymbol{\omega}_\varepsilon(t) = \tfrac{1}{\varepsilon^2}\mathbf{1}_{D_t^\varepsilon},$$

where $\varepsilon \in (0,1)$ is small enough, the domain $D_t^\varepsilon$ is given by

$$D_t^\varepsilon \triangleq \varepsilon O_t^\varepsilon + \xi_\lambda(t)$$

and $O_t^\varepsilon$ is a simply connected domain localized around the unit disc. The core of the vortex follows the dynamics of a point vortex, that is,

$$\dot{\overline{\xi}}_\lambda(t) = -\tfrac{\mathrm{i}}{2}\partial_z\mathcal{R}_{\mathbf{D}}\big(\xi_\lambda(t)\big).$$

We assume that the orbit $t \mapsto \xi_\lambda(t)$ is $\mathtt{T}(\lambda)-$periodic with $\lambda$ its energy level. The goal is to construct $\mathtt{T}(\lambda)$-periodic solution meaning that

$$\forall t \in \mathbb{R}, \quad O^\varepsilon_{t+\mathtt{T}(\lambda)} = O_t,$$

with $O_t^\varepsilon$ being a simply connected domain localized around the unit disc. To provide a more precise description, we will parametrize the boundary $\partial O^\varepsilon_{t,1}$ as follows

$$\theta \in \mathbb{T} \mapsto \sqrt{1+2\varepsilon r(\omega(\lambda)t,\theta)}\, e^{\mathrm{i}\theta}, \qquad \omega(\lambda) = \tfrac{2\pi}{\mathtt{T}(\lambda)},$$

with $r : (\varphi,\theta) \in \mathbb{T}^2 \mapsto r(\varphi,\theta) \in \mathbb{R}$ being a smooth periodic function. We need to reparamterize the point trajectory as follows

$$\xi_\lambda(t) = p\big(\omega(\lambda)t\big), \quad \text{with} \quad p : \mathbb{T} \mapsto \mathbb{C}.$$

Then from the contour dynamics equation, see (3.14), we find by taking $G \triangleq \frac{\mathbf{G}}{\varepsilon}$

$$\begin{aligned} G(r)(\varphi,\theta) &\triangleq \varepsilon^2\omega(\lambda)\partial_\varphi r(\varphi,\theta) - \tfrac{1}{2}\partial_\theta \mathrm{Re}\Big\{\partial_z\mathcal{R}_{\mathbf{D}}\big(p(\varphi)\big)R(\varphi,\theta)e^{\mathrm{i}\theta}\Big\} \\ &\quad + \tfrac{1}{\varepsilon}\partial_\theta\Big[\Psi_1\big(r,z(\varphi,\theta)\big) + \Psi_2\big(r,z(\varphi,\theta)\big)\Big] = 0, \end{aligned} \tag{1.5}$$

where $\Psi_1$ describes the induced effect, whereas $\Psi_2$ describes the boundary interaction. The linearization around a small state $r$ is described in Proposition 7.2. Actually, we get the following asymptotic structure,

$$d_rG(r)[h] = \varepsilon^2\omega(\lambda)\partial_\varphi h + \partial_\theta\big[\mathbf{V}^\varepsilon(r)h\big] - \tfrac{1}{2}\mathbf{H}[h] + \varepsilon^2\partial_\theta\mathbf{Q}_1[h] + \varepsilon^2\partial_\theta\mathcal{R}_1^\varepsilon[h] + \varepsilon^3\partial_\theta\mathcal{R}_2^\varepsilon[h],$$

where $\mathbf{H}$ denotes the toroidal Hilbert transform and the function $\mathbf{V}^\varepsilon(r)$ decomposes as follows

$$\mathbf{V}^\varepsilon(r) \triangleq \tfrac{1}{2} - \tfrac{\varepsilon}{2}r + \varepsilon^2\big(\tfrac{1}{2}\mathtt{g} + V_1^\varepsilon(r)\big) + \varepsilon^3 V_2^\varepsilon(r),$$

with

$$\mathtt{g}(\varphi,\theta) \triangleq \mathrm{Re}\Big\{\mathtt{w}_2\big(p(\varphi)\big)e^{2\mathrm{i}\theta}\Big\}, \qquad \mathtt{w}_2(p) \triangleq \big(\partial_z\mathcal{R}_{\mathbf{D}}(p)\big)^2 + \tfrac{1}{3}S(\Phi)(p).$$

In the expression above $S(\Phi)$ is the classical Schwazian derivative, defined in (2.16). However, the real-valued function $r\mapsto V_1^\varepsilon(r)$ is quadratic while $r\mapsto V_2^\varepsilon(r)$ is affine. As to the operator $\mathbf{Q}_1$, it is a of finite rank localizing in the spatial modes $\pm1$ and takes the integral form

$$\mathbf{Q}_1[h](\varphi,\theta)\triangleq\int_{\mathbb{T}}h(\eta)\left[\frac{\cos(\theta-\eta)}{r_{\mathbf{D}}^2\big(p(\varphi)\big)}+\frac{1}{6}\mathrm{Re}\left\{e^{\mathrm{i}(\theta+\eta)}S(\Phi)\big(p(\varphi)\big)\right\}\right]d\eta,\qquad r_{\mathbf{D}}\triangleq e^{-\mathcal{R}_{\mathbf{D}}}.$$

The operator $\mathcal{R}_1^\varepsilon(r)$ exhibits a smoothing effect in space and depends quadratically on $r$. On the other hand, the operator $\mathcal{R}_2^\varepsilon(r)$ also smoothes in space, but its dependence on $r$ is affine. The main challenge, in inverting the linearized operator, involves a small divisor problem caused by the time-space resonance, where the time direction degenerates as $\varepsilon$ approaches zero. To solve this issue, it is necessary to work within Cantor sets on the parameter $\lambda$ that satisfy Diophantine conditions, which also degenerate with rate of order $\varepsilon^{2+\delta},\delta>0$. This allows to ensure an almost full Lebesgue measure to these sets after the multiple steps applied in the Nash-Moser scheme. Note that the rate of degeneracy in $\varepsilon$ has significant consequences in constructing a right inverse for the linearized operator, which results in a loss of regularity and leads to a divergent control of order $\varepsilon^{-2-\delta}$. Nevertheless, when initializing the Nash-Moser scheme with $r=0$, the subsequent iteration corresponds to the term $(d_rG(0))^{-1}[G(0)]$. However, from Lemma 7.3, we infer that $G(0)$ is of size $O(\varepsilon)$. As a result, the estimate of $(d_rG(0))^{-1}[G(0)]$ will exhibit a divergent behavior as $\varepsilon$ becomes small, making this approach unsuitable for the current scheme. Therefore, before proving the invertibility of the linearized operator and applying the Nash-Moser scheme, we need to construct a better approximate solution for (1.5). This construction is provided by Lemma 7.4, which ensures the existence of a function $r_\varepsilon:[\lambda_*,\lambda^*]\times\mathbb{T}^2\to\mathbb{R}$ satisfying

$$\|r_\varepsilon\|_s^{\mathrm{Lip}(\gamma)}\lesssim1\qquad\text{and}\qquad\|G(\varepsilon r_\varepsilon)\|_s^{\mathrm{Lip}(\gamma)}\lesssim\varepsilon^4.$$

The proof uses the fact that $G(0)$ is localized outside the modes $\pm1$, as established in Lemma 7.3. Then, we introduce a rescaling of the function $G$ as stated in (7.25) that takes the form

$$\mathbf{F}(\rho)\triangleq\tfrac{1}{\varepsilon^{1+\mu}}G(\varepsilon r_\varepsilon+\varepsilon^{1+\mu}\rho),\tag{1.6}$$

where $\mu\in(0,1)$ is a free parameter. Note that the parameter $\delta$ will be carefully selected later in the Nash-Moser iteration, in relation to $\mu$ and the measure of the final Cantor set.

❷ *Construction of an approximate right inverse.* In Proposition 7.13, we present the formulation of a Nash-Moser iteration tailored to our problem, based on [59, Prop. 6.1]. The implementation of this iteration requires the construction an approximate right inverse for the linearized operator of the functional $\mathbf{F}$, defined in (1.6), at small state $\rho$. The process follows as the following plan. First, we expand the linearized operator in terms of $\varepsilon$. Next, we use the KAM method, combined with an appropriate preliminary steps, to conjugate the transport equation into an operator with constant coefficients. This step makes appearing a Cantor-type set. Then, we reduce a truncated operator at a suitable order in $\varepsilon$, addressing the challenge posed by the degeneracy of the spatial modes $\pm1$, utilizing the monodromy matrix. The inversion on the remaining modes requires a second Cantor-type set. Finally, the approximate right inverse is derived through a perturbative argument.

❸ *Nash-Moser scheme and measure of the final Cantor set.* In the Section 7.6, we build a non-trivial solution to the equation (1.5) by means of Nash-Moser iteration scheme. The method is now classical and we borrow the result from [59, Prop. 6.1]. As stated in Corollary 7.14, the solutions are obtained modulo a suitable choice of the energy level $\lambda$ among a Cantor set. Using the non degeneracy of the point vortex frequency $\lambda\mapsto\omega(\lambda)$ we prove in Lemma 7.15 a lower bound for the Lebesgue measure of this final Cantor set.

## 2 Motion of a single vortex

The point vortex system in bounded domains is well-explored in the literature and a lot of important facts illustrating the boundary effects have been established. We refer for instance to [53, 85]. Our primary goal here is to examine certain results related to the motion of a single vortex and analyze its orbit. Specifically, we are interested in the existence of periodic orbits, which are closely linked to the geometry of the domain.

## 2.1 Green and Robin functions

In this section, we will gather some fundamental results on Green functions associated with bounded simply connected domains $\mathbf{D}$. This topic is well-documented in the literature; for instance, see [34] for more detailed information. To begin, let's recall that the stream function $\psi$ associated with a given vorticity $\boldsymbol{\omega}$ is the unique solution to the elliptic problem

$$\begin{cases} \Delta\psi = \boldsymbol{\omega}, & \text{in } \mathbf{D}, \\ \psi = 0, & \text{for all } z \in \partial\mathbf{D}. \end{cases}$$

Then $\psi = \Delta^{-1}\boldsymbol{\omega}$ is linked to $\boldsymbol{\omega}$ through the following integral formula

$$\psi(t,z) = \frac{1}{2\pi}\int_{\mathbf{D}} G_{\mathbf{D}}(z,w)\boldsymbol{\omega}(t,w)dA(w), \tag{2.1}$$

where $G_{\mathbf{D}}$ denotes the *Green function* in $\mathbf{D}$, which is defined as the unique solution to the elliptic equation

$$\begin{cases} \Delta_z G_{\mathbf{D}}(z,w) = 2\pi\delta_w(z), & \text{in } \mathbf{D}, \\ G_{\mathbf{D}}(z,w) = 0, & \text{for all } z \in \partial\mathbf{D}. \end{cases} \tag{2.2}$$

In addition, the function $G_{\mathbf{D}}$ decomposes as follows,

$$G_{\mathbf{D}}(z,w) = \log|z-w| + K(z,w), \tag{2.3}$$

where $K : \mathbf{D}\times\mathbf{D} \to \mathbb{R}$ is the regular part of the Green function, which is harmonic in each variable $z$ and $w$. In the particular case of the unit disc $\mathbb{D}$, one gets

$$G_{\mathbb{D}}(z,w) = \log\left|\frac{z-w}{1-z\overline{w}}\right|, \qquad \text{i.e.} \qquad K(z,w) = -\log|1-z\overline{w}|.$$

It is a classical fact [3, Chap. 6], that the Green function is linked to the conformal mappings $\Phi : \mathbf{D} \to \mathbb{D}$ as follows

$$G_{\mathbf{D}}(z,w) = G_{\mathbb{D}}\big(\Phi(z),\Phi(w)\big). \tag{2.4}$$

Therefore, we easily deduce from (2.3)

$$\forall z \neq w \in \mathbf{D}, \quad K(z,w) = \log\left|\frac{\frac{\Phi(z)-\Phi(w)}{z-w}}{1-\Phi(z)\overline{\Phi(w)}}\right|. \tag{2.5}$$

The *Robin function* is defined by

$$\forall z \in \mathbf{D}, \quad \mathcal{R}_{\mathbf{D}}(z) \triangleq K(z,z) = \lim_{w\to z} K(z,w). \tag{2.6}$$

Thus we get from (2.5) that the Robin function can be explicitly linked to the conformal mapping as follows,

$$\begin{aligned} \forall z \in \mathbf{D}, \quad \mathcal{R}_{\mathbf{D}}(z) &= \log\left(\frac{|\Phi'(z)|}{1-|\Phi(z)|^2}\right) \\ &= -\log\big(r_{\mathbf{D}}(z)\big), \end{aligned} \tag{2.7}$$

where $r_{\mathbf{D}}$ is the conformal radius at a point $z \in \mathbf{D}$ is defined by

$$r_{\mathbf{D}}(z) \triangleq \frac{1 - |\Phi(z)|^2}{|\Phi'(z)|}. \tag{2.8}$$

In particular, $\mathcal{R}_{\mathbf{D}}$ is real analytic on $\mathbf{D}$ and, according to [9, 40, 53], one may deduce from a direct computation that $\mathcal{R}_{\mathbf{D}}$ satisfies the Liouville equation

$$\Delta \mathcal{R}_{\mathbf{D}} = 4e^{2\mathcal{R}_{\mathbf{D}}} \quad \text{in } \mathbf{D}. \tag{2.9}$$

Notice that Robin function satisfies the global bounds, see for instance [54]

$$\forall z \in \mathbf{D}, \quad -\log\big(4\delta(z)\big) \leqslant \mathcal{R}_{\mathbf{D}}(z) \leqslant -\log \delta(z), \tag{2.10}$$

with $\delta$ the distance to the boundary $\partial\mathbf{D}$ defined by

$$\delta(z) \triangleq \inf_{w \in \partial\mathbf{D}} |z - w|.$$

We point out that when the domain $\mathbf{D}$ is bounded and convex, the Robin function is strictly convex and admits only one non-degenerate critical point, see [15, 54]. Later, we will need the following identity which follows from a direct differentiation of (2.7)

$$\partial_z \mathcal{R}_{\mathbf{D}}(z) = \frac{1}{2}\frac{\Phi''(z)}{\Phi'(z)} + \frac{\Phi'(z)\overline{\Phi(z)}}{1 - |\Phi(z)|^2}. \tag{2.11}$$

## 2.2 Hamiltonian structure and periodic orbits

In this section we continue to assume that the domain $\mathbf{D}$ is a simply connected bounded domain. The dynamics of a single vortex $\omega = \Gamma\delta_{\xi(t)}$ inside the domain $\mathbf{D}$ is governed by the Robin function, introduced in (2.6), through the Hamiltonian complex equation, see for instance [54, 85],

$$\dot{\xi}(t) = \mathrm{i}\tfrac{\Gamma}{2\pi}\partial_{\overline{z}}\mathcal{R}_{\mathbf{D}}\big(\xi(t)\big). \tag{2.12}$$

Notice that this equation can be recast in terms of a 2d real Hamiltonian system,

$$\dot{\xi}(t) = \nabla^{\perp} H_{\mathbf{D}}\big(\xi(t)\big), \qquad \nabla^{\perp} \triangleq \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix}, \qquad H_{\mathbf{D}} \triangleq \tfrac{\Gamma}{4\pi}\mathcal{R}_{\mathbf{D}}. \tag{2.13}$$

Therefore the trajectories $t \mapsto \xi(t)$ are globally well-defined and lie in the level sets of the Hamiltonian $H_{\mathbf{D}}$. However, exploring critical points, such as their number and structure, together with the shape of level sets within general domains remains a captivating pursuit, with only a sparse collection of results currently available. Notably, as indicated by (2.10), the growth of the Hamiltonian near the boundary implies the existence of at least one critical point. In convex domains, it is shown in [15, 54] that this point is unique and the Hamiltonian is strictly convex. As a by-product, the phase space is foliated by periodic orbits surrounding convex subdomains. The uniqueness of critical points has been extended to more general domains. Actually, if we denote the inverse conformal mapping $F \triangleq \Phi^{-1} : \mathbb{D} \to \mathbf{D}$, then we infer from (2.11) that the critical points $F(\xi)$ to Robin function are given by the $\xi \in \mathbb{D}$ which are the roots of Grakhov's equation [41]

$$\frac{F''(\xi)}{F'(\xi)} = \frac{2\overline{\xi}}{1 - |\xi|^2}. \tag{2.14}$$

According to [23], the uniqueness is established under the Nehari univalence criterion

$$\forall z \in \mathbb{D}, \quad |S(F)(z)| \leqslant \tfrac{2}{(1-|z|^2)^2}, \tag{2.15}$$

where $S$ is the Schwarzian derivative defined by

$$S(F) \triangleq \left(\tfrac{F''}{F'}\right)' - \tfrac{1}{2}\left(\tfrac{F''}{F'}\right)^2 = \tfrac{F^{(3)}}{F'} - \tfrac{3}{2}\left(\tfrac{F''}{F'}\right)^2. \tag{2.16}$$

We emphasize that the condition (2.15) is satisfied by convex domains, see [24, 84]. More results in the same spirit leading to the uniqueness of critical points can be found in [73, 74]. In general domains, a little is known on the critical points. For instance, by invoking Sard theorem we can deduce weak results such as the set of critical values

$$\mathcal{C} \triangleq \big\{ H_{\mathbf{D}}(z) \quad \text{s.t.} \quad \nabla H_{\mathbf{D}}(z) = 0 \big\} \tag{2.17}$$

is compact with zero Lebesgue measure. Furthermore, the range of the Hamiltonian $H_{\mathbf{D}}$ is a connected set of $\mathbb{R}$ and takes the form

$$H_{\mathbf{D}}(\mathbf{D}) \triangleq [\lambda_{\min}, \infty). \tag{2.18}$$

Remark that $\lambda_{\min} \in \mathcal{C}$ and the set $H_{\mathbf{D}}(\mathbf{D}) \setminus \mathcal{C}$ is open. For each regular energy value $\lambda \in H_{\mathbf{D}}(\mathbf{D}) \setminus \mathcal{C}$ all the connected components of $H_{\mathbf{D}}^{-1}(\{\lambda\})$ are periodic orbits diffeomorphic to a circle, see for instance [79, Prop. 2.1]. We denote by $\mathtt{T}(\lambda)$ the minimal period of each periodic orbit $\mathcal{E}_\lambda$. This period can be recovered from area $A(\lambda)$ of the domain enclosed by $\mathcal{E}_\lambda$ as follows

$$\mathtt{T}(\lambda) = \frac{dA(\lambda)}{d\lambda}\cdot$$

In the following proposition we shall collect some basic classical results on planar Hamiltonian dynamical system. They follow from the fact that the Hamiltonian $\mathcal{R}_{\mathbf{D}}$ is real analytic inside the domain $\mathbf{D}$.

**Proposition 2.1.** *Let $[\lambda_*, \lambda^*] \subset H_{\mathbf{D}}(\mathbf{D}) \setminus \mathcal{C}$, and consider a continuous family of periodic orbits $(\mathcal{E}_\lambda)_{\lambda \in [\lambda_*, \lambda^*]}$. Then the following results hold true.*

1. *The period map $\lambda \in [\lambda_*, \lambda^*] \mapsto \mathtt{T}(\lambda)$ is analytic.*
2. *Each periodic orbit $\mathcal{E}_\lambda$ admits an analytic parametrization $s \in \mathbb{T} \mapsto p_\lambda(s)$.*
3. *The map $\lambda \in [\lambda_*, \lambda^*] \mapsto p_\lambda$ is analytic.*

## 2.3 Period asymptotics

The purpose of this section is to analyze the asymptotic behaviors of the period of point vortex orbits in two extreme regimes: near an elliptic critical point and close to the domain boundary. Near an elliptic critical point, the vortex exhibits slow motion, whereas near the boundary, it rotates rapidly.

### 2.3.1 Near an elliptic critical point

Let us consider $\xi_0 \in \mathbf{D}$ an elliptic critical point of the Robin function, such as the point associated with the global minimum. Then

$$\partial_z \mathcal{R}_{\mathbf{D}}(\xi_0) = 0. \tag{2.19}$$

We can always choose (and this is done in a unique way) the conformal mapping so that

$$\Phi(\xi_0) = 0 \qquad \text{and} \qquad \Phi'(\xi_0) > 0.$$

According to (2.11), the condition (2.19) is equivalent to

$$\Phi''(\xi_0) = 0.$$

Recall from (2.12) that the point vortex motion writes

$$\dot{\xi} = \tfrac{\mathrm{i}\Gamma}{2\pi} \partial_{\overline{z}} \mathcal{R}_{\mathbf{D}}(\xi). \tag{2.20}$$

By means of Morse's Lemma, locally the trajectory of the point vortex will be close to an ellipse. Therefore, one can choose a parametrization of the orbit using polar formulation as follows

$$\xi(t) = \xi_0 + \mathtt{R}\big(\lambda, \Theta(t)\big) e^{\mathrm{i}\Theta(t)}, \tag{2.21}$$

where the energy level $\lambda \in H_{\mathbf{D}}(\mathbf{D})$ is taken close to $\lambda_\star \triangleq H_{\mathbf{D}}(\xi_0)$. In particular

$$\mathtt{R}(\lambda_\star, \Theta) = 0.$$

Inserting the ansatz (2.21) into (2.20) gives

$$\dot{\Theta} = \frac{\Gamma}{2\pi \mathtt{R}(\lambda, \Theta)} \mathrm{Re}\big\{(\partial_z \mathcal{R}_{\mathbf{D}})\big(\xi_0 + \mathtt{R}(\lambda, \Theta) e^{\mathrm{i}\Theta}\big) e^{\mathrm{i}\Theta}\big\}.$$

It follows that

$$\mathtt{T}(\lambda) = \frac{2\pi}{\Gamma} \int_0^{2\pi} \frac{\mathtt{R}(\lambda, \Theta)}{\mathrm{Re}\big\{(\partial_\xi \mathcal{R}_{\mathbf{D}})\big(\xi_0 + \mathtt{R}(\lambda, \Theta) e^{\mathrm{i}\Theta}\big) e^{\mathrm{i}\Theta}\big\}} d\Theta.$$

Performing a Taylor expansion, we find

$$\partial_z \mathcal{R}_{\mathbf{D}}\big(\xi_0 + \mathtt{R}\big(\lambda, \Theta(t)\big) e^{\mathrm{i}\Theta(t)}\big) = \mathtt{R}(\lambda, \Theta) e^{\mathrm{i}\Theta} \tfrac{\Phi^{(3)}(\xi_0)}{2\Phi'(\xi_0)} + \mathtt{R}(\lambda, \Theta) e^{-\mathrm{i}\Theta} \big(\Phi'(\xi_0)\big)^2 + O\big(\mathtt{R}^2(\lambda, \Theta)\big).$$

Therefore, we deduce that

$$\begin{aligned}\mathtt{T}(\lambda_\star) \triangleq \lim_{\lambda \to \lambda_\star} \mathtt{T}(\lambda) &= \frac{2\pi}{\Gamma} \int_0^{2\pi} \frac{d\Theta}{\mathtt{a} + |\mathtt{b}| \cos(2\Theta + \arg(\mathtt{b}))} \\ &= \frac{4\pi}{\Gamma} \int_0^{\pi} \frac{d\Theta}{\mathtt{a} + |\mathtt{b}| \cos(\Theta)},\end{aligned}$$

where

$$\mathtt{a} \triangleq \big(\Phi'(\xi_0)\big)^2 \qquad \text{and} \qquad \mathtt{b} \triangleq \frac{\Phi^{(3)}(\xi_0)}{2\Phi'(\xi_0)} \cdot \tag{2.22}$$

Finally, using [50, p. 402] or a direct computation based on the residue Theorem, we find

$$\mathtt{T}(\lambda_\star) = \frac{4\pi^2}{\Gamma \sqrt{\mathtt{a}^2 - |\mathtt{b}|^2}} \cdot \tag{2.23}$$

Notice that this result was established in a different way in [53].

**Remark 2.2.** *1. In* (2.23) *the condition* $\mathtt{a} > |\mathtt{b}|$ *is required. This corresponds to the ellipticity of the critical point* $\xi_0$ *and it is necessary for the dynamics to be confined locally near* $\xi_0$*, as explained in* [31].

*2. A more refined analysis based on the expansion of the orbit leads to the first order expansion*

$$\begin{aligned}\mathtt{T}(\lambda) = \frac{4\pi^2}{\Gamma\sqrt{\mathtt{a}^2 - |\mathtt{b}|^2}} + \frac{16\pi^2(\lambda - \lambda_\star)}{\Gamma(\sqrt{\mathtt{a}^2 - |\mathtt{b}|^2})^7} \Big[ &\tfrac{2}{3}\mathtt{a}^3|\mathtt{c}|^2 + \mathtt{a}|\mathtt{b}|^2|\mathtt{c}|^2 + \tfrac{5}{3}\mathtt{a}\mathrm{Re}\{\mathtt{c}^2\overline{\mathtt{b}}^2\} - \tfrac{3}{2}(\mathtt{a} - |\mathtt{b}|^2)\mathrm{Re}\{\mathtt{d}\overline{\mathtt{b}}^2\} \\ &- \tfrac{3}{2}\mathtt{a}^4|\mathtt{b}|^2 - \mathtt{a}^6 + \tfrac{5}{2}\mathtt{a}^2|\mathtt{b}|^4 \Big] + O\big((\lambda - \lambda_\star)^{\frac{3}{2}}\big),\end{aligned}$$

*where*

$$\mathtt{c} \triangleq \frac{\Phi^{(4)}(\xi_0)}{4\Phi'(\xi_0)}, \qquad \mathtt{d} \triangleq \frac{\Phi^{(5)}(\xi_0)}{12\Phi'(\xi_0)} - \left(\frac{\Phi^{(3)}(\xi_0)}{2\Phi'(\xi_0)}\right)^2.$$

### 2.3.2 Near the boundary

In what follows, we intend to study the asymptotic of $\mathtt{T}(\lambda)$ as $\lambda$ approaches infinity. We will begin with a weaker result, which we believe remains valid even at a significant distance from the boundary.

**Lemma 2.3.** *Let* $\mathbf{D}$ *be a simply-connected bounded domain. Then, all the orbits near the boundary are closed and the period function cannot be constant near the boundary* $\partial\mathbf{D}$.

*Proof.* First, close to the boundary all the orbits are periodic since there is no critical points for Robin function near the boundary. Assume by contraction that the period function is constant on some interval $[\overline{\lambda}, \infty)$. Then from $\mathtt{T} = A'$, we deduce that the area is an affine function in this range of $\lambda$, namely there exists $(\alpha, \beta) \in \mathbb{R}^2$ such that

$$\forall \lambda \in [\overline{\lambda}, \infty) \quad A(\lambda) = \alpha\lambda + \beta.$$

Since $A(\lambda)$ is bounded by the area of $\mathbf{D}$, then taking the limit $\lambda \to \infty$ forces $\alpha = 0$. Hence, the area is constant in $\lambda$. Now, if $\overline{\lambda} \leqslant \lambda_1 < \lambda_2$, then the domain enclosed by the periodic orbit $\mathcal{E}_{\lambda_1}$ is strictly embedded in the domain enclosed by the periodic orbit $\mathcal{E}_{\lambda_2}$, implying in turn $A(\lambda_1) < A(\lambda_2)$. This gives the contradiction. □

We emphasize that the previous argument applies to any simply connected bounded domain without any regularity assumption on the boundary. When the boundary is supposed to be slightly smooth, specifically more regular than $C^1$, then we obtain a more precise estimate of the period. This result, was obtained for instance in [53].

**Proposition 2.4.** *Let $\mathbf{D}$ be a simply-connected bounded domain with boundary of regularity $C^2$. Then, near the boundary, we have the following asymptotic of the period*

$$\mathtt{T}(\lambda) \underset{\lambda\to\infty}{=} \frac{2\pi L_{\partial\mathbf{D}}}{\Gamma} e^{-\frac{4\pi\lambda}{\Gamma}} + O(e^{-\frac{8\pi\lambda}{\Gamma}}), \tag{2.24}$$

*where $L_{\partial\mathbf{D}}$ denotes the length of the boundary $\partial\mathbf{D}$.*

*Proof.* According to [87], the conformal mapping $\mathbf{F} : \mathbb{D} \to \mathbf{D}$ admits a smooth extension to the boundary, still denoted $\mathbf{F}$. In addition,

$$\mathbf{F}(\mathbb{T}) = \partial\mathbf{D}.$$

This allows us to consider the following parametrization of the boundary of $\partial\mathbf{D}$

$$\forall \theta \in \mathbb{T}, \quad X(\theta) \triangleq \mathbf{F}(e^{\mathrm{i}\theta}).$$

Referring to [36, Cor. 14], the Robin function admits the following behavior close to a point $b \in \partial\mathbf{D}$

$$\mathcal{R}_{\mathbf{D}}\big(b - s\nu(b)\big) \underset{s\to 0}{=} -\log(2s) + O(s^2),$$

where $\nu(b)$ denotes the unit outward normal vector to the boundary $\partial\mathbf{D}$ at the point $b$. For a periodic orbit corresponding to the energy level $\lambda$, we have in view of (2.13)

$$s = s(\lambda, b) \underset{\lambda\to\infty}{=} \tfrac{1}{2} e^{-\frac{4\pi\lambda}{\Gamma}} + O(e^{-\frac{8\pi\lambda}{\Gamma}}),$$

uniformly in $b \in \partial\mathbf{D}$. We obtain a parametrization of the periodic orbit by setting

$$\gamma_\lambda(\theta) \triangleq X(\theta) - s\big(\lambda, X(\theta)\big)\frac{\mathrm{i}X'(\theta)}{|X'(\theta)|} = \mathbf{F}(e^{\mathrm{i}\theta}) + s\big(\lambda, X(\theta)\big)\frac{\mathbf{F}'(e^{\mathrm{i}\theta})e^{\mathrm{i}\theta}}{|\mathbf{F}'(e^{\mathrm{i}\theta})|}. \tag{2.25}$$

The Gauss-Green formula gives the area of the enclosed domain

$$A(\lambda) = -\frac{1}{2}\int_0^{2\pi} \mathrm{Im}\left\{\overline{\gamma_\lambda(\theta)}\partial_\theta\gamma_\lambda(\theta)\right\} d\theta. \tag{2.26}$$

Inserting (2.25) into (2.26), we obtain the following asymptotic

$$A(\lambda) \underset{s\to 0}{=} A_{\mathbf{D}} - \tfrac{1}{4}e^{-\frac{4\pi\lambda}{\Gamma}}\mathrm{Im}\left\{\int_0^{2\pi} \overline{F(e^{\mathrm{i}\theta})}\partial_\theta\left(\frac{\mathbf{F}'(e^{\mathrm{i}\theta})e^{\mathrm{i}\theta}}{|\mathbf{F}'(e^{\mathrm{i}\theta})|}\right) + \frac{\overline{\mathbf{F}'(e^{\mathrm{i}\theta})}e^{-\mathrm{i}\theta}}{|\mathbf{F}'(e^{\mathrm{i}\theta})|}\mathrm{i}e^{\mathrm{i}\theta}\mathbf{F}'(e^{\mathrm{i}\theta})d\theta\right\} + O(e^{-\frac{8\pi\lambda}{\Gamma}}),$$

where $A_{\mathbf{D}}$ is the area of the domain $\mathbf{D}$. Integrating by parts yields

$$A(\lambda) \underset{s\to 0}{=} A_{\mathbf{D}} - \tfrac{1}{2}e^{-\frac{4\pi\lambda}{\Gamma}}\int_0^{2\pi}|\mathbf{F}'(e^{\mathrm{i}\theta})|d\theta + O(e^{-\frac{8\pi\lambda}{\Gamma}}).$$

On the other hand, as

$$L_{\partial\mathbf{D}} \triangleq \int_0^{2\pi} |\mathbf{F}'(e^{\mathrm{i}\theta})|d\theta$$

is the length of the boundary $\partial\mathbf{D}$, then we deduce that

$$A(\lambda) \underset{\lambda\to\infty}{=} A_{\mathbf{D}} - \tfrac{1}{2}L_{\partial\mathbf{D}}e^{-\frac{4\pi\lambda}{\Gamma}} + O(e^{-\frac{8\pi\lambda}{\Gamma}}).$$

From this, we derive the following asymptotic of the period,

$$\mathtt{T}(\lambda) \underset{\lambda\to\infty}{=} \frac{2\pi L_{\partial\mathbf{D}}}{\Gamma}e^{-\frac{4\pi\lambda}{\Gamma}} + O(e^{-\frac{8\pi\lambda}{\Gamma}}).$$

This concludes the proof of Proposition 2.4. □

**Remark 2.5.** *The asymptotic* (2.24) *only involves the length* $L_{\partial\mathbf{D}}$. *This suggests that (at least at first order) this asymptotic remains valid for less regular domains, namely only rectifiable.*

# 3 Main results

In this section, we expanded upon the main results partially discussed in the introduction, providing a broader generalization covering more than convex domains. We shall first see how to desingularize a single vortex using the contour dynamics. Afterwards, we will state our main results on the existence of time periodic solutions near the single vortex.

## 3.1 Desingularization of a single vortex

In what follows, we shall consider the motion of a single vortex inside a simply connected bounded domain at a non degenerate energy level $\lambda \in H_{\mathbf{D}}(\mathbf{D})\backslash\mathcal{C}$ of the Hamiltonian $H_{\mathbf{D}}$ as described in Section 2. Without loss of generality, we can make the normalization

$$\Gamma = \pi \tag{3.1}$$

and we assume that the orbit is closed with a minimal period $\mathtt{T}(\lambda)$. We denote by $t \mapsto \xi(t)$ the vortex orbit which satisfies according to (2.12) the Hamiltonian equation

$$\dot{\overline{\xi}}(t) = -\tfrac{\mathrm{i}}{2}\partial_z\mathcal{R}_{\mathbf{D}}\big(\xi(t)\big). \tag{3.2}$$

The goal is to construct $\mathtt{T}(\lambda)$-periodic solution in the vortex patch form whose core follows asymptotically the point vortex motion. Given $\varepsilon \in (0,1)$, we will look for solutions to (1.1) in the form

$$\boldsymbol{\omega}_\varepsilon(t) = \tfrac{1}{\varepsilon^2}\mathbf{1}_{D_t^\varepsilon}, \tag{3.3}$$

where $D_t^\varepsilon$ is given by

$$D_t^\varepsilon \triangleq \varepsilon O_t^\varepsilon + \xi(t)$$

and $O_t^\varepsilon$ is a simply connected domain localized around the unit disc. Due to the mass conservation, one has

$$|D_t^\varepsilon| = |D_0^\varepsilon|.$$

We will normalize the area of the initial patch to $\pi$ so that

$$\lim_{\varepsilon\to 0}\boldsymbol{\omega}_\varepsilon(t) = \pi\delta_{\xi(t)} \quad \text{in} \quad \mathcal{D}'(\mathbf{D}),$$

which is coherent with the choice of the point vortex circulation (3.1). In this particular case, and in view of (2.1) and (2.3), the stream function takes the form

$$\psi(t,z) = \frac{1}{2\pi\varepsilon^2}\int_{D_t^\varepsilon}\log(|z-\zeta|)dA(\zeta) + \frac{1}{2\pi\varepsilon^2}\int_{D_t^\varepsilon}K(z,\zeta)dA(\zeta). \tag{3.4}$$

Let $\gamma(t,\cdot):\mathbb{T}\mapsto\partial O_t^\varepsilon$ be any smooth parametrization of the boundary $\partial O_t^\varepsilon$, that gives in turn a parametrization of the boundary $\partial D_t^\varepsilon$ through

$$w(t,\cdot)\triangleq\varepsilon\gamma(t,\cdot)+\xi(t). \tag{3.5}$$

According to [70, p. 174], the vortex patch equation writes

$$\partial_t w(t,\theta)\cdot\mathbf{n}\big(t,w(t,\theta)\big)+\partial_\theta\big[\psi\big(t,w(t,\theta)\big)\big]=0, \tag{3.6}$$

where $\mathbf{n}(t,\cdot)$ refers to an outward normal vector to the boundary $\partial D_t^\varepsilon$. Note that, from (3.4), we readily get

$$\partial_\theta\big[\psi\big(t,w(t,\theta)\big)\big]=\frac{1}{2\pi\varepsilon^2}\partial_\theta\bigg[\int_{D_t^\varepsilon}\log(|w(t,\theta)-\zeta|)dA(\zeta)+\int_{D_t^\varepsilon}K(w(t,\theta),\zeta)dA(\zeta)\bigg].$$

Therefore, using (3.5) with a suitable change of variable we conclude that

$$\begin{aligned}\partial_\theta\big[\psi\big(t,w(t,\theta)\big)\big]&=\frac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}\log(|\gamma(t,\theta)-\zeta|)dA(\zeta)\\&\quad+\frac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}K\big(\varepsilon\gamma(t,\theta)+\xi(t),\varepsilon\zeta+\xi(t)\big)dA(\zeta).\end{aligned} \tag{3.7}$$

Now, identifying $\mathbb{C}$ with $\mathbb{R}^2$ and making the choice $\mathbf{n}\big(t,w(t,\theta)\big)=-\mathrm{i}\partial_\theta w(t,\theta)$ we get

$$\begin{aligned}\partial_t w(t,\theta)\cdot\mathbf{n}(t,w(t,\theta))&=\mathrm{Im}\Big\{\partial_t\overline{w(t,\theta)}\partial_\theta w(t,\theta)\Big\}\\&=\varepsilon^2\mathrm{Im}\Big\{\partial_t\overline{\gamma(t,\theta)}\partial_\theta\gamma(t,\theta)\Big\}+\varepsilon\mathrm{Im}\Big\{\partial_t\overline{\xi(t)}\partial_\theta\gamma(t,\theta)\Big\}.\end{aligned} \tag{3.8}$$

Thus, combining (3.6), (3.7) and (3.8) we obtain the equation

$$\begin{aligned}&\varepsilon^2\mathrm{Im}\Big\{\partial_t\overline{\gamma(t,\theta)}\partial_\theta\gamma(t,\theta)\Big\}+\varepsilon\mathrm{Im}\Big\{\partial_t\overline{\xi(t)}\partial_\theta\gamma(t,\theta)\Big\}\\&+\frac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}\log(|\gamma(t,\theta)-\zeta|)dA(\zeta)+\frac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}K\big(\varepsilon\gamma(t,\theta)+\xi(t),\varepsilon\zeta+\xi(t)\big)dA(\zeta)=0.\end{aligned} \tag{3.9}$$

Consequently, using (3.2), the relation (3.9) writes

$$\begin{aligned}\varepsilon^2\mathrm{Im}\Big\{\partial_t\overline{\gamma(t,\theta)}\partial_\theta\gamma(t,\theta)\Big\}-\tfrac{\varepsilon}{2}\mathrm{Re}\Big\{\partial_z\mathcal{R}_{\mathbf{D}}\big(\xi(t)\big)\partial_\theta\gamma(t,\theta)\Big\}+\frac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}\log(|\gamma(t,\theta)-\zeta|)dA(\zeta)\\+\frac{1}{2\pi}\partial_\theta\int_{O_t^\varepsilon}K\big(\varepsilon\gamma(t,\theta)+\xi(t),\varepsilon\zeta+\xi(t)\big)dA(\zeta)=0.\end{aligned} \tag{3.10}$$

Recall that we are looking for $\mathtt{T}(\lambda)$-periodic solution whose frequency is

$$\omega(\lambda)\triangleq\frac{2\pi}{\mathtt{T}(\lambda)}\cdot$$

Therefore, we will work with the structure

$$\xi(t)=p\big(\omega(\lambda)t\big),\qquad\gamma(t,\theta)=z\big(\omega(\lambda)t,\theta\big), \tag{3.11}$$

where $p:\mathbb{T}\to\mathbb{C}$ is $2\pi$-periodic function and the boundary is parametrized as

$$\begin{array}{rccl}z(\varphi): & \mathbb{T} & \mapsto & \partial O_\varphi^\varepsilon\\ & \theta & \mapsto & z(\varphi,\theta)\end{array}\qquad\text{with}\qquad\begin{array}{rl}z(\varphi,\theta) & \triangleq R(\varphi,\theta)e^{\mathrm{i}\theta},\\ R(\varphi,\theta) & \triangleq\big(1+2\varepsilon r(\varphi,\theta)\big)^{\frac{1}{2}}.\end{array} \tag{3.12}$$

Thus, using (3.10) and polar coordinates, the curve $z$ satisfies the equation

$$\begin{aligned}&\varepsilon^2\omega(\lambda)\mathrm{Im}\Big\{\partial_\varphi\overline{z(\varphi,\theta)}\partial_\theta z(\varphi,\theta)\Big\}-\tfrac{\varepsilon}{2}\partial_\theta\mathrm{Re}\Big\{\partial_\xi\mathcal{R}_{\mathbf{D}}\big(p(\varphi)\big)R(\varphi,\theta)e^{\mathrm{i}\theta}\Big\}\\&+\partial_\theta\Big[\Psi_1\big(r,z(\varphi,\theta)\big)+\Psi_2\big(r,z(\varphi,\theta)\big)\Big]=0,\end{aligned}$$

where

$$
\begin{aligned}
\Psi_1(r,z) &\triangleq \int_{\mathbb{T}}\int_0^{R(t,\eta)} \log\big(\big|z-le^{\mathrm{i}\eta}\big|\big)ldld\eta,\\
\Psi_2(r,z) &\triangleq \int_{\mathbb{T}}\int_0^{R(\varphi,\eta)} K\big(p(\varphi)+\varepsilon z, p(\varphi)+\varepsilon le^{\mathrm{i}\eta}\big)ldld\eta.
\end{aligned}
\tag{3.13}
$$

Here, we are using the notation

$$
\int_{\mathbb{T}} .. = \frac{1}{2\pi}\int_0^{2\pi} ..
$$

Straightforward computations, using (3.12), lead to

$$
\mathrm{Im}\Big\{\partial_\varphi\overline{z(\varphi,\theta)}\partial_\theta z(\varphi,\theta)\Big\} = \varepsilon\partial_\varphi r(\varphi,\theta).
$$

It follows that the radial deformation $r$, defined through (3.12), satisfies the following nonlinear transport equation,

$$
\begin{aligned}
\mathbf{G}(r)(\varphi,\theta) \triangleq \varepsilon^3\omega(\lambda)\partial_\varphi r(\varphi,\theta) &- \tfrac{\varepsilon}{2}\partial_\theta \mathrm{Re}\Big\{\partial_z\mathcal{R}_{\mathbf{D}}\big(p(\varphi)\big)R(\varphi,\theta)e^{\mathrm{i}\theta}\Big\}\\
&+\partial_\theta\Big[\Psi_1\big(r,z(\varphi,\theta)\big)+\Psi_2\big(r,z(\varphi,\theta)\big)\Big]=0.
\end{aligned}
\tag{3.14}
$$

This is the final form of the contour dynamics equation to which we have to show the existence of solutions for small values of $\varepsilon$.

### 3.2 General statement

In this paragraph, we intend to give our main result from which we can deduce several interesting consequences. For this we aim, we need to introduce some basic objects stemming from the periodic single vortex described by the periodic parametrization $\varphi \in \mathbb{T} \mapsto p(\varphi)$ as in (3.11). Notice that this orbit is associated with the level energy $\lambda$ and it admits a minimal period $\mathtt{T}(\lambda)$. We define the following matrix with $2\pi-$ periodic coefficients,

$$
\mathbb{A}_\lambda(\varphi) = \begin{pmatrix}\mathtt{u}_\lambda(\varphi) & \mathtt{v}_\lambda(\varphi)\\ \mathtt{v}_\lambda(\varphi) & \mathtt{u}_\lambda(\varphi)\end{pmatrix}, \qquad \mathtt{u}_\lambda(\varphi) = -\frac{\mathrm{i}\mathtt{T}(\lambda)}{4\pi}r_{\mathbf{D}}^{-2}\big(p_\lambda(\varphi)\big), \qquad \mathtt{v}_\lambda(\varphi) = \frac{\mathrm{i}\mathtt{T}(\lambda)}{8\pi}\big(\partial_z\mathcal{R}_{\mathbf{D}}\big(p_\lambda(\varphi)\big)\big)^2,
\tag{3.15}
$$

where $\mathcal{R}_{\mathbf{D}}$ denotes the Robin function stated in (2.6) and $r_{\mathbf{D}}$ is the conformal radius defined in (2.8). To this matrix $\mathbb{A}_\lambda$, we associate the fundamental matrix $\mathscr{M}_\lambda$ which satisfies the linear ODE

$$
\partial_\varphi\mathscr{M}_\lambda(\varphi) = \mathbb{A}_\lambda(\varphi)\mathscr{M}_\lambda(\varphi), \qquad \mathscr{M}_\lambda(0) = \mathrm{Id}.
\tag{3.16}
$$

Since $\mathrm{Tr}(\mathbb{A}_\lambda(\varphi)) = 0$, then from Abel's Theorem $\det\mathscr{M}_\lambda(\varphi) = 1$. Hence, one can easily show the equivalence relation on the monodromy matrix $\mathscr{M}_\lambda(2\pi)$

$$
\mathrm{Tr}\big(\mathscr{M}_\lambda(2\pi)\big) \neq 2 \qquad \Longleftrightarrow \qquad 1\notin \mathrm{sp}(\mathscr{M}_\lambda(2\pi)),
$$

where $\mathrm{sp}(\mathscr{M}_\lambda(2\pi))$ denotes the spectrum of the matrix $\mathscr{M}_\lambda(2\pi)$. Now, we are in a position to state our main result.

**Theorem 3.1.** *Let $\mathbf{D}$ be a simply-connected bounded domain in $\mathbb{R}^2$. Let $\lambda_* < \lambda^*$ such that $[\lambda_*,\lambda^*] \subset H_{\mathbf{D}}(\mathbf{D})\setminus\mathcal{C}$, with the definition* (2.17)*. Assume the following conditions.*

1. *Non-degeneracy of the period:*

$$
\min_{\lambda\in[\lambda_*,\lambda^*]}\big|T'(\lambda)\big| > 0.
\tag{3.17}
$$

2. *Spectral assumption on the monodromy matrix:*

$$
\forall\lambda\in[\lambda_*,\lambda^*],\quad 1\notin\mathrm{sp}\big(\mathscr{M}_\lambda(2\pi)\big).
\tag{3.18}
$$

*Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists a Cantor set $\mathscr{C}_\varepsilon \subset [\lambda_*, \lambda^*]$, with*

$$\lim_{\varepsilon \to 0} |\mathscr{C}_\varepsilon| = \lambda^* - \lambda_*,$$

*such that for any $\lambda \in \mathscr{C}_\varepsilon$, we can construct a time-periodic solution to (1.1) in the form*

$$\boldsymbol{\omega}(t) = \frac{1}{\varepsilon^2} \mathbf{1}_{D_t^\varepsilon}, \qquad D_t^\varepsilon = p_\lambda\left(\tfrac{2\pi}{\mathtt{T}(\lambda)} t\right) + \varepsilon O_t^\varepsilon,$$

*with*

$$O_t^\varepsilon = \left\{ \ell e^{\mathrm{i}\theta}, \quad \theta \in [0, 2\pi], \quad 0 \leqslant \ell < \sqrt{1 + \varepsilon r\left(\tfrac{2\pi}{\mathtt{T}(\lambda)} t, \theta\right)} \right\},$$

*where $r \in H^s(\mathbb{T}^2)$ for $s$ large enough and admits the asymptotics*

$$r(\varphi, \theta) = -\varepsilon \mathrm{Re}\left\{ \left[ \Big(\partial_z \mathcal{R}_{\mathbf{D}}\big(p_\lambda(\varphi)\big)\Big)^2 + \tfrac{1}{3} S(\Phi)\big(p_\lambda(\varphi)\big) \right] e^{2\mathrm{i}\theta} \right\} + O(\varepsilon^{1+\mu}),$$

*for some $\mu \in (0,1)$ and $S(\Phi)$ stands for the Schwarzian derivative of the conformal mapping $\Phi : \mathbf{D} \to \mathbb{D}$ defined through* (2.16).

**Remark 3.2.** *We believe that the non-degeneracy of the period in Theorem* 3.1 *holds true for most energy levels. It would be surprising if the period were constant over a non-trivial open interval. Nevertheless, partial results have been obtained in Lemma* 2.3 *and Remark* 2.23 *that support this fact. As to the monodromy condition stated in theorem* 3.1*, it sounds to be generic and in the same time very challenging to check for a given geometry. In Section* 5*, we will provide sufficient conditions to ensure the validity of this condition*

**Corollary 3.3.** *Theorem* 3.1 *holds true under the following assumptions*

1. *The set defined by* (2.17) *contains only one critical point, that is, $\mathcal{C} = \{\xi_0\}$.*
2. *The conformal mapping $F : \mathbb{D} \to \mathbf{D}$ with $F(0) = \xi_0$ satisfies*

$$\left| \tfrac{F^{(3)}(0)}{F'(0)} \right| \notin \left\{ 2\sqrt{1 - \tfrac{1}{n^2}}, \ n \in \mathbb{N}^* \cup \{\infty\} \right\}. \tag{3.19}$$

Notice that this corollary generalizes Corollary 1.3, seen in the introduction, as any bounded convex domain has only one critical point, as discussed in [15, 54]. As we will see in Section 5, assumption 2 in the preceding corollary holds for almost all rectangles and ellipses. Now, we intend to prove Corollary 3.3.

*Proof.* As the domain $\mathbf{D}$ is assumed to be a simply-connected bounded domain with only one critical point. Then, the phase portrait of the single vortex motion is foliated by periodic orbits. More precisely, for any level energy $\lambda > \lambda_{\min}$, see (2.18), the orbit $H_{\mathbf{D}}^{-1}(\{\lambda\})$ is a compact trajectory. Applying Proposition 2.1, we infer that the map $\lambda \in (\lambda_{\min}, \infty) \mapsto \mathtt{T}(\lambda)$ is analytic. On the other hand, Lemma 2.3 asserts that the period map is not constant near the boundary. Therefore the set of zeroes of the map $\lambda \in (\lambda_{\min}, \infty) \mapsto \mathtt{T}'(\lambda)$ is a discrete set denoted by $\mathcal{Z}$. Hence, for any compact interval $[\lambda_*, \lambda^*] \subset (\lambda_{\min}, \infty) \backslash \mathcal{Z}$ we have

$$\min_{\lambda \in [\lambda_*, \lambda^*]} \left| \mathtt{T}'(\lambda) \right| > 0.$$

which guarantees the validity of the assumption (3.17). Next, let us move to the second assumption (3.18). Combining (3.15), (3.16) with Proposition 2.1, we infer that the mapping

$$\psi : \lambda \in (\lambda_{\min}, \infty) \mapsto \mathrm{Tr}\big(S_\lambda(2\pi)\big) - 2 \tag{3.20}$$

is real analytic. Now, we shall show that this map is not identically zero under the assumption 2. of Corollary 3.3. According to (3.15), (2.22), (2.8) and (2.23), we have at the critical point ($\lambda = \lambda_{\min}$),

$$\mathtt{u}_{\lambda_{\min}}(\varphi) = -\frac{\mathrm{i}}{4\pi} \frac{\mathtt{T}(\lambda_{\min})}{r_{\mathbf{D}}^2(\xi_0)} = -\mathrm{i} \frac{\mathtt{a}}{\sqrt{\mathtt{a}^2 - |\mathtt{b}|^2}} \qquad \text{and} \qquad \mathtt{v}_{\lambda_{\min}}(\varphi) = \frac{\mathrm{i}}{2\sqrt{\mathtt{a}^2 - |\mathtt{b}|^2}} \Big(\partial_z \mathcal{R}_{\mathbf{D}}(\xi_0)\Big)^2 = 0,$$

with

$$\mathtt{a}=\frac{1}{|F'(0)|^2}\qquad\text{and}\qquad \mathtt{b}=-\frac{F^{(3)}(0)}{2[F'(0)]^3}.$$

This implies that

$$\mathtt{u}_{\lambda_{\min}}(\varphi)=-\frac{2\mathrm{i}}{\sqrt{4-|S(F)(0)|^2}},\qquad S(F)(0)=\tfrac{F^{(3)}(0)}{F'(0)}$$

and

$$\mathbb{A}_{\lambda_{\min}}=\tfrac{2\mathrm{i}}{\sqrt{4-|S(F)(0)|^2}}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

which is a constant matrix. From the equation (3.16), we obtain

$$\partial_\varphi\mathscr{M}_{\lambda_{\min}}(\varphi)=\mathbb{A}_{\lambda_{\min}}\mathscr{M}_{\lambda_{\min}}(\varphi),\qquad \mathscr{M}_{\lambda_{\min}}(0)=\mathrm{Id}.$$

Consequently,

$$\forall\varphi\in\mathbb{R},\quad \mathscr{M}_{\lambda_{\min}}(\varphi)=e^{\varphi\mathbb{A}_{\lambda_{\min}}}=\begin{pmatrix}e^{-\frac{2\mathrm{i}\varphi}{\sqrt{4-|S(F)(0)|^2}}} & 0\\ 0 & e^{\frac{2\mathrm{i}\varphi}{\sqrt{4-|S(F)(0)|^2}}}\end{pmatrix}.$$

Then taking the trace, we infer

$$\mathrm{Tr}\big(\mathscr{M}_{\lambda_{\min}}(2\pi)\big)=2\cos\Big(\tfrac{4\pi}{\sqrt{4-|S(F)(0)|^2}}\Big).$$

Thus,

$$\mathrm{Tr}\big(\mathscr{M}_{\lambda_{\min}}(2\pi)\big)=2\qquad\Longleftrightarrow\qquad |S(F)(0)|\in\left\{2\sqrt{1-\tfrac{1}{n^2}},\ n\in\mathbb{N}^*\cup\{\infty\}\right\}.$$

Therefore, in view of (3.20), we get

$$\psi(\lambda_{\min})\neq 0\qquad\Longleftrightarrow\qquad |S(F)(0)|\notin\left\{2\sqrt{1-\tfrac{1}{n^2}},\ n\in\mathbb{N}^*\cup\{\infty\}\right\}.\tag{3.21}$$

Hence, the condition of the r.h.s. of (3.21) together with the analyticity of $\psi$ ensures that the set of zeroes of $\psi$ denoted by $\mathcal{Z}'$ is discrete. Hence by imposing $[\lambda_*,\lambda^*]\subset(\lambda_{\min},\infty)\backslash(\mathcal{Z}\cup\mathcal{Z}')$ the conditions (3.17) and (3.18) are satisfied and Theorem 3.1 can be applied. This ends the proof of Corollary 3.3. □

# 4 Unit disc and rigidity

In this section, we will focus on the specific case of the unit disc $\mathbf{D}=\mathbb{D}$. As we shall see below, the conditions (3.17) and (3.18) are satisfied ensuring the applicability of Theorem 3.1. Furthermore, by leveraging the radial symmetry of the disc, we show that it is possible to desingularize all the admissible energy levels without involving Cantor sets, and achieve this through rigid patch motion.

## 4.1 Validity of Theorem 3.1.

We will study the point vortex motion in the unit disc and show that the particle follows a circular orbit. Afterwards, we will investigate the assumptions of Theorem 3.1. First, recall that the conformal mapping here is just the identity. In particular,

$$\Phi_{\mathbb{D}}(z)=z,\qquad \Phi'_{\mathbb{D}}(z)=1,\qquad \Phi''_{\mathbb{D}}(z)=0.\tag{4.1}$$

In addition, from the discussion of Sections 2.1 and 2 we know that the Hamiltonian $H_{\mathbb{D}}$ expresses as

$$H_{\mathbb{D}}(z)=\tfrac14\mathcal{R}_{\mathbb{D}}(z),\qquad \mathcal{R}_{\mathbb{D}}(z)=-\log(1-|z|^2).$$

Therefore, from this structure, we observe that this Hamiltonian admits a unique critical point located at the origin and its range $H_{\mathbb{D}}(\mathbb{D}) = [0, \infty)$. Let us consider a point vortex orbit $\pi\delta_{\xi(t)}$ on an energy level $\lambda > 0$, parametrized as follows

$$\xi(t) = q(t)e^{\mathrm{i}\Theta(t)}.$$

In view of (2.12), the point vortex dynamics is given by

$$\tfrac{\mathrm{i}}{2}\,\overline{\partial_z \mathcal{R}_{\mathbb{D}}\big(\xi(t)\big)} = \dot{\xi}(t) = \big(\dot{q}(t) + \mathrm{i}\dot{\Theta}(t)q(t)\big)e^{\mathrm{i}\Theta(t)}.$$

According to (4.1) and (2.11), we have

$$\partial_z \mathcal{R}_{\mathbb{D}}\big(\xi(t)\big) = \frac{\overline{\xi(t)}}{1 - |\xi(t)|^2} = \frac{q(t)e^{-\mathrm{i}\Theta(t)}}{1 - q^2(t)}.$$

Putting together the previous two identities and identifying real/imaginary parts yield

$$\dot{q}(t) = 0 \qquad \text{and} \qquad \dot{\Theta}(t) = \frac{1}{2(1 - q^2(t))}.$$

Integrating, we infer

$$q \equiv q_0(\lambda) \triangleq \sqrt{1 - e^{-4\lambda}} \qquad \text{and} \qquad \Theta(t) = \omega(\lambda)t, \qquad \omega(\lambda) \triangleq \frac{1}{2\big(1 - q_0^2(\lambda)\big)} = \frac{e^{4\lambda}}{2}.$$

Hence, the orbit is a circle of radius $q_0(\lambda)$ and with minimal period

$$\mathtt{T}(\lambda) = \frac{2\pi}{\omega(\lambda)} = 4\pi e^{-4\lambda}.$$

Notice that this formulae is compatible with the asymptotic (2.24), given that $L_{\partial\mathbb{D}} = 2\pi$ and $\Gamma = \pi$. Clearly the period function is strictly monotone ensuring the condition (3.17) is met. It remains to check the assumption (3.18). Before doing so, we should emphasize that Corollary 3.3 cannot be applied here because the second assumption is not satisfied.
From (3.15) and (3.16), one can check by uniqueness of the Cauchy problem that the fundamental matrix $\mathscr{M}_\lambda$ enjoys the same structure as the generator $\mathbb{A}_\lambda$, that is,

$$\mathscr{M}_\lambda(\varphi) = \begin{pmatrix} a_\lambda(\varphi) & b_\lambda(\varphi) \\ \overline{b_\lambda(\varphi)} & \overline{a_\lambda(\varphi)} \end{pmatrix}, \qquad a_\lambda(0) = 1, \qquad b_\lambda(0) = 0. \tag{4.2}$$

By straightforward computation we get

$$0 = \partial_\varphi \mathscr{M}_\lambda - \mathbb{A}_\lambda \mathscr{M}_\lambda = \begin{pmatrix} a_\lambda' - \mathtt{u}_\lambda a_\lambda - \mathtt{v}_\lambda \overline{b_\lambda} & b_\lambda' - \mathtt{u}_\lambda b_\lambda - \mathtt{v}_\lambda \overline{a_\lambda} \\ \overline{b_\lambda' - \mathtt{u}_\lambda b_\lambda - \mathtt{v}_\lambda \overline{a_\lambda}} & \overline{a_\lambda' - \mathtt{u}_\lambda a_\lambda - \mathtt{v}_\lambda \overline{b_\lambda}} \end{pmatrix},$$

which is equivalent to solve the following system

$$\begin{cases} a' - \mathtt{u}_\lambda a_\lambda - \mathtt{v}_\lambda \overline{b_\lambda} = 0, \\ b_\lambda' - \mathtt{u}_\lambda b_\lambda - \mathtt{v}_\lambda \overline{a_\lambda} = 0. \end{cases}$$

We set

$$U(\varphi) \triangleq a_\lambda(\varphi)W_{\mathtt{u}_\lambda}^-(\varphi), \qquad V(\varphi) \triangleq b_\lambda(\varphi)W_{\mathtt{u}_\lambda}^-(\varphi), \qquad W_{\mathtt{u}_\lambda}^\pm(\varphi) \triangleq e^{\pm\int_0^\varphi \mathtt{u}_\lambda(\tau)d\tau}. \tag{4.3}$$

From the initial dat in (4.2) we deduce that

$$U(0) = 1, \qquad V(0) = 0. \tag{4.4}$$

From straightforward computations we infer

$$U' = \mathtt{v}_\lambda \overline{V} W_{-2\mathrm{iIm}(\mathtt{u}_\lambda)}^{-1}, \qquad V' = \mathtt{v}_\lambda \overline{U} W_{-2\mathrm{iIm}(\mathtt{u}_\lambda)}^{-1},$$

with

$$U'(0)=0,\qquad V'(0)=\frac{\mathrm{i}}{4\omega(\lambda)}\Big(\partial_z\mathcal{R}_{\mathbf{D}}\big(p(0)\big)\Big)^2.$$

Therefore, we deduce that both $U$ and $V$ solve the following homogeneous linear scalar differential equation of second order

$$y''+\Big(2\mathrm{i}\mathrm{Im}(\mathtt{u}_\lambda)-\tfrac{\mathtt{v}_\lambda'}{\mathtt{v}_\lambda}\Big)y'-|\mathtt{v}_\lambda|^2y=0. \tag{4.5}$$

Solving this equation allows to recover the complex coefficients of the fundamental matrix. In the particular case of the unit disc, the computations can be made explicitly, since the entries of the matrix $\mathbb{A}_\lambda(\varphi)$ in (7.88) are

$$\mathtt{u}_\lambda=-\frac{\mathrm{i}}{1-q_0^2(\lambda)}=-\mathrm{i}e^{4\lambda},\qquad \mathtt{v}_\lambda(\varphi)=\mathrm{i}\zeta(\lambda)e^{-2\mathrm{i}\varphi},$$

with

$$\zeta(\lambda)\triangleq\frac{q_0^2(\lambda)}{2(1-q_0^2(\lambda))}=\frac{e^{4\lambda}-1}{2}\in(0,\infty).$$

Inserting this into (4.5), we find that $U$ solves the second order EDO with constant coefficients,

$$y''-4\mathrm{i}\zeta(\lambda)y'-\zeta^2(\lambda)y=0.$$

Solving explicitly, we infer

$$U(\varphi)=\tfrac{\sqrt{3}-2}{2\sqrt{3}}e^{\varphi\mu_+(\lambda)}+\tfrac{\sqrt{3}+2}{2\sqrt{3}}e^{\varphi\mu_-(\lambda)},\qquad \mu_\pm(\lambda)\triangleq\mathrm{i}\zeta(\lambda)(2\pm\sqrt{3}).$$

Coming back to (4.3), we find after simplifications

$$\mathrm{Re}\big(a_\lambda(2\pi)\big)=\cos\big(2\pi\zeta(\lambda)\sqrt{3}\big).$$

Finally,

$$\mathrm{Tr}\big(\mathscr{M}_\lambda(2\pi)\big)=2\mathrm{Re}\big(a_\lambda(2\pi)\big)\neq 2\qquad\Longleftrightarrow\qquad \lambda\notin\Big\{\tfrac14\log\Big(1+\tfrac{2k}{\sqrt{3}}\Big),\ k\in\mathbb{N}^*\Big\}. \tag{4.6}$$

Thus, under the condition

$$[\lambda_*,\lambda^*]\subset(0,\infty)\backslash\Big\{\tfrac14\log\Big(1+\tfrac{2k}{\sqrt{3}}\Big),\ k\in\mathbb{N}^*\Big\},$$

Theorem 3.1 applies. However, as we shall see below, using the radial symmetry of of the unit disc we may derive a better result. In fact, we can remove this restriction on the energy levels and desingularize all the positive energy levels with rigid rotating patches.

## 4.2 Rigid rotation

Now that we have described the periodic orbits of point vortices within the circular domain, we turn our attention to their desingularization into uniformly rotating patch motion. To achieve this, we insert the ansatz

$$\xi(t)=qe^{\mathrm{i}\Omega t}\quad\text{ and }\quad\gamma(t,\theta)=e^{\mathrm{i}\Omega t}\gamma_0(\theta) \tag{4.7}$$

into the equation (3.9), leading to

$$\begin{aligned}&-\varepsilon^2\tfrac{\Omega}{2}\partial_\theta\big(|\gamma_0(\theta)|^2\big)-\varepsilon q\Omega\partial_\theta\mathrm{Re}\big\{\gamma_0(\theta)\big\}+\frac{1}{2\pi}\partial_\theta\int_{O_0^\varepsilon}\log(|\gamma_0(\theta)-\zeta|)dA(\zeta)\\&-\frac{1}{2\pi}\partial_\theta\int_{O_0^\varepsilon}\log\big(\big|1-\big(\varepsilon\overline{\gamma_0(\theta)}+q\big)\big(\varepsilon\zeta+q\big)\big)\big|dA(\zeta)=0.\end{aligned} \tag{4.8}$$

We shall parametrize the boundary of the domain $O_0^\varepsilon$ as follows,

$$\gamma_0(\theta)=R(\theta)e^{\mathrm{i}\theta},\qquad R(\theta)\triangleq\sqrt{1+2\varepsilon qr(\theta)}.$$

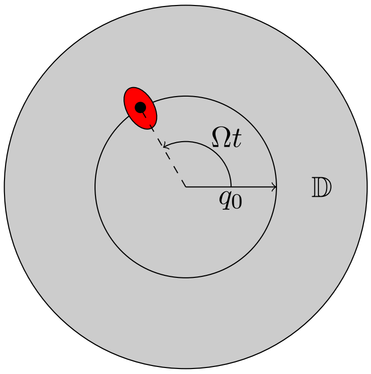


Figure 2: Rigid rotation in the unit disc.

Hence, the contour equation takes the form

$$\begin{aligned}\mathcal{G}(\varepsilon,q,\Omega,r)\triangleq\varepsilon^2\Omega\partial_\theta r(\theta)+\Omega\partial_\theta\big[R(\theta)\cos(\theta)\big]-\frac{1}{2\pi\varepsilon q}\partial_\theta\int_{O_0^\varepsilon}\log(|R(\theta)e^{\mathrm{i}\theta}-\zeta|)dA(\zeta)\\+\frac{1}{2\pi\varepsilon q}\partial_\theta\int_{O_0^\varepsilon}\log\left(\left|1-\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon\zeta+q\big)\right|\right)dA(\zeta)=0.\end{aligned}\tag{4.9}$$

The goal is to establish solutions to this nonlinear equation. To achieve this, we plan to apply the implicit function theorem within appropriate function spaces. Specifically, we will work with Hölder spaces, given for $s\in(0,1)$ by

$$\mathbb{Y}^s_{\mathrm{odd}}\triangleq\left\{r:\mathbb{T}\to\mathbb{R}\quad\text{s.t.}\quad\forall\theta\in\mathbb{T},\ r(\theta)=\sum_{n=1}^{\infty}\alpha_n\sin(n\theta),\quad\alpha_n\in\mathbb{R},\quad\|r\|_{C^s(\mathbb{T})}<\infty\right\}$$

equipped with the norm

$$\|r\|_{C^s(\mathbb{T})}\triangleq\|r\|_{L^\infty(\mathbb{T})}+\sup_{\substack{(\theta_1,\theta_2)\in\mathbb{T}^2\\ \theta_1\neq\theta_2}}\frac{|r(\theta_1)-r(\theta_2)|}{|\theta_1-\theta_2|^s}$$

and

$$\mathbb{X}^{1,s}_{\mathrm{even}}\triangleq\left\{r:\mathbb{T}\to\mathbb{R}\quad\text{s.t.}\quad\forall\theta\in\mathbb{T},\ r(\theta)=\sum_{n=1}^{\infty}\alpha_n\cos(n\theta),\quad\alpha_n\in\mathbb{R},\quad\|r\|_{C^{1,s}(\mathbb{T})}<\infty\right\},$$

endowed with the norm

$$\|r\|_{C^{1,s}(\mathbb{T})}\triangleq\|r\|_{L^\infty(\mathbb{T})}+\|r'\|_{C^s(\mathbb{T})}.$$

For any $a>0$, we define the open ball

$$B_{\mathbb{X}^{1,s}_{\mathrm{even}}}(a)\triangleq\left\{r\in\mathbb{X}^{1,s}_{\mathrm{even}}\quad\text{s.t.}\quad\|r\|_{C^{1,s}(\mathbb{T})}<a\right\}.$$

Now, we state the main result of this section.

**Theorem 4.1.** *Let $s\in(0,1)$. For any $\delta\in(0,1)$, there exist positive numbers $\varepsilon_0,a_0>0$, such that*

*(i) The functional $\mathcal{G}:(-\varepsilon_0,\varepsilon_0)\times(-1+\delta,1-\delta)\times\mathbb{R}\times B_{\mathbb{X}^{1,s}_{\mathrm{even}}}(a_0)\to\mathbb{Y}^s_{\mathrm{odd}}$ is well-defined and of class $C^1$.*

*(ii) We have the equivalence*

$$\forall q\in(-1+\delta,1-\delta),\quad\mathcal{G}(0,q,\Omega,0)=0\qquad\Leftrightarrow\qquad\Omega=\Omega_0\triangleq\frac{1}{2(1-q^2)}.$$

*(iii) The linear operator* $d_{(\Omega,r)}\mathcal{G}(0,q,\Omega_0,0):\mathbb{R}\times\mathbb{X}^{1,s}_{\text{even}}\to\mathbb{Y}^s_{\text{odd}}$ *is an isomorphism.*

*(iv) There exist* $C^1-$*functions*

$$\Omega:(-\varepsilon_0,\varepsilon_0)\times(-1+\delta,1-\delta)\to\mathbb{R}\quad\text{and}\quad r:(-\varepsilon_0,\varepsilon_0)\times(-1+\delta,1-\delta)\to B_{\mathbb{X}^{1,s}_{\text{even}}}(a_0)$$

*satisfying*

$$\Omega(0,q)=\Omega_0\qquad\text{and}\qquad r(0,q)=0,$$

*such that*

$$\forall(\varepsilon,q)\in(-\varepsilon_0,\varepsilon_0)\times(-1+\delta,1-\delta),\quad \mathcal{G}\big(\varepsilon,q,\Omega(\varepsilon,q),r(\varepsilon,q)\big)=0.$$

Before giving the proof, we shall make some comments.

**Remark 4.2.** *1. Following exactly the same lines as [61, Sec. 5.4], we actually can show that the boundary of the uniformly rotating vortex patches inside the unit disc is analytic.*

*2. Observe that*

$$\Omega(0,0)=\tfrac{1}{2}\cdot\tag{4.10}$$

*Now, according to [28], in the case* $\overline{q}_0=0$*, the uniformly rotating solutions with amplitude* 1 *bifurcate from the disc of radius* $\varepsilon\in(0,1)$ *at the values*

$$\Omega_n=\frac{n-1+\varepsilon^{2n}}{2n}\cdot$$

*Thus, for a disc with amplitude* $\frac{1}{\varepsilon^2}$ *the spectrum is given by*

$$\Omega_n=\frac{n-1+\varepsilon^{2n}}{2n\varepsilon^2},$$

*which converges when* $\varepsilon\to 0$ *only for* $n=1$ *(corresponding to* 1*-fold solutions) and the limiting value is* $\frac{1}{2}$*, which is consistant with* (4.10) *and Theorem 4.1.*

*Proof.* $(i)$ We denote

$$\mathcal{I}_1(\varepsilon,q,r)(\theta)\triangleq\frac{1}{2\pi\varepsilon q}\partial_\theta\int_{O_0^\varepsilon}\log(|R(\theta)e^{\mathrm{i}\theta}-\zeta|)dA(\zeta),$$
$$\mathcal{I}_2(\varepsilon,q,r)(\theta)\triangleq\frac{1}{2\pi\varepsilon q}\partial_\theta\int_{O_0^\varepsilon}\log\left(\left|1-\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon\zeta+q\big)\right|\right)dA(\zeta).$$

According to (4.9), it suffices to study the well-posedness and regularity of the terms $\mathcal{I}_1$ and $\mathcal{I}_2$, the other terms being obvious. In view of [62, Lemma 2.1] we may write

$$\mathcal{I}_1(\varepsilon,q,r)(\theta)=\frac{1}{\varepsilon q}\int_{\mathbb{T}}\log(|R(\theta)e^{\mathrm{i}\theta}-R(\eta)e^{\mathrm{i}\eta}|)\partial^2_{\theta\eta}\big[R(\theta)R(\eta)\sin(\theta-\eta)\big]d\eta\tag{4.11}$$

and

$$\begin{aligned}\mathcal{I}_2(\varepsilon,q,r)(\theta)=&-\frac{1}{2}\partial_\theta r(\theta)\int_{\mathbb{T}}\tfrac{R^2(\eta)}{R^2(\theta)}d\eta\\&-\frac{1}{2\varepsilon q}\int_{\mathbb{T}}\log\left(\left|1-\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon R(\eta)e^{\mathrm{i}\eta}+q\big)\right|\right)\partial^2_{\theta\eta}\Big[\tfrac{R(\eta)}{R(\theta)}\sin(\theta-\eta)\Big]d\eta.\end{aligned}\tag{4.12}$$

First, we mention that the regularity with respect to the variable $r$ and the parity property of $\mathcal{I}_1$ have already been studied in previous works, see [66, Sec. 3.3]. As for $\mathcal{I}_2$, since $\overline{q}_0<1$, then taking $\varepsilon_0$ and $a_0$ small enough, the integrand inside (4.12) is not singular. This gives the desired regularity in $r$ with respect to the desired function spaces (loss of only one derivative). For the parity property, it is obtained by immediate changes of variables. Thus, it remains to proves that the quantities (4.11)-(4.12) are actually not singular in $\varepsilon q$. From the identities

$$\log(|R(\theta)e^{\mathrm{i}\theta}-R(\eta)e^{\mathrm{i}\eta}|)=\log\big(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|\big)+\log\big(|1+\tfrac{(R(\theta)-1)e^{\mathrm{i}\theta}-(R(\eta)-1)e^{\mathrm{i}\eta}}{e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}}|\big),$$
$$\forall|z|<1,\quad \log|1+z|=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\mathrm{Re}\big\{z^k\big\},$$

we obtain the following decomposition

$$\mathcal{I}_1=\mathcal{I}_{1,1}+\mathcal{I}_{1,2}+\mathcal{I}_{1,3},\tag{4.13}$$

with

$$\begin{aligned}
\mathcal{I}_{1,1}(\varepsilon,q)(\theta)&\triangleq\frac{1}{\varepsilon q}\int_{\mathbb{T}}\log(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|)\sin(\theta-\eta)d\eta,\\
\mathcal{I}_{1,2}(\varepsilon,q,r)(\theta)&\triangleq\frac{1}{\varepsilon q}\int_{\mathbb{T}}\log(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|)\partial_{\theta\eta}^2\big[\big(R(\theta)R(\eta)-1\big)\sin(\theta-\eta)\big]d\eta,\\
\mathcal{I}_{1,3}(\varepsilon,q,r)(\theta)&\triangleq\frac{1}{\varepsilon q}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\int_{\mathbb{T}}\mathrm{Re}\Big\{\Big(\tfrac{(R(\theta)-1)e^{\mathrm{i}\theta}-(R(\eta)-1)e^{\mathrm{i}\eta}}{e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}}\Big)^k\Big\}\partial_{\theta\eta}^2\big[R(\theta)R(\eta)\sin(\theta-\eta)\big]d\eta.
\end{aligned}\tag{4.14}$$

First observe that by parity argument,

$$\mathcal{I}_{1,1}(\varepsilon,q,r)=0.\tag{4.15}$$

Moreover, expanding the second term we get

$$\begin{aligned}
\mathcal{I}_{1,2}(\varepsilon,q,r)(\theta)&=\varepsilon q\int_{\mathbb{T}}\log(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|)\frac{\partial_\theta r(\theta)\partial_\eta r(\eta)}{R(\theta)R(\eta)}\sin(\theta-\eta)d\eta\\
&\quad+\int_{\mathbb{T}}\log(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|)\left(\frac{\partial_\theta r(\theta)}{R(\theta)}R(\eta)-\frac{\partial_\eta r(\eta)}{R(\eta)}R(\theta)\right)\cos(\theta-\eta)d\eta\\
&\quad+\frac{1}{\varepsilon q}\int_{\mathbb{T}}\log\big(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|\big)\big(R(\theta)R(\eta)-1\big)\sin(\theta-\eta)d\eta.
\end{aligned}$$

Remark that one may write

$$R(\theta)R(\eta)-1=\big(R(\theta)-1\big)\big(R(\eta)-1\big)+\big(R(\theta)-1\big)+\big(R(\eta)-1\big)$$

and, by Taylor formula,

$$R(\theta)-1=\varepsilon q\int_0^1\frac{(\partial_\theta r)(s\theta)}{R(s\theta)}ds.\tag{4.16}$$

Therefore, we get

$$\begin{aligned}
\mathcal{I}_{1,2}(\varepsilon,q,r)(\theta)&=\varepsilon q\int_{\mathbb{T}}\log(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|)\frac{\partial_\theta r(\theta)\partial_\eta r(\eta)}{R(\theta)R(\eta)}\sin(\theta-\eta)d\eta\\
&\quad+\int_{\mathbb{T}}\log(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|)\left(\frac{\partial_\theta r(\theta)}{R(\theta)}R(\eta)-\frac{\partial_\eta r(\eta)}{R(\eta)}R(\theta)\right)\cos(\theta-\eta)d\eta\\
&\quad+\varepsilon q\int_{\mathbb{T}}\int_0^1\int_0^1\log\big(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|\big)\frac{(\partial_\theta r)(s_1\theta)}{R(s_1\theta)}\frac{(\partial_\eta r)(s_2\eta)}{R(s_2\eta)}\sin(\theta-\eta)ds_1ds_2d\eta\\
&\quad+\int_{\mathbb{T}}\int_0^1\log\big(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|\big)\frac{(\partial_\theta r)(s_1\theta)}{R(s_1\theta)}\sin(\theta-\eta)ds_1d\eta\\
&\quad+\int_{\mathbb{T}}\int_0^1\log\big(|e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}|\big)\frac{(\partial_\eta r)(s_2\eta)}{R(s_2\eta)}\sin(\theta-\eta)ds_2d\eta.
\end{aligned}\tag{4.17}$$

This implies that $\mathcal{I}_{1,2}$ is not singular in $\varepsilon q$ and actually smooth with respect to $\varepsilon$ and $q$. In addition, one obtains from (4.17) that

$$\mathcal{I}_{1,2}(0,q,0)=0.\tag{4.18}$$

As for the last term in (4.13) we use the identity

$$\begin{aligned}\mathrm{Re}\Big\{\Big(\tfrac{(R(\theta)-1)e^{\mathrm{i}\theta}-(R(\eta)-1)e^{\mathrm{i}\eta}}{e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}}\Big)^k\Big\} &= \mathrm{Re}\Big\{\Big(R(\theta)-1+\tfrac{R(\theta)-R(\eta)}{e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}}e^{\mathrm{i}\eta}\Big)^k\Big\}\\ &= \mathrm{Re}\Big\{\Big(R(\theta)-1+\tfrac{R(\theta)-R(\eta)}{2\mathrm{i}\sin(\frac{\theta-\eta}{2})}e^{\mathrm{i}\frac{\eta-\theta}{2}}\Big)^k\Big\}.\end{aligned}$$

Using one more time Taylor formula, we can write

$$\begin{aligned}R(\theta)-R(\eta) &= \varepsilon q\big(r(\theta)-r(\eta)\big)\mathtt{I}(\varepsilon,q,r,\theta,\eta),\\ \mathtt{I}(\varepsilon,q,r,\theta,\eta) &\triangleq \int_0^1 \frac{1}{\sqrt{1+2\varepsilon q\,(r(\eta)+s(r(\theta)-r(\eta)))}}ds.\end{aligned} \tag{4.19}$$

Combining (4.16), (4.19) and (4.14), we infer

$$\begin{aligned}&\mathcal{I}_{1,3}(\varepsilon,q,r)(\theta)\\ &=\sum_{k=1}^{\infty}\tfrac{(-1)^{k+1}}{k}(\varepsilon q)^{k-1}\int_{\mathbb{T}}\mathrm{Re}\left\{\left(\int_0^1\tfrac{(\partial_\theta r)(s\theta)}{R(s\theta)}ds+\tfrac{r(\theta)-r(\eta)}{2\mathrm{i}\sin(\frac{\theta-\eta}{2})}\mathtt{I}(\varepsilon,q,r,\theta,\eta)\right)^k\right\}\partial^2_{\theta\eta}\big[R(\theta)R(\eta)\sin(\theta-\eta)\big]d\eta.\end{aligned}$$

Therefore, $\mathcal{I}_{1,3}$ is not singular in $\varepsilon q$ and actually smooth with respect to $\varepsilon$ and $q$. In addition,

$$\mathcal{I}_{1,3}(0,q,0)=0. \tag{4.20}$$

The previous calculations show that $\mathcal{I}_1$ is not singular in $\varepsilon q$ and can actually been prolongated in $\varepsilon$ on an interval of the form $(-\varepsilon_0,\varepsilon_0)$ and in $q$ in an interval of the form $(\overline{q}_0-a_0,\overline{q}_0+a_0)$. Moreover, gathering (4.13), (4.15), (4.18) and (4.20) gives

$$\mathcal{I}_1(0,q,0)=0. \tag{4.21}$$

For $\mathcal{I}_2(\varepsilon,q,r)$ we write

$$\mathcal{I}_2=\mathcal{I}_{2,1}+\mathcal{I}_{2,2}+\mathcal{I}_{2,3},$$

with

$$\begin{aligned}\mathcal{I}_{2,1}(r)(\theta) &\triangleq -\frac{1}{2}\frac{\partial_\theta r(\theta)}{R^2(\theta)},\\ \mathcal{I}_{2,2}(\varepsilon,q,r)(\theta) &\triangleq \frac{1}{\varepsilon q}\sum_{k=1}^{\infty}\frac{1}{k}\int_{\mathbb{T}}\mathrm{Re}\Big\{\Big(\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon R(\eta)e^{\mathrm{i}\eta}+q\big)\Big)^k\Big\}\sin(\theta-\eta)d\eta,\\ \mathcal{I}_{2,3}(\varepsilon,q,r)(\theta) &\triangleq \frac{1}{\varepsilon q}\sum_{k=1}^{\infty}\frac{1}{k}\int_{\mathbb{T}}\mathrm{Re}\Big\{\Big(\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon R(\eta)e^{\mathrm{i}\eta}+q\big)\Big)^k\Big\}\partial^2_{\theta\eta}\Big[\Big(\tfrac{R(\eta)}{R(\theta)}-1\Big)\sin(\theta-\eta)\Big]d\eta.\end{aligned}$$

First observe that

$$\mathcal{I}_{2,1}(0)=0. \tag{4.22}$$

Notice that for any $k\in\mathbb{N}^*$,

$$\begin{aligned}\mathcal{J}_k(\varepsilon,q,r)(\theta) &\triangleq \frac{1}{\varepsilon q}\int_{\mathbb{T}}\mathrm{Re}\Big\{\Big(\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon R(\eta)e^{\mathrm{i}\eta}+q\big)\Big)^k\Big\}\sin(\theta-\eta)d\eta\\ &=\frac{1}{\varepsilon q}\sum_{n=0}^{k}\sum_{m=0}^{k}\binom{k}{n}\binom{k}{m}q^{2k-m-n}\varepsilon^{n+m}R^n(\theta)\mathrm{Re}\Big\{e^{-\mathrm{i}n\theta}\int_{\mathbb{T}}R^m(\eta)e^{\mathrm{i}m\eta}\sin(\theta-\eta)d\eta\Big\}\\ &=\frac{1}{\varepsilon q}\sum_{n=0}^{k}\sum_{m=0}^{k}\binom{k}{n}\binom{k}{m}q^{2k-m-n}\varepsilon^{n+m}R^n(\theta)\mathrm{Re}\Big\{e^{-\mathrm{i}n\theta}\int_{\mathbb{T}}e^{\mathrm{i}m\eta}\sin(\theta-\eta)d\eta\Big\}\\ &\quad+\frac{1}{\varepsilon q}\sum_{n=0}^{k}\sum_{m=0}^{k}\binom{k}{n}\binom{k}{m}q^{2k-m-n}\varepsilon^{n+m}R^n(\theta)\mathrm{Re}\Big\{e^{-\mathrm{i}n\theta}\int_{\mathbb{T}}(R^m(\eta)-1)e^{\mathrm{i}m\eta}\sin(\theta-\eta)d\eta\Big\}.\end{aligned}$$

Also, straightforward computations give

$$\mathrm{Re}\Big\{e^{-\mathrm{i}n\theta}\int_{\mathbb{T}}e^{\mathrm{i}m\eta}\sin(\theta-\eta)d\eta\Big\}=\begin{cases}0, & \text{if}\quad m\neq 1,\\ -\frac{1}{2}\sin\big((n-1)\theta\big), & \text{if}\quad m=1.\end{cases}$$

Therefore,

$$\begin{aligned}\mathcal{J}_k(\varepsilon,q,r)(\theta)&=\frac{k}{2}q^{2(k-1)}\sin(\theta)+\mathcal{J}_{k,1}(\varepsilon,q,r)(\theta),\\ \mathcal{J}_{k,1}(\varepsilon,q,r)(\theta)&\triangleq-\frac{k}{2}\sum_{n=2}^{k}\binom{k}{n}q^{2(k-1)-n}\varepsilon^nR^n(\theta)\sin\big((n-1)\theta\big)\\ &\quad+\frac{1}{\varepsilon q}\sum_{n=0}^{k}\sum_{m=0}^{k}\binom{k}{n}\binom{k}{m}q^{2k-m-n}\varepsilon^{n+m}R^n(\theta)\mathrm{Re}\Big\{e^{-\mathrm{i}n\theta}\int_{\mathbb{T}}(R^m(\eta)-1)e^{\mathrm{i}m\eta}\sin(\theta-\eta)d\eta\Big\}.\end{aligned}$$

Observe that

$$R^m(\eta)-1=\big(R(\eta)-1\big)\sum_{\ell=0}^{m-1}R^{m-1-\ell}(\eta).$$

Hence, making appeal to (4.16), we infer

$$\begin{aligned}\mathcal{J}_{k,1}(\varepsilon,q,r)(\theta)&=-\frac{k}{2}\sum_{n=2}^{k}\binom{k}{n}q^{2(k-1)-n}\varepsilon^nR^n(\theta)\sin\big((n-1)\theta\big)\\ &+\sum_{n=0}^{k}\sum_{m=0}^{k}\sum_{\ell=0}^{m-1}\binom{k}{n}\binom{k}{m}q^{2k-m-n}\varepsilon^{n+m}R^n(\theta)\mathrm{Re}\Big\{e^{-\mathrm{i}n\theta}\int_{\mathbb{T}}\int_0^1\tfrac{(\partial_\eta r)(s\eta)}{R(s\eta)}dsR^{m-1-\ell}(\eta)e^{\mathrm{i}m\eta}\sin(\theta-\eta)d\eta\Big\}.\end{aligned}$$

We have removed the singularity and it is immediate that

$$\mathcal{J}_{k,1}(0,q,0)=0. \tag{4.23}$$

Thus, we have the following decomposition

$$\begin{aligned}\mathcal{I}_{2,2}(\varepsilon,q,r)(\theta)&=\tfrac{1}{2}\sum_{k=1}^{\infty}q^{2(k-1)}\sin(\theta)+\sum_{k=1}^{\infty}\tfrac{1}{k}\mathcal{J}_{k,1}(\varepsilon,q,r)(\theta)\\ &\triangleq\frac{1}{2(1-q^2)}\sin(\theta)+\widetilde{\mathcal{I}}_{2,2}(\varepsilon,q,r)(\theta).\end{aligned}$$

From what preceeds, $\mathcal{I}_{2,2}$ and $\widetilde{\mathcal{I}}_{2,2}$ are smooth in $\varepsilon$ and $q$. Moreover, (4.23) implies

$$\widetilde{\mathcal{I}}_{2,2}(0,q,0)=0. \tag{4.24}$$

Besides, another use of (4.16) yields

$$\begin{aligned}\mathcal{I}_{2,3}(\varepsilon,q,r)(\theta)&=\frac{1}{\varepsilon q}\sum_{k=1}^{\infty}\frac{1}{k}\int_{\mathbb{T}}\mathrm{Re}\Big\{\Big(\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon R(\eta)e^{\mathrm{i}\eta}+q\big)\Big)^k\Big\}\partial^2_{\theta\eta}\Big[\Big(\tfrac{R(\eta)}{R(\theta)}-1\Big)\sin(\theta-\eta)\Big]d\eta\\ &=-\sum_{k=1}^{\infty}\frac{1}{k}\int_{\mathbb{T}}\mathrm{Re}\Big\{\Big(\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon R(\eta)e^{\mathrm{i}\eta}+q\big)\Big)^k\Big\}\partial_\theta\Big[\int_0^1\tfrac{(\partial_\eta r)(s\eta)}{R(\theta)R(s\eta)}ds\cos(\theta-\eta)\Big]d\eta\\ &\quad+\sum_{k=1}^{\infty}\frac{1}{k}\int_{\mathbb{T}}\mathrm{Re}\Big\{\Big(\big(\varepsilon R(\theta)e^{-\mathrm{i}\theta}+q\big)\big(\varepsilon R(\eta)e^{\mathrm{i}\eta}+q\big)\Big)^k\Big\}\partial_\theta\Big[\tfrac{(\partial_\eta r)(\eta)}{R(\theta)R(\eta)}\sin(\theta-\eta)\Big]d\eta.\end{aligned}$$

One readily sees that $\mathcal{I}_{2,3}$ is smooth and that

$$\mathcal{I}_{2,3}(0,q,0)=0. \tag{4.25}$$

Combining the foregoing calculations, we find that

$$\mathcal{G}(\varepsilon,q,\Omega,r)=\varepsilon^2\Omega\partial_\theta r(\theta)+\Omega\partial_\theta\big[R(\theta)\cos(\theta)\big]+\frac{1}{2(1-q^2)}\sin(\theta)+\mathcal{I}_3(\varepsilon,q,r)(\theta), \tag{4.26}$$

where

$$\mathcal{I}_3\triangleq\mathcal{I}_1+\mathcal{I}_{2,1}+\widetilde{\mathcal{I}}_{2,2}+\mathcal{I}_{2,3}$$

is a smooth function. Finally, putting together (4.21), (4.22), (4.24) and (4.25), we get

$$\mathcal{I}_3(0,q,0)=0. \tag{4.27}$$

*(ii)* From (4.26) and (4.27), we find

$$\mathcal{G}(0,q,\Omega,0)=-\Big[\Omega-\tfrac{1}{2(1-q^2)}\Big]\sin(\theta). \tag{4.28}$$

This gives the desired equivalence.

*(iii)* The linearized operator of $\mathcal{G}$ with respect to $(\Omega,r)$ at $(\varepsilon,\Omega,r)=(0,\Omega_0,0)$ is given by

$$d_{(\Omega,r)}\mathcal{G}(0,q,\Omega_0,0)[(\widehat{\Omega},h)]=-\widehat{\Omega}\sin(\theta)-\tfrac{1}{2}\big[\partial_\theta-\mathbf{H}\big]h(\theta).$$

Given $g\in\mathbb{Y}^s_{\textnormal{odd}}$ in the form

$$g(\theta)=\sum_{n=1}^{\infty}g_n\sin(n\theta),\quad g_n\in\mathbb{R}.$$

Then, we choose

$$\widehat{\Omega}=-g_1\qquad\text{and}\qquad\forall n\geqslant 2,\quad h_n=-\frac{g_n}{nB_n},\qquad B_n\triangleq\frac{n-1}{2n}$$

so that setting

$$h(\theta)=\sum_{n=2}^{\infty}h_n\cos(n\theta),$$

we find

$$d_{(\Omega,r)}\mathcal{G}(0,q,\Omega_0,0)[(\widehat{\Omega},h)]=g.$$

Moreover, since for any $n\geqslant 2$, we have $B_n\in(\frac14,\frac12)$, we immediately get from Cauchy-Schwarz and Bessel inequalities

$$\|h\|_{L^\infty(\mathbb{T})}\lesssim\sum_{n=2}^{\infty}\frac{|g_n|}{n}\lesssim\|g\|_{L^2(\mathbb{T})}\lesssim\|g\|_{C^s(\mathbb{T})}.$$

We can also write

$$h'(\theta)=\sum_{n=2}^{\infty}\frac{g_n}{B_n}\sin(n\theta)=(\widetilde{g}*g)(\theta)+2\big[g(\theta)-g_1\sin(\theta)\big],$$

where

$$\widetilde{g}(\theta)\triangleq-\sum_{n=2}^{\infty}\frac{2}{n-1}\cos(n\theta),\qquad(\widetilde{g}*g)(\theta)\triangleq 2\int_{\mathbb{T}}\widetilde{g}(\theta-\eta)g(\eta)d\eta.$$

Notice that $\widetilde{g}\in L^2(\mathbb{T})\subset L^1(\mathbb{T})$. Thus, using that $L^1(\mathbb{T})*C^s(\mathbb{T})\to C^s(\mathbb{T})$, we obtain $\widetilde{g}*g\in C^s(\mathbb{T})$. Hence $h'\in C^s(\mathbb{T})$. This proves that $h\in\mathbb{X}^{1,s}_{\textnormal{even}}$ and therefore that $d_{(\Omega,r)}\mathcal{G}(0,q,\Omega_0,0):\mathbb{X}^{1,s}_{\textnormal{even}}\to\mathbb{Y}^s_{\textnormal{odd}}$ is an isomorphism.

*(iv)* It follows from the previous points by applying the Implicit Function Theorem.
The proof of Theorem 4.1 is now complete. □

# 5 Applications of Corollary 3.3

In this section, we focus on the application of Corollary 3.3 to specific domains. Since the domains we will consider are convex, the first assumption of this corollary is automatically satisfied. Therefore, we only need to verify the second assumption in Corollary 3.3 for the admissibility of these domains. This condition requires knowledge of the critical point and the conformal mapping. We have two strategies to check this condition. The first strategy involves providing the conformal mapping that satisfies the required constraint, from which we can derive the geometry. The second strategy involves defining the geometry first and then verifying the constraint. This second approach that we will develop here, which is particularly challenging due to the complex structure of the conformal mapping, even for simple geometries. We will focus on ellipses, rectangles, and more generally, polygonal domains. All these examples are convex bounded domains making Corollary 3.3 applicable whenever its second assumption holds true, that is,

$$\left|\tfrac{F^{(3)}(0)}{F'(0)}\right| \notin \left\{2\sqrt{1-\tfrac{1}{n^2}},\ n\in\mathbb{N}^*\right\}. \tag{5.1}$$

with $F:\mathbb{D}\to\mathbf{D}$ the conformal mapping such that $F(0)=\xi_0$, is the critical point of the Robin function. In terms of the conformal mapping $\Phi=F^{-1}:\mathbf{D}\to\mathbb{D}$, this condition is equivalent to

$$\left|\tfrac{\Phi^{(3)}(0)}{(\Phi'(0))^3}\right| \notin \left\{2\sqrt{1-\tfrac{1}{n^2}},\ n\in\mathbb{N}^*\right\}. \tag{5.2}$$

To check one of these conditions, we need to get access to the conformal mapping structures.

## 5.1 Elliptic domains

We want to check the validity of the Corollary 3.3 with one of the most simplest domain shapes, namely ellipses. As we have mentioned before, since ellipses are convex, the first assumption of this corollary is automatically satisfied. Thus it remains to check the second assumption reformulated in (5.1). Consider $\mathbf{D}=\mathbf{E}_a$ the domain delimited by the renormalized ellipse with semi-axis $a>1$ and $b=1$,

$$\mathbf{E}_a \triangleq \left\{(x,y)\in\mathbb{R}^2 \quad \text{s.t.} \quad \tfrac{x^2}{a^2}+y^2<1\right\}.$$

Using the symmetry of this ellipse, we can show that the critical point of Robin function will be given by $\xi_0=0$. On the other hand, it is well known [72] (see also [83, p. 296], that the conformal mapping $\Phi:\mathbf{E}_a\to\mathbb{D}$ is given by

$$\Phi(z)=\sqrt{k}\,\mathtt{sn}\left(\frac{2K(k)}{\pi}\sin^{-1}\left(\frac{z}{\sqrt{a^2-1}}\right);k\right), \tag{5.3}$$

where $K:[0,1)\to[\frac{\pi}{2},\infty)$ is the complete Legendre elliptic integral of first kind defined by

$$K(x)\triangleq\int_0^{\frac{\pi}{2}}\tfrac{d\varphi}{\sqrt{1-x^2\sin^2(\varphi)}}. \tag{5.4}$$

The application $\mathtt{sn}$ is the Jacobi elliptic sinus amplitudinus function and $k$ is defined through

$$G(k)=\frac{2}{\pi}\sinh^{-1}\left(\frac{2a}{a^2-1}\right), \tag{5.5}$$

with $G:(0,1]\to\mathbb{R}$ the strictly decreasing function given by

$$G(x)\triangleq\frac{K\big(\sqrt{1-x^2}\big)}{K(x)}. \tag{5.6}$$

The previous relation gives a bijective correspondance between $k\in(0,1)$ and $a>1$, with

$$a\to1 \ \leftrightarrow\ k\to0 \qquad \text{and} \qquad a\to\infty\ \leftrightarrow\ k\to1.$$

Our result reads as follows.

**Proposition 5.1.** *Let $a > 1$ and $k = k(a) \in (0,1)$ defined through* (5.5)*. Introduce the function*

$$g(k) \triangleq \left(1 - \frac{1}{4k^2}\left((1+k^2) - \left(\frac{\pi}{2K(k)}\right)^2\right)^2\right)^{-\frac{1}{2}}.$$

*The function $g : [0,1) \to \mathbb{R}_+$ is continuous, strictly increasing and satisfies*

$$g(0) = 1 \qquad \text{and} \qquad \lim_{k\to 1} g(k) = \infty.$$

*For any $n \in \mathbb{N}^*$, there exists a unique $k_n \in [0,1)$ (and therefore a unique $a_n > 1$) such that*

$$g(k_n) = n.$$

*Finally, for any $a \in (1,\infty) \setminus \{a_n,\, n \in \mathbb{N}\}$, the non-degeneracy condition* (5.2) *holds.*

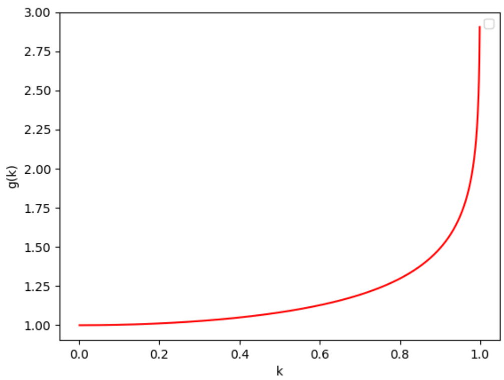


Figure 3: Graph of the function $g$.

*Proof.* First, we recall the following power series expansions, see for instance [1, p. 81],

$$\sin^{-1}(z) = \sum_{n=0}^{\infty} \frac{(2k-1)!!}{(2k)!!}\frac{z^{2k+1}}{2k+1} = z + \tfrac{1}{6}z^3 + \sum_{n=2}^{\infty} \frac{(2k-1)!!}{(2k)!!}\frac{z^{2k+1}}{2k+1}$$

and, see [90],

$$\mathtt{sn}(z,k) = z - \tfrac{1}{6}(1+k^2)z^3 + \sum_{n=2}^{\infty} b_n(k) z^{2n+1}, \qquad b_n(k) \in \mathbb{R}.$$

Inserting this into (5.3), we find

$$\Phi(z) = a_1(k)z + \sum_{n=3}^{\infty} a_n(k) z^n,$$

and in particular, we have

$$a_1(k) = \frac{2K(k)\sqrt{k}}{\pi\sqrt{a^2-1}}, \qquad a_3(k) = \frac{K(k)\sqrt{k}}{3\pi(a^2-1)^{\frac{3}{2}}}\left[1 - \frac{4K^2(k)}{\pi^2}(1+k^2)\right].$$

Recall that the mapping $x \mapsto K(x)$ is continuous increasing on $[0,1)$ with

$$K(0) = \tfrac{\pi}{2}, \qquad \lim_{x\to 1} K(x) = \infty. \tag{5.7}$$

In particular,

$$K(k) > \tfrac{\pi}{2}\cdot$$

Hence,

$$\begin{aligned}\frac{|a_3(k)|}{a_1^3(k)} &= \frac{\pi^2}{24kK^2(k)}\left(\frac{4K^2(k)}{\pi^2}(1+k^2)-1\right)\\ &= \frac{1}{6k}\left((1+k^2)-\left(\frac{\pi}{2K(k)}\right)^2\right).\end{aligned}$$

Denote by $M$ the arithmetic-geometric mean function defined on $\{(\alpha,\beta)\in(0,\infty)^2\}$ by

$$M(\alpha,\beta) \triangleq \lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\beta_n,$$

where the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ are given by

$$(\alpha_0,\beta_0)=(\alpha,\beta) \qquad \text{and} \qquad \forall n\in\mathbb{N},\quad (\alpha_{n+1},\beta_{n+1}) = \left(\tfrac{\alpha_n+\beta_n}{2},\sqrt{\alpha_n\beta_n}\right).$$

According to [82, p. 66-67], we have

$$K(k) = \frac{\pi}{2M(1,\sqrt{1-k^2})}\cdot \tag{5.8}$$

Now, we turn to condition (5.2) which is equivalent to

$$\frac{|a_3(k)|}{a_1^3(k)} \neq \frac{1}{3}\sqrt{1-\tfrac{1}{n^2}} \Longleftrightarrow \qquad n \neq \left(1-\frac{1}{4k^2}\left((1+k^2)-\left(\frac{\pi}{2K(k)}\right)^2\right)^2\right)^{-\frac{1}{2}} \triangleq g(k). \tag{5.9}$$

The function $K$ admits the following power series expansion, see [1, p. 591]

$$K(k) = \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{(2n)!}{2^{2n}(n!)^2}\right)^2 k^{2n}.$$

Consequently, the map $g:[0,1)\to\mathbb{R}_+$ is continuous and satisfies, in view of (5.7),

$$g(0)=1 \qquad \text{and} \qquad \lim_{k\to 1} g(k) = \infty. \tag{5.10}$$

Now, let us turn to the monotonicity of $g$. The derivative of $g$ is

$$\begin{aligned}g'(k) &= \frac{g^3(k)}{2}\left((1+k^2)-\left(\frac{\pi}{2K(k)}\right)^2\right)\left[\frac{\left(k+\frac{\pi^2K'(k)}{4K^3(k)}\right)}{k^2} - \frac{(1+k^2)-\left(\frac{\pi}{2K(k)}\right)^2}{2k^3}\right]\\ &= \frac{g^3(k)}{4k^3}\left((1+k^2)-\left(\frac{\pi}{2K(k)}\right)^2\right)\left[k^2-1+\left(\frac{\pi}{2K(k)}\right)^2\left(1+\frac{2kK'(k)}{K(k)}\right)\right].\end{aligned}$$

From (5.8), we deduce that

$$K(k) < \frac{\pi}{2\sqrt{1-k^2}},$$

which implies

$$(1-k^2)\left(\frac{2K(k)}{\pi}\right)^2 < 1.$$

Since $K$ is positive and strictly increasing, we have proved that

$$(1-k^2)\left(\frac{2K(k)}{\pi}\right)^2 < 1+\frac{2kK'(k)}{K(k)}\cdot \tag{5.11}$$

Finally, (5.11) implies $g'(k)>0$, and therefore $g$ is strictly increasing on $[0,1)$. Combined with (5.10), we obtain that $g:[0,1)\to[1,\infty)$ is a bijection. Hence, from (5.9) we get the proof of Proposition 5.1. □

## 5.2 Polygonal domains

The main goal in this section is to explore some polygonal shapes that satisfy the assumption (5.1). This will be implemented through the Schwarz-Christoffel mapping. The Schwarz-Christoffel transformation is a powerful tool in complex analysis that describes the conformal mapping of unit disc onto the interior of a polygon. This mapping provides a useful framework for examining the geometric properties and behaviors of various polygonal domains, particularly in the context of the non-degeneracy condition (5.1). We begin this section with the following remark showing that for highly symmetric polygons, the condition (5.1) fails. This observation underscores the necessity of exploring less symmetric, more irregular polygonal shapes to satisfy the given assumption.

**Remark 5.2.** *Let $m \geqslant 3$ be an integer and consider a regular polygon $\mathbf{P}_m$ with $m$ sides, centered at $0$ and with length side*

$$\frac{2^{1-\frac{4}{m}}\Gamma^2\left(\frac{1}{2}-\frac{1}{m}\right)}{m\Gamma\left(1-\frac{2}{m}\right)}.$$

*It is well-known, see for instance [83, p.196], that the associated conformal mapping $F:\mathbb{D}\to\mathbf{P}_m$ is given by*

$$F(z)=\int_0^z\frac{d\xi}{(1-\xi^m)^{\frac{2}{m}}}. \tag{5.12}$$

*From this we deduce that*

$$F(0)=F''(0)=F^{(3)}(0)=0,\qquad F'(0)=1.$$

*Consequently, $0$ is the unique (since $\mathbf{P}_m$ is convex) critical point of the Robin function $\mathcal{R}_{\mathbf{P}_m}$ and*

$$S(F)(0)=0.$$

*Thus, the condition (5.1) fails.*

The previous remark teaches us that we need to look for less symmetric configurations. This is what we will explore in the next subsections. Before entering into details, let us discuss the general theory of conformal mapping for polygonal domains discovered by Schwarz and Christoffel. We refer the reader to [83, p. 189] for the theory. The Schwarz-Christoffel mapping provides a way to map the unit disk conformally onto the interior of the polygon $\mathbf{P}$. This transformation is essential for understanding the geometric properties of the polygonal domain. Consider $\mathbf{P}$ a polygon with $m \geqslant 3$ vertices labelled $z_1$, $z_2$, ..., $z_m$. We denote $\pi\alpha_1$, $\pi\alpha_2$, ..., $\pi\alpha_m$ as its interior angles, see Figure 4. The associated exterior angles $\pi\mu_1$, $\pi\mu_2$, ..., $\pi\mu_m$ are defined by

$$\forall k\in[\![1,m]\!],\quad \mu_k\triangleq 1-\alpha_k\in(-1,1)$$

and they satisfy the identity

$$\sum_{k=1}^m\mu_k=2. \tag{5.13}$$

Then, there exist $\alpha,\beta\in\mathbb{C}$, determining respectively the size and the position of the polygon $\mathbf{P}$, and distinct angles $\theta_1$, $\theta_2$, ... , $\theta_m\in[0,2\pi)$ such that the following application maps conformally the unit disc $\mathbb{D}$ onto the interior of the polygon $\mathbf{P}$

$$F(z)=\alpha\int_0^z\prod_{k=1}^m\left(\xi-e^{\mathrm{i}\theta_k}\right)^{-\mu_k}d\xi+\beta, \tag{5.14}$$

with the additional property

$$\forall k\in[\![1,m]\!],\quad F\big(e^{\mathrm{i}\theta_k}\big)=z_k.$$

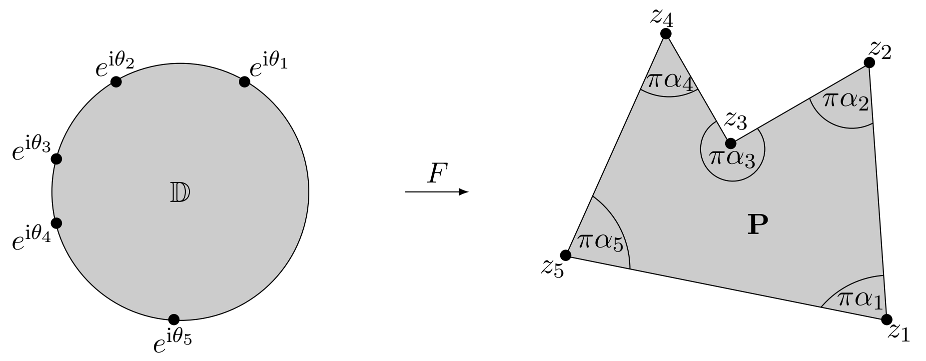


Figure 4: Schwarz Christoffel conformal mapping for polygonal domains.

Observe that the map in (5.12) is nothing but the Schwarz-Christoffel conformal mapping with

$$\alpha = 1, \qquad \beta = 0 \qquad \text{and} \qquad \forall k \in [\![1, m]\!], \quad \theta_k = \tfrac{2k\pi}{m}.$$

It is readily seen from the expression of $F$ that

$$\frac{F''(z)}{F'(z)} = -\sum_{k=1}^{m} \frac{\mu_k}{z - e^{\mathrm{i}\theta_k}}. \tag{5.15}$$

Differentiating (5.15), we get

$$\frac{F^{(3)}(z)}{F'(z)} - \left(\frac{F''(z)}{F'(z)}\right)^2 = \sum_{k=1}^{m} \frac{\mu_k}{(z - e^{\mathrm{i}\theta_k})^2}.$$

Therefore, by vitue of (2.16), the Schwarzian derivative of $F$ is given by

$$S(F)(z) = \sum_{k=1}^{m} \frac{\mu_k}{(z - e^{\mathrm{i}\theta_k})^2} - \frac{1}{2}\left(\sum_{k=1}^{m} \frac{\mu_k}{z - e^{\mathrm{i}\theta_k}}\right)^2. \tag{5.16}$$

Notice that the convexity of the polygon $\mathbf{P}$ is equivalent to require

$$\forall k \in [\![1, m]\!], \quad \mu_k > 0.$$

### 5.2.1 Symmetric convex polygons

Here we will explore the non-degeneracy condition (5.1) for symmetric polygons and see a concrete application for rectangles. We consider a polygon $\mathbf{P}$ with $2m$ vertices ($m \geqslant 2$) denoted $(z_k)_{1\leqslant k\leqslant 2m}$ subject to the following configuration,

1. (Symmetry with respect to 0) The vertices satisfy the following properties

$$\forall k \in [\![1, m]\!], \quad z_{m+k} = -z_k, \qquad \mathrm{Im}(z_k) \geqslant 0. \tag{5.17}$$

2. (Convexity) The exterior angles satisfy

$$\forall k \in [\![1, 2m]\!], \quad \mu_k > 0. \tag{5.18}$$

The symmetry property (5.17) imposes that

$$\forall k \in [\![1, m]\!], \quad \mu_{m+k} = \mu_k. \tag{5.19}$$

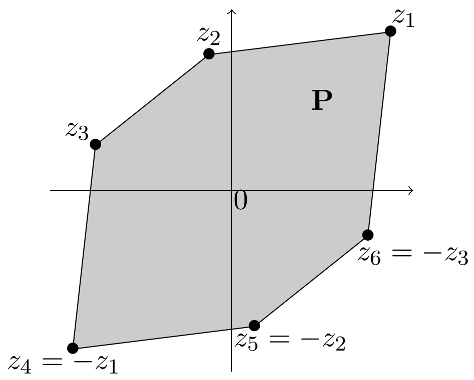


Figure 5: Symmetric convex polygons.

Putting together (5.19) and (5.13) (with $m$ replaced by $2m$), we deduce that

$$\sum_{k=1}^{m} \mu_k = 1. \tag{5.20}$$

Recall from (5.14) that the Schwarz-Christoffel mapping satisfying $F(0) = 0$ and $F'(0) > 0$ takes the form

$$F(z) = \alpha \int_0^z \prod_{k=1}^{2m} \left(\xi - e^{\mathrm{i}\theta_k}\right)^{-\mu_k} d\xi, \qquad \alpha = \widetilde{\alpha} \prod_{k=0}^{2m-1} \left(-e^{\mathrm{i}\theta_k}\right)^{\mu_k}, \qquad \widetilde{\alpha} > 0.$$

Due to the symmetry, we can impose to the angles $\theta_k$ the constraints

$$0 \leqslant \theta_1 < \theta_2 < ... < \theta_m < \pi \qquad \text{and} \qquad \forall k \in [\![1, m]\!], \quad \theta_{k+m} = \theta_k + \pi \tag{5.21}$$

leading, with (5.17), to

$$F(z) = \widetilde{\alpha} \int_0^z \prod_{k=1}^{m} \left(\xi^2 - e^{2\mathrm{i}\theta_k}\right)^{-\mu_k} d\xi. \tag{5.22}$$

Next, we will show the following result.

**Proposition 5.3.** *Consider* **P** *a symmetric convex polygonal domain associated with the Schwarz-Christoffel conformal mapping* (5.22)*. Assume that*

$$\sum_{k=1}^{m} \mu_k \cos(2\theta_k) \notin \left\{ \sqrt{1 - \tfrac{1}{n^2}},\, n \in \mathbb{N}^* \right\}. \tag{5.23}$$

*Then, the origin* 0 *is the unique critical point of the Robin function* $\mathcal{R}_{\mathbf{P}}$ *and the condition* (5.1) *is satisfied.*

*Proof.* Recall that by construction $F(0) = 0$ and $F'(0) > 0$. Moreover, a direct differentiation in (5.22) gives

$$F''(0) = 0. \tag{5.24}$$

It follows that $z = 0$ is a solution to Grakhov's equation (2.14). Therefore $z = 0$ is a critical point to the Robin function associated with the domain **P**. As the domain **P** is convex, then this critical point is unique. Now, using (5.16), (5.15), (5.24), (5.19) and (5.21) yield

$$\begin{aligned} S(F)(0) = \frac{F^{(3)}(0)}{F'(0)} &= \sum_{k=1}^{2m} \mu_k e^{-2\mathrm{i}\theta_k} \\ &= \sum_{k=1}^{m} \mu_k e^{-2\mathrm{i}\theta_k} + \sum_{k=1}^{m} \mu_{m+k} e^{-2\mathrm{i}\theta_{m+k}} \\ &= 2 \sum_{k=1}^{m} \mu_k \cos(2\theta_k). \end{aligned} \tag{5.25}$$

Thus the condition (5.1) is equivalent to (5.23). This concludes the proof of Proposition 5.3. $\square$

### 5.2.2 Rectangles

As an application of the previous proposition, we intend to study rectangular domains for which the condition (5.1) is explicitly described. In this context, the non-degeneracy condition (5.1) will be related to the aspect ratio which should avoid a discrete set of values. Our main result reads as follows.

**Proposition 5.4.** *Let* $\mathbf{D}$ *be a rectangle with sides* $0 < l < L$. *Then the condition* (5.1) *is satisfied if and only if*

$$\tfrac{l}{L} \notin \{G_n,\, n \in \mathbb{N}^\star\}, \qquad G_n \triangleq G\left(\sqrt{\tfrac{1+\sqrt{1-\frac{1}{n^2}}}{2}}\right), \tag{5.26}$$

*where* $G$ *has been introduced in* (5.6).

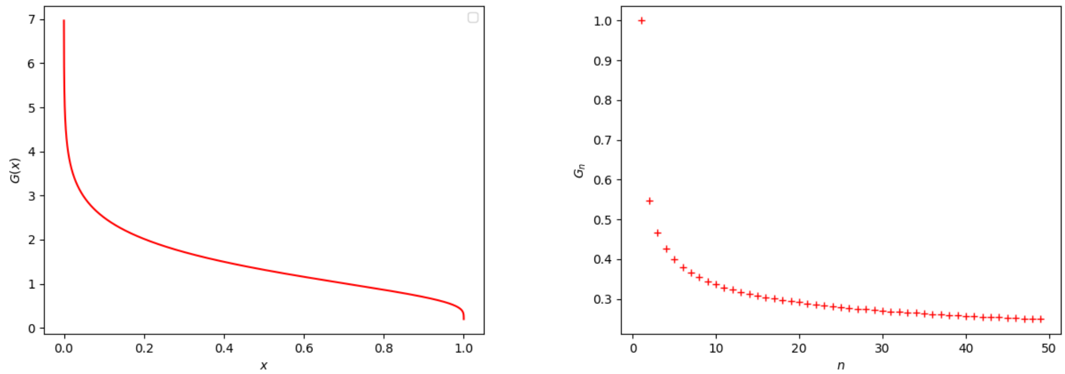


Figure 6: Graph of the function $G$ and the sequence $(G_n)_{n\in\mathbb{N}^*}$.

*Proof.* Without loss of generality, we can shall chose the rectangle $\mathbf{D}$ to be symmetric with respect to the real and imaginary axis,

$$\theta_1 \in (0, \tfrac{\pi}{4}], \qquad \theta_2 = \pi - \theta_1, \qquad \theta_3 = \theta_1 + \pi, \qquad \theta_4 = \theta_2 + \pi. \tag{5.27}$$

Moreover,

$$\mu_1 = \mu_2 = \frac{1}{2}, \tag{5.28}$$

implying in turn

$$\sum_{k=1}^{2} \mu_k \cos(2\theta_k) = \cos(2\theta_1).$$

Therefore, the condition (5.23) becomes

$$\cos(2\theta_1) \notin \left\{\sqrt{1-\tfrac{1}{n^2}},\, n \in \mathbb{N}^*\right\},$$

which is equivalent to

$$\cos(\theta_1) \notin \left\{\sqrt{\tfrac{1+\sqrt{1-\frac{1}{n^2}}}{2}},\, n \in \mathbb{N}^*\right\}. \tag{5.29}$$

According to (5.22), (5.27) and (5.28), we have

$$F(z) = \widetilde{\alpha} \int_0^z \frac{d\xi}{\sqrt{(\xi^2 - e^{2i\theta_1})(\xi^2 - e^{-2i\theta_1})}}, \qquad \widetilde{\alpha} > 0.$$

The length of the side joining $z_1$ and $z_2$ is given by

$$L = \int_{\theta_1}^{\pi-\theta_1} |F'(e^{it})| dt. \tag{5.30}$$

From straightfoward computations we infer

$$\begin{aligned} |F'(e^{it})| &= \frac{\widetilde{\alpha}}{\sqrt{|e^{2it} - e^{2i\theta_1}||e^{2it} - e^{-2i\theta_1}|}} \\ &= \frac{\widetilde{\alpha}}{2\sqrt{|\sin(t-\theta_1)||\sin(t+\theta_1)|}} \\ &= \frac{\widetilde{\alpha}}{\sqrt{2}\sqrt{|\cos(2\theta_1) - \cos(2t)|}}. \end{aligned}$$

Therefore, by change of variables, we get with $\beta = \pi - 2\theta_1$,

$$\begin{aligned} \frac{L}{\widetilde{\alpha}} &= \frac{1}{\sqrt{2}} \int_{\theta_1}^{\pi-\theta_1} \frac{dt}{\sqrt{|\cos(2\theta_1) - \cos(2t)|}} \\ &= \frac{1}{2\sqrt{2}} \int_{2\theta_1}^{\pi} \frac{dt}{\sqrt{|\cos(2\theta_1) - \cos(t)|}} + \frac{1}{2\sqrt{2}} \int_{\pi}^{2\pi-2\theta_1} \frac{dt}{\sqrt{|\cos(2\theta_1) - \cos(t)|}} \\ &= \int_0^{\beta} \frac{dt}{\sqrt{2\cos(t) - 2\cos(\beta)}} \\ &= \frac{\pi}{2} P_{-\frac{1}{2}}\big(\cos(\beta)\big) = \frac{\pi}{2} P_{-\frac{1}{2}}\big(-\cos(2\theta_1)\big), \end{aligned}$$

where $P_\nu$ is the Legendre functions of degree $\nu$ related to the hypergeometric function $\mathtt{F}$ through

$$P_\nu(z) = \mathtt{F}\left(-\nu, \nu+1; 1; \tfrac{1-z}{2}\right).$$

We refer the reader to [1, p. 337 and 561] for the justification of the previous identities involving special functions. Hence

$$\begin{aligned} \frac{L}{\widetilde{\alpha}} &= \frac{\pi}{2} \mathtt{F}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1+\cos(2\theta_1)}{2}\right) \\ &= \frac{\pi}{2} \mathtt{F}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \cos^2(\theta_1)\right) \\ &= K\big(\cos(\theta_1)\big), \end{aligned}$$

with $K$ the complete elliptic function defined in (5.4). The last equality comes from [1, p. 591]. The length $l$ of the side $[z_1, z_4]$ satisfies

$$\begin{aligned} \frac{l}{\widetilde{\alpha}} &= \frac{1}{\sqrt{2}} \int_{-\theta_1}^{\theta_1} \frac{dt}{\sqrt{|\cos(2\theta_1) - \cos(2t)|}} \\ &= \int_0^{2\theta_1} \frac{dt}{\sqrt{2\cos(t) - 2\cos(2\theta_1)}} \\ &= \frac{\pi}{2} P_{-\frac{1}{2}}\big(\cos(\theta_1)\big). \end{aligned}$$

Hence

$$\begin{aligned} \frac{l}{\widetilde{\alpha}} &= \frac{\pi}{2} \mathtt{F}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \tfrac{1-\cos(2\theta_1)}{2}\right) \\ &= \frac{\pi}{2} \mathtt{F}\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; \sin^2(\theta_1)\right) \\ &= K\big(\sin(\theta_1)\big). \end{aligned}$$

Therefore for $\theta_1 \in (0, \frac{\pi}{4}]$

$$\frac{l}{L} = \frac{K\big(\sin(\theta_1)\big)}{K\big(\cos(\theta_1)\big)} = G\big(\cos(\theta_1)\big), \tag{5.31}$$

where $G$ has been introduced in (5.6). We can easily check that $G$ is strictly decreasing. It follows that (5.29) is satisfied if and only if

$$\tfrac{l}{L} \notin \big\{G_n, n \in \mathbb{N}^*\big\}, \qquad G_n \triangleq G\left(\sqrt{\tfrac{1+\sqrt{1-\frac{1}{n^2}}}{2}}\right).$$

This concludes the proof of Proposition 5.4. □

# 6 Vortex duplication

The aim of this section is to explore the duplication method achievable through a standard reflection scheme. This mechanism will be employed to construct multi-vortex configurations from a single point vortex, ultimately leading to vortex synchronization. To illustrate this, we will discuss two cases. The first case involves a rectangular domain divided into several identical rectangular cells, each filled with a time-periodic vortex patch and exhibiting alternating circulations. The second case involves a disc divided into several identical sectors.

## 6.1 Duplication method

Let $\mathbf{D}$ be a bounded simply connected domain such that its boundary $\partial\mathbf{D}$ contains a non-trivial segment $[z, w]$ with $z \neq w$ and

$$\partial\mathbf{D} \cap (z, w) = [z, w], \tag{6.1}$$

where $(z, w)$ denotes the infinite line joining the points $z$ and $w$. Now, denote by $\mathtt{S}$ the reflection mapping through the line $(z, w)$, namely

$$\forall\, x \in \mathbb{C}, \quad \mathtt{S}(x) = 2\mathrm{Re}\big\{x\overline{(z-w)}\big\}\frac{z-w}{|z-w|} - x.$$

Define

$$\mathbf{D}' \triangleq \mathtt{S}(\mathbf{D}) = \big\{\mathtt{S}(x),\, x \in \mathbf{D}\big\}.$$

Notice that $\mathbf{D} \cap \mathbf{D}' = [z, w]$. We set

$$\mathbf{D}^\star \triangleq \mathrm{Int}\big(\overline{\mathbf{D}} \cup \overline{\mathbf{D}'}\big).$$

Let $\omega_0 \in L^\infty_c(\mathbf{D})$ and $\omega_0^\star \in L^\infty_c(\mathbf{D}^\star)$ and consider Euler equations,

$$\begin{cases} \partial_t\omega + u \cdot \nabla\omega = 0, \quad x \in \mathbf{D}, \\ u = \nabla^\perp(\Delta_{\mathbf{D}})^{-1}\omega, \\ \omega(0, x) = \omega_0 \end{cases} \tag{6.2}$$

and

$$\begin{cases} \partial_t\omega^\star + u^\star \cdot \nabla\omega^\star = 0, \quad x \in \mathbf{D}^\star, \\ u^\star = \nabla^\perp(\Delta_{\mathbf{D}^\star})^{-1}\omega^\star, \\ \omega^\star(0, x) = \omega_0^\star. \end{cases} \tag{6.3}$$

For a function $f : \mathbf{D} \to \mathbb{R}$ compactly supported in $\mathbf{D}$, we define its reflection $f^\# : \mathbf{D}^\star \to \mathbb{R}$ as

$$f^\#(x) \triangleq \begin{cases} f(x), & \text{if } x \in \mathbf{D}, \\ -f(\mathtt{S}x), & \text{if } x \in \mathbf{D}'. \end{cases} \tag{6.4}$$

We intend to prove the following result.

**Proposition 6.1.** *Let* $\mathbf{D}$ *be a simply connected bounded domain satisfying* (6.1)*. Given* $\omega_0 \in L^\infty_c(\mathbf{D})$ *such that the system* (6.2) *admits a global weak solution* $\omega \in L^\infty\big(\mathbb{R}; L^\infty_c(\mathbf{D})\big)$*. Let* $\omega_0^\star = \omega_0^\#$*, then the system* (6.3) *admits a weak global solution* $\omega^\star \in L^\infty\big(\mathbb{R}; L^\infty_c(\mathbf{D}^\star)\big)$*, such that,*

$$\forall t \in \mathbb{R}, \quad \omega^\star(t) = \omega(t)^\#.$$

*In particular, if* $\omega$ *is time periodic then* $\omega^\star$ *is a time periodic counter-rotating pairs.*

**Remark 6.2.** *The existence of at least one solution to* (6.2) *satisfying* $\omega \in L^\infty\big(\mathbb{R}; L^\infty_c(\mathbf{D})\big)$*, particularly with compact support for each time, can be guaranteed with a piecewise smooth boundary* $\partial\mathbf{D}$ *that has a finite number of corners with angles smaller than* $\pi$*, see [78]. Uniqueness can be achieved in slightly smoother domains, as detailed in [2, 47, 55, 56, 78]. It is important to note that in Theorem* 3.1 *and Corollary* 3.3*, we do not impose any regularity conditions on the boundary. In fact, the time-periodic solutions we construct are inherently compactly supported in space due to the construction scheme.*

*Proof.* Let $\omega$ be a compactly supported solution of (6.2) whose stream function is $\psi$. Let us show that the function $\omega^\#$ is a solution to (6.3). Define

$$\begin{aligned}\forall x \in \mathbf{D}^\star, \quad \psi^\#(t,x) &\triangleq \int_{\mathbf{D}^\star} G_{\mathbf{D}^\star}(x,y)\omega^\#(t,y)dy \\ &= \int_{\mathbf{D}} \big[G_{\mathbf{D}^\star}(x,y) - G_{\mathbf{D}^\star}(x,\mathtt{S}y)\big]\omega(t,y)dy. \end{aligned}\tag{6.5}$$

Assume for a while that

$$\forall x,y \in \mathbf{D}, \quad G_{\mathbf{D}^\star}(x,y) - G_{\mathbf{D}^\star}(x,\mathtt{S}y) = G_{\mathbf{D}}(x,y)\tag{6.6}$$

and

$$\forall x \in \mathbf{D}', \quad \forall y \in \mathbf{D}, \quad G_{\mathbf{D}^\star}(x,y) - G_{\mathbf{D}^\star}(x,\mathtt{S}y) = -G_{\mathbf{D}}(\mathtt{S}x,y).\tag{6.7}$$

Then, we deduce from (6.5) that

$$\begin{aligned}\forall x \in \mathbf{D}, \quad \psi^\#(t,x) &= \int_{\mathbf{D}} G_{\mathbf{D}}(x,y)\omega(t,y)dy \\ &= \psi(t,x)\end{aligned}$$

and

$$\begin{aligned}\forall x \in \mathbf{D}', \quad \psi^\#(t,x) &= -\int_{\mathbf{D}} G_{\mathbf{D}}(\mathtt{S}x,y)\omega(t,y)dy \\ &= -\psi(t,\mathtt{S}x).\end{aligned}$$

Therefore, we get

$$\nabla^\perp\psi^\#(t,x)\cdot\nabla\omega^\#(t,x) = \begin{cases}(\nabla^\perp\psi\cdot\nabla\omega)(t,x), & x \in \mathbf{D}, \\ -(\nabla^\perp\psi\cdot\nabla\omega)(t,\mathtt{S}x), & x \in \mathbf{D}'.\end{cases}$$

Consequently, we find

$$\partial_t\omega^\# + \nabla^\perp\psi^\#\cdot\nabla\omega^\# = 0, \quad \text{in} \quad \mathbf{D}^\star$$

implying that $\omega^\#$ is a solution to (6.3). This gives the desired result. It remains to check the identities (6.6) and (6.7). Before checking these identities, we will first show

$$\forall\, x,y \in \mathbf{D}^\star, \quad G_{\mathbf{D}^\star}(x,\mathtt{S}y) = G_{\mathbf{D}^\star}(\mathtt{S}x,y).\tag{6.8}$$

Its proof follows from the fact that

$$\begin{aligned}\Delta_x[G_{\mathbf{D}^\star}(\mathtt{S}x,\mathtt{S}y)] &= [\Delta_x G_{\mathbf{D}^\star}](\mathtt{S}x,\mathtt{S}y)\\ &= \delta_{\mathtt{S}x}(\mathtt{S}y)\\ &= \delta_x(y).\end{aligned}$$

In addition, as $\mathtt{S}\mathbf{D}^\star = \mathbf{D}^\star$ and for $x \in \partial\mathbf{D}^\star$ we have $\mathtt{S}x \in \partial\mathbf{D}^\star$, then

$$\forall y \in \mathbf{D}, \quad \forall\, x \in \partial\mathbf{D}, \quad G_{\mathbf{D}^\star}(\mathtt{S}x,\mathtt{S}y) = 0.$$

By uniqueness of the Green function, we find that

$$\forall\, x,y \in \mathbf{D}^\star, \quad G_{\mathbf{D}^\star}(\mathtt{S}x,\mathtt{S}y) = G_{\mathbf{D}^\star}(x,y),$$

which implies (6.8) in view of the identity $\mathtt{S}^2 = \mathrm{Id}$. Let's start with checking the identity (6.6). For this aim, we introduce the auxiliary function

$$\forall\, x,y \in \mathbf{D}, \quad G_1(x,y) \triangleq G_{\mathbf{D}^\star}(x,y) - G_{\mathbf{D}^\star}(x,\mathtt{S}y). \tag{6.9}$$

First, for $x,y \in \mathbf{D}$, as $\mathtt{S}y \notin \mathbf{D}$, we have

$$\begin{aligned}\Delta_x[G_1(x,y)] &= \Delta_x G_{\mathbf{D}^\star}(x,y) - \Delta_x G_{\mathbf{D}^\star}(x,\mathtt{S}y)\\ &= \delta_x(y) - \delta_x(\mathtt{S}y)\\ &= \delta_x(y).\end{aligned}$$

One may easily verify that

$$\partial\mathbf{D} = A \cup [z,w],$$

where $A \subset \partial\mathbf{D}^\star$. As $\mathtt{S}(\mathbf{D}^\star) = \mathbf{D}^\star$, then for $y \in \mathbf{D}$, we obtain from (6.9) and (2.2)

$$\forall\, x \in A, \quad G_1(x,y) = 0 - 0 = 0.$$

However, for $x \in [z,w]$, we have $\mathtt{S}x = x$ and thus we find once again from (6.9) and (6.8)

$$\begin{aligned}\forall\, y \in \mathbf{D}, \quad G_1(x,y) &= G_{\mathbf{D}^\star}(x,y) - G_{\mathbf{D}^\star}(\mathtt{S}x,y)\\ &= 0.\end{aligned}$$

Therefore $G_1$ satisfies the same elliptic equation as $G_{\mathbf{D}}$. By uniqueness, we find

$$\forall x,y \in \mathbf{D}, \quad G_1(x,y) = G_{\mathbf{D}}(x,y).$$

Together with (6.8), it yields the identity (6.6). It remains to prove (6.7). We write by virtue of (6.8)

$$\begin{aligned}\forall x \in \mathbf{D}', \quad \forall y \in \mathbf{D}, \quad G_{\mathbf{D}^\star}(x,y) - G_{\mathbf{D}^\star}(x,\mathtt{S}y) &= G_{\mathbf{D}^\star}(\mathtt{S}^2x,y) - G_{\mathbf{D}^\star}(\mathtt{S}^2x,\mathtt{S}y)\\ &= G_{\mathbf{D}^\star}(\mathtt{S}x,\mathtt{S}y) - G_{\mathbf{D}^\star}(\mathtt{S}x,y).\end{aligned} \tag{6.10}$$

On the other hand, as $\mathtt{S}x \in \mathbf{D}$, then applying (6.6)

$$G_{\mathbf{D}^\star}(\mathtt{S}x,y) - G_{\mathbf{D}^\star}(\mathtt{S}x,\mathtt{S}y) = G_{\mathbf{D}}(\mathtt{S}x,y). \tag{6.11}$$

Putting together (6.11) with (6.10) yields

$$\forall x \in \mathbf{D}', \quad \forall y \in \mathbf{D}, \quad G_{\mathbf{D}^\star}(x,y) - G_{\mathbf{D}^\star}(x,\mathtt{S}y) = -G_{\mathbf{D}}(\mathtt{S}x,y).$$

This achieves the proof of (6.7).

□

Combining Theorem 3.1 together with Proposition 6.1 allows to generate synchronized pairs of counter-rotating time periodic patches. More precisely, we obtain the following result.

**Corollary 6.3.** *Let* $\mathbf{D}$ *be a simply-connected bounded domain such that Theorem 3.1 holds true. Assume the existence of a non-trivial segment* $[z,w]$ *such that*

$$\partial\mathbf{D}\cap(z,w)=[z,w].$$

*Then, counter-rotating time periodic vortex patches exist as solutions to Euler equations* (6.3) *in the domain* $\mathbf{D}^{\star}$*, generated in the spirit of Proposition 6.1 by duplication.*

In the remainder of this section, we shall explore how to iterate the duplication process when the boundary of the initial domain $\mathbf{D}$ contains more than one segment. Let $\mathbf{D}$ be a domain such that its boundary $\partial\mathbf{D}$ contains two non-trivial segments $[z_1,w_1]$ and $[z_2,w_2]$ and

$$\forall j\in\{1,2\},\quad \partial\mathbf{D}\cap(z_j,w_j)=[z_j,w_j].$$

Notice that the line $(z_1,w_1)$ will necessary intersect (at infinity when they are parallel) the line $(z_2,w_2)$ at a point $I$ outside the segment $[z_1,w_1]$. We denote by $\theta$ the geometric angle $\widehat{z_1Iz_2}$, which belongs to $[0,\pi)$. We denote by $\mathtt{S}_j$ the reflection with respect to the axis $(z_j,w_j)$. Denote

$$\mathbf{D}_{\star,1}\triangleq\operatorname{Int}\big(\overline{\mathbf{D}}\cup\overline{\mathtt{S}_1\mathbf{D}}\big).$$

Then, for $\theta\in[0,\frac{\pi}{2})$ we still get that

$$\forall j\in\{1,2\},\quad \partial\mathbf{D}_{\star,1}\cap(z_2,w_2)=[z_2,w_2].$$

Therefore, we can generate a new duplication by introducing

$$\mathbf{D}_{\star,2}\triangleq\operatorname{Int}\big(\overline{\mathbf{D}_{\star,1}}\cup\overline{\mathtt{S}_2\mathbf{D}_{\star,1}}\big).$$

The new domain is simply connected. Notice that we have excluded $\theta=\frac{\pi}{2}$ because in that case the resulting domain would not be simply connected and would contain a hole. However, we can reach $\theta=\frac{\pi}{2}$ if we allow $w_1=w_2$ which permits the duplication of rectangles.

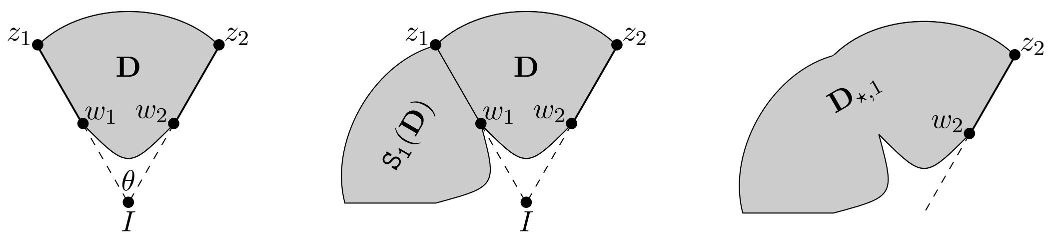


Figure 7: Duplication iteration : 1st step ($\theta=\frac{\pi}{3}$)

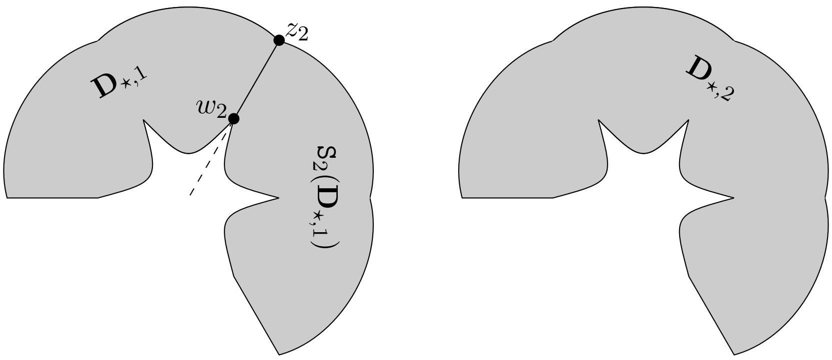


Figure 8: Duplication iteration : 2nd step

### 6.2 Application to vortex synchronization

Our aim here is to apply the duplication method explained in the Section 6 to study the desingularization of a highly symmetric system of $2^m$ point vortices. Due to the symmetry, the problem reduces to the evolution a single point vortex within a primitive fundamental cell, in the application it will be a rectangle or an $m$-sector.

#### 6.2.1 Time periodic vortex choreography within rectangular cells

Let us consider a rectangular fundamental domain $\mathbf{R}$ with length $L$ and width $l$. Assume that

$$\tfrac{l}{L} \notin \{G_n,\, n \in \mathbb{N}^*\},$$

where the sequence $(G_n)_{n\in\mathbb{N}^*}$ has been introduced in (5.26). Therefore, Proposition 5.4 applies and by virtue of Corollary 3.3 *most* of the point vortex periodic orbits can be desingularized into periodic vortex patch solutions. Given such dynamics, one can apply and iterate the duplication method to construct a simply connected domain formed by $2^m$ cells (copies of $\mathbf{R}$) within which time periodic multi-vortex patch motion occurs. Therefore one obtains time periodic symmetric choreography of $2^m$ patches as illustrated in Figure 9.

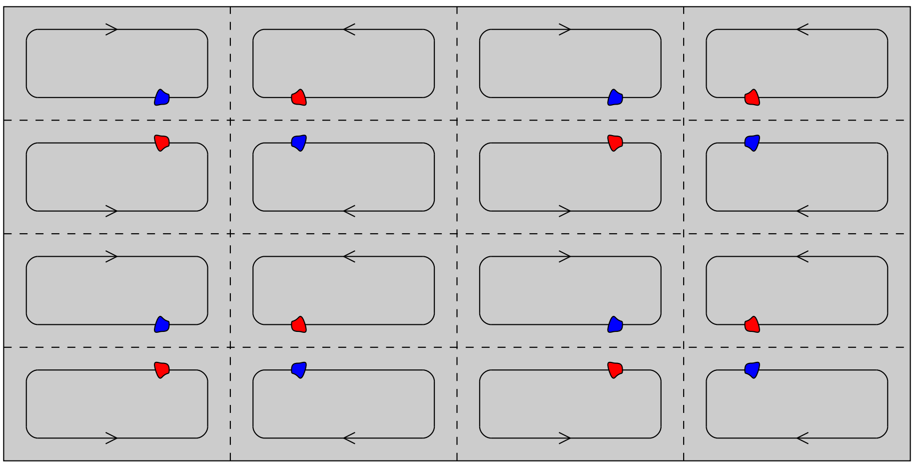

Figure 9: Rectangle cells choreography

#### 6.2.2 Duplication of $m$-sectors: application to the unit disc

Here, we aim to investigate time-periodic multi-vortex motion in sectors, similar to our previous discussion on rectangles. Let $\mathbf{D}$ be a simply connected bounded domain centered in 0 such that there exists $m \in \mathbb{N}^*$ with

$$\mathbf{D} = \mathrm{Int}\left(\bigcup_{k=0}^{2m-1} e^{\frac{ik\pi}{m}}\overline{\mathbf{D}_m}\right), \tag{6.12}$$

where $\mathbf{D}_m$ is the fundamental cell given by the $m$-sector

$$\mathbf{D}_m \triangleq \left\{z \in \mathbf{D} \quad \text{s.t.} \quad 0 < \arg(z) < \tfrac{\pi}{m}\right\}. \tag{6.13}$$

As examples, we can take discs, ellipses with $m = 2$ or regular polygons of $2^m$ sides.

Using the image method, we can establish a connection between the Green functions of the cell $\mathbf{D}_m$ and the full domain $\mathbf{D}$. The result is as follows, with a proof analogous to the initial part of the Lemma. 6.5.

**Proposition 6.4.** *The Green function of the domain* $\mathbf{D}_m$ *defined through* (6.13) *takes the form*

$$G_{\mathbf{D}_m}(z,w) = \sum_{k=0}^{m-1}\Big[G_{\mathbf{D}}\big(z,\omega_m^k w\big) - G_{\mathbf{D}}\big(z,\omega_m^k \overline{w}\big)\Big], \quad \omega_m \triangleq e^{\frac{2i\pi}{m}}.$$

Now, we shall focus on the particular case of the unit disc $\mathbb{D}$. Define the sets

$$\mathbb{D}_+ \triangleq \left\{z \in \mathbb{D} \quad \text{s.t.} \quad \mathrm{Im}(z) > 0\right\}, \qquad \mathbb{H}_+ \triangleq \left\{z \in \mathbb{C} \quad \text{s.t.} \quad \mathrm{Im}(z) > 0\right\} \tag{6.14}$$

and for $m \in \mathbb{N}^*$ we define the unit $m$-sector

$$\mathbb{D}_m \triangleq \left\{z \in \mathbb{D} \quad \text{s.t.} \quad 0 < \arg(z) < \tfrac{\pi}{m}\right\}. \tag{6.15}$$

For the specific case of the disc, the computations are explicit.

**Lemma 6.5.** *The Green function of the unit m-sector* $\mathbb{D}_m$ *defined in* (6.15) *takes the form*

$$G_{\mathbb{D}_m}(z,w) = \sum_{k=0}^{m-1}\Big[G_{\mathbb{D}}\big(z,\omega_m^k w\big) - G_{\mathbb{D}}\big(z,\omega_m^k \overline{w}\big)\Big].$$

*Its associated Robin function writes*

$$\mathcal{R}_{\mathbb{D}_m}(z) = -\log\left(1-|z|^2\right) - \log\left|\tfrac{z-\overline{z}}{1-z^2}\right| + \sum_{k=1}^{m-1}\Big[G_{\mathbb{D}}\big(z,\omega_m^k z\big) - G_{\mathbb{D}}\big(z,\omega_m^k \overline{z}\big)\Big]$$

*and admits a unique critical point* $\xi_m \in \mathbb{D}_m$ *given by*

$$\xi_m = t_m e^{\frac{i\pi}{2m}}, \qquad t_m \triangleq \left(2m + \sqrt{4m^2+1}\right)^{-\frac{1}{2m}}. \tag{6.16}$$

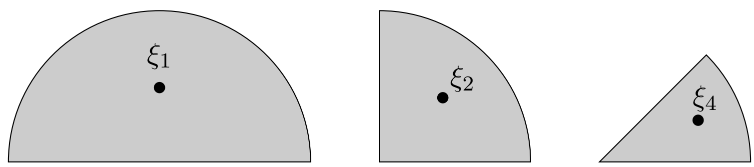


Figure 10: Position of the critical point $\xi_m$ in the circular sector $D_m$ for $m \in \{1,2,4\}$.

*Proof.* By the image method [75], finding the Green function in $\mathbb{D}_m$ amounts to solve in $\mathbb{D}$ the equation

$$\begin{cases} \Delta_z G_{\mathbb{D}_m}(z,w) = 2\pi \displaystyle\sum_{k=0}^{m-1}\delta_{\omega_m^k w}(z) - 2\pi\sum_{k=0}^{m-1}\delta_{\omega_m^k \overline{w}}(z), & \text{in } \mathbb{D}, \\ G_{\mathbb{D}_m}(z,w) = 0, & \text{for all } z \in \partial\mathbb{D}. \end{cases} \tag{6.17}$$

By a uniqueness argument, we conclude that

$$G_{\mathbb{D}_m}(z,w)=\sum_{k=0}^{m-1}\Big[G_{\mathbb{D}}\big(z,\omega_m^k w\big)-G_{\mathbb{D}}\big(z,\omega_m^k\overline{w}\big)\Big].$$

By isolating the term $k=0$, we can write

$$G_{\mathbb{D}_m}(z,w)=\log\left|\tfrac{z-w}{1-z\overline{w}}\right|-\log\left|\tfrac{z-\overline{w}}{1-zw}\right|+\sum_{k=1}^{m-1}\Big[G_{\mathbb{D}}\big(z,\omega_m^k w\big)-G_{\mathbb{D}}\big(z,\omega_m^k\overline{w}\big)\Big].$$

Thus Robin Function takes the form

$$\mathcal{R}_{\mathbb{D}_m}(z)=-\log\big(1-|z|^2\big)-\log\left|\tfrac{z-\overline{z}}{1-z^2}\right|+\sum_{k=1}^{m-1}\Big[G_{\mathbb{D}}\big(z,\omega_m^k z\big)-G_{\mathbb{D}}\big(z,\omega_m^k\overline{z}\big)\Big].$$

Straightforward computations yield for any $k\in[\![1,m-1]\!]$,

$$\partial_z\Big(G_{\mathbb{D}}\big(z,\omega_m^k z\big)\Big)=\frac{1}{2z}+\overline{z}\,\mathrm{Re}\left(\tfrac{\omega_m^k}{1-\omega_m^k|z|^2}\right)$$

and for any $k\in[\![0,m-1]\!]$,

$$\partial_z\Big(G_{\mathbb{D}}\big(z,\omega_m^k\overline{z}\big)\Big)=\tfrac{1}{z-\omega_m^k\overline{z}}+\tfrac{\omega_m^{-k}z}{1-\omega_m^{-k}z^2}.$$

Hence

$$\partial_z\mathcal{R}_{\mathbb{D}_m}(z)=\frac{m-1}{2z}+\sum_{k=0}^{m-1}\overline{z}\,\mathrm{Re}\left(\tfrac{\omega_m^k}{1-\omega_m^k|z|^2}\right)+\tfrac{1}{\omega_m^k\overline{z}-z}+\tfrac{\omega_m^{-k}z}{\omega_m^{-k}z^2-1}.$$

The critical point $\xi_m\in\mathbb{D}_m$ is unique since $\mathbb{D}_m$ is convex and it solves the equation

$$\frac{m-1}{2\xi_m}+\sum_{k=0}^{m-1}\overline{\xi_m}\,\mathrm{Re}\left(\tfrac{\omega_m^k}{1-\omega_m^k|\xi_m|^2}\right)+\tfrac{1}{\omega_m^k\overline{\xi_m}-\xi_m}+\tfrac{\omega_m^{-k}\xi_m}{\omega_m^{-k}\xi_m^2-1}=0. \tag{6.18}$$

By symmetry, we expect

$$\xi_m=t_m\,\omega_{4m},\qquad t_m\in(0,1). \tag{6.19}$$

Notice that

$$\omega_{4m}=e^{\frac{\mathrm{i}\pi}{2m}},\qquad \omega_{4m}^4=\omega_m,\qquad \omega_{4m}^{4m}=1. \tag{6.20}$$

Inserting (6.19) into (6.18) and using (6.20) yields

$$\frac{m-1}{2t_m}+\sum_{k=0}^{m-1}t_m\,\mathrm{Re}\left(\tfrac{\omega_{4m}^{4k}}{1-\omega_{4m}^{4k}t_m^2}\right)+\tfrac{1}{t_m(\omega_{4m}^{4k-2}-1)}+\tfrac{\omega_{4m}^{2-4k}t_m}{\omega_{4m}^{2-4k}t_m^2-1}=0.$$

In addition, the third identity in (6.20) implies

$$\begin{aligned}\sum_{k=0}^{m-1}\mathrm{Re}\left(\tfrac{\omega_{4m}^{4k}}{1-\omega_{4m}^{4k}t_m^2}\right)&=\tfrac{1}{2}\sum_{k=0}^{m-1}\tfrac{\omega_{4m}^{4k}}{1-\omega_{4m}^{4k}t_m^2}+\tfrac{1}{2}\sum_{k=0}^{m-1}\tfrac{\omega_{4m}^{-4k}}{1-\omega_{4m}^{-4k}t_m^2}\\&=\sum_{k=0}^{m-1}\tfrac{\omega_{4m}^{4k}}{1-\omega_{4m}^{4k}t_m^2}\end{aligned}$$

and

$$\sum_{k=0}^{m-1}\tfrac{\omega_{4m}^{2-4k}t}{\omega_{4m}^{2-4k}t_m^2-1}=\sum_{k=0}^{m-1}\tfrac{\omega_{4m}^{2+4k}t_m}{\omega_{4m}^{2+4k}t_m^2-1}.$$

Therefore,

$$\left(\frac{m-1}{2}+\sum_{k=0}^{m-1}\frac{1}{\omega_{4m}^{4k-2}-1}\right)+\sum_{k=0}^{m-1}\frac{\omega_{4m}^{4k}t_m^2}{1-\omega_{4m}^{4k}t_m^2}+\frac{\omega_{4m}^{2+4k}t_m^2}{\omega_{4m}^{2+4k}t_m^2-1}=0,$$

which is equivalent to

$$\left(\frac{m-1}{2}+\sum_{k=0}^{m-1}\frac{1}{\omega_{4m}^{4k-2}-1}\right)+\sum_{k=0}^{m-1}\frac{1}{1-\omega_{4m}^{4k}t_m^2}+\frac{1}{\omega_{4m}^{4k+2}t_m^2-1}=0.$$

Writing

$$\begin{aligned}\frac{1}{1-\omega_{4m}^{4k}t_m^2}&=\sum_{j=0}^{\infty}\omega_{4m}^{4kj}t_m^{2j}\\&=\sum_{\ell=0}^{\infty}\sum_{j=\ell m}^{(\ell+1)m-1}\omega_{4m}^{4kj}t_m^{2j}\\&=\sum_{\ell=0}^{\infty}t_m^{2\ell m}\sum_{j=0}^{m-1}\omega_{4m}^{4kj}t_m^{2j}\\&=\frac{P_k(t_m)}{1-t_m^{2m}},\end{aligned}$$

with

$$P_k(x)\triangleq\sum_{j=0}^{m-1}\omega_{4m}^{4kj}x^{2j}.$$

Then

$$\frac{1}{\omega_{4m}^{4k+2}t_m^2-1}=\frac{P_k(\omega_{4m}t_m)}{\omega_{4m}^{2m}t_m^{2m}-1}=-\frac{P_k(\omega_{4m}t_m)}{t_m^{2m}+1}.$$

In addition, notice that from (6.20), we have

$$\begin{aligned}P_k\left(\omega_{4m}^{-1}\right)&=\sum_{j=0}^{m-1}\left(\omega_{4m}^{4k-2}\right)^j\\&=\frac{1-\omega_{4m}^{(4k-2)m}}{1-\omega_{4m}^{4k-2}}\\&=\frac{2}{1-\omega_{4m}^{4k-2}}.\end{aligned}$$

Thus

$$\left(\frac{m-1}{2}-\frac{1}{2}\sum_{k=0}^{m-1}P_k\left(\omega_{4m}^{-1}\right)\right)-\sum_{k=0}^{m-1}\frac{P_k(t_m)}{t^{2m}-1}+\frac{P_k(\omega_{4m}t_m)}{t^{2m}+1}=0.$$

By exchanging finite sums, we can check that for any $x\in\mathbb{C}$,

$$\begin{aligned}\sum_{k=0}^{m-1}P_k(x)&=\sum_{k=0}^{m-1}\sum_{j=0}^{m-1}\omega_{4m}^{4kj}x^{2j}\\&=m+\sum_{j=1}^{m-1}x^{2j}\sum_{k=0}^{m-1}\omega_{4m}^{4kj}\\&=m,\end{aligned}$$

since from (6.20) we have for any $j \in [\![1, m-1]\!]$,

$$\sum_{k=0}^{m-1} \omega_{4m}^{4kj} = \frac{1-\omega_{4m}^{4jm}}{1-\omega_{4m}^{4j}} = 0.$$

It follows that

$$\tfrac{1}{2} + \tfrac{m}{t_m^{2m}-1} + \tfrac{m}{t_m^{2m}+1} = 0.$$

Thus, $t_m$ is solution to the equation

$$t_m^{4m} + 4mt_m^{2m} - 1 = 0.$$

Hence, under the constraint $t_m \in (0,1)$, we find

$$t_m = \left(2m + \sqrt{4m^2+1}\right)^{-\frac{1}{2m}}.$$

This achieves the proof of Lemma 6.5. □

Now, we prove the following result.

**Proposition 6.6.** *Let $m \in \mathbb{N}^*$ and consider $\Phi_m : \mathbb{D}_m \to \mathbb{D}$ the unique Riemann mapping such that*

$$\Phi_m(\xi_m) = 0 \qquad \Phi_m'(\xi_m) > 0,$$

*where $\xi_m \in \mathbb{D}_m$ is the unique critical point of the Robin function $\mathcal{R}_{\mathbb{D}_m}$ given in* (6.16)*. Then, the condition* (3.19) *is satisfied, with $F = \Phi_m^{-1}$, and Corollary* 3.3 *holds true. As a consequence, for any $w \in \mathbb{D}_m$, the initial configuration*

$$\omega_0 = \pi \sum_{k=0}^{m-1} \delta_{w\omega_m^k}(z) - \pi \sum_{k=0}^{m-1} \delta_{\overline{w}\omega_m^k}(z)$$

*generates time-periodic solution to the point vortex system, which can be desingularized with time periodic vortex patches.*

*Proof.* It is known that the following mappings are conformal

$$\begin{array}{rccc} \phi_+ : & \mathbb{D}_+ & \to & \mathbb{H}_+, \\ & z & \mapsto & -\frac{1}{2}\left(z + \frac{1}{z}\right) \end{array} \qquad \begin{array}{rccc} \phi_0 : & \mathbb{H}_+ & \to & \mathbb{D} \\ & z & \mapsto & \frac{z-\mathrm{i}}{z+\mathrm{i}}, \end{array}$$

where we refer to (6.14) for the definition of $\mathbb{D}_+$ and $\mathbb{H}_+$. Therefore $\phi \triangleq \phi_0 \circ \phi_+ : \mathbb{D}_+ \to \mathbb{D}$ is conformal. One can check that

$$\phi(z) = \tfrac{z^2+2iz+1}{z^2-2iz+1}.$$

On the other hand, the mapping $z \in \mathbb{D}_m \mapsto z^m \in \mathbb{D}_+$ is conformal. Therefore the mapping

$$\begin{array}{rccc} \phi_m : & \mathbb{D}_m & \to & \mathbb{D} \\ & z & \mapsto & \phi(z^m) = \frac{z^{2m}+2iz^m+1}{z^{2m}-2iz^m+1} \end{array}$$

is conformal. We want to construct the unique conformal mapping $\Phi_m : \mathbb{D}_m \mapsto \mathbb{D}$ such that

$$\Phi_m(\xi_m) = 0, \qquad \Phi_m'(\xi_m) > 0,$$

where $\xi_m$ is the critical point defined in (6.16). Now, set

$$a_m \triangleq \phi_m(\xi_m) = \tfrac{t_m^{2m}+2t_m^m-1}{t_m^{2m}-2t_m^m-1} \in (-1,1)$$

and denote by $T_m : \mathbb{D} \to \mathbb{D}$ the conformal map (Moebius transform)

$$T_m(z) = e^{i\theta_m} \frac{z - a_m}{1 - a_m z}$$

for some $\theta_m \in \mathbb{R}$ that will be fixed later. Then set

$$\Phi_m \triangleq T_m \circ \phi_m.$$

From direct computations we show that

$$\Phi_m(\xi_m) = 0$$

and, from (6.16),

$$\phi_m'(\xi_m) = \alpha_m e^{i(\pi - \frac{\pi}{2m})}, \qquad \alpha_m \triangleq \frac{4m t_m^{m-1}(1 + t_m^{2m})}{(1 + 2t_m^m - t_m^{2m})^2} > 0.$$

Therefore

$$\Phi_m'(\xi_m) = \frac{\alpha_m}{1 - a_m^2} e^{i(\theta + \pi - \frac{\pi}{2m})}.$$

By taking $\theta_m \triangleq \frac{\pi}{2m} - \pi$, we infer that

$$\Phi_m'(\xi_m) = \frac{\alpha_m}{1 - a_m^2} > 0.$$

By Faà di Bruno formula, we have that

$$\Phi_m' = \phi_m' \cdot T_m' \circ \phi_m, \qquad \Phi_m^{(3)} = (\phi_m')^3 \cdot T_m^{(3)} \circ \phi_m + 3\phi_m' \phi_m'' \cdot T_m'' \circ \phi_m + \phi_m^{(3)} \cdot T_m' \circ \phi_m.$$

Helped by Mathematica, we find that

$$\left| \tfrac{F^{(3)}(0)}{F'(0)} \right|^2 = \left| \frac{\Phi_m^{(3)}(\xi_m)}{\left(\Phi_m'(\xi_m)\right)^3} \right|^2 = \mathtt{A}_m \sqrt{4m^2 + 1} + \mathtt{B}_m,$$

where

$$\mathtt{A}_m \triangleq \frac{-8}{m^8}\big(2m^6 + 9m^4 + 6m^2 + 1\big) \in \mathbb{Q}^*, \qquad \mathtt{B}_m \triangleq \frac{4}{m^8}\big(m^8 + 24m^6 + 38m^4 + 8m^2 + 2\big) \in \mathbb{Q}.$$

Notice that for any integer $m \geqslant 1$,

$$\sqrt{4m^2 + 1} \notin \mathbb{Q}.$$

Hence,

$$\forall n \in \mathbb{N}^*, \quad \mathtt{A}_m \sqrt{4m^2 + 1} + \mathtt{B}_m \neq 4\left(1 - \tfrac{1}{n^2}\right).$$

This proves the condition (3.19), which concludes the proof of Proposition 6.6. □

# 7 Proof of Theorem 3.1

The goal of this section is to prove Theorem 3.1 by employing a modified Nash-Moser scheme tin conjunction with KAM tools. Specifically, we will utilize the framework developed in [59] to address the leapfrogging phenomenon in the context of Euler equations involving two symmetric pairs of vortices. Our approach begins with an introduction to the functional setting, followed by a detailed presentation of the asymptotic structure of the linearized operator. Subsequently, we will construct a suitable approximate solution to the nonlinear equation (3.14). Next, we rescale the functional **G**, defined via (3.14), and transform the new linear operator into a Fourier multiplier, with a regularizing remainder. This transformation enables us to identify an approximate right inverse of the linearized operator, which is a critical component of the Nash-Moser scheme.

## 7.1 Functional setting

We denote by $\mathbb{T}^2$ the two-dimensional torus and any $h : \mathbb{T}^2 \to \mathbb{C}$ in the class $L^2(\mathbb{T}^2, \mathbb{C})$ admits the Fourier expansion,

$$h = \sum_{(l,j)\in\mathbb{Z}^2} h_{l,j}\,\mathbf{e}_{l,j}, \qquad \mathbf{e}_{l,j}(\varphi,\theta) \triangleq e^{\mathrm{i}(l\varphi+j\theta)}, \qquad h_{l,j} \triangleq \big\langle h, \mathbf{e}_{l,j}\big\rangle_{L^2(\mathbb{T}^2,\mathbb{C})}.$$

The Hilbert $L^2(\mathbb{T}^2, \mathbb{C})$ is equipped with the scalar product

$$\big\langle h_1, h_2\big\rangle_{L^2(\mathbb{T}^2,\mathbb{C})} \triangleq \int_{\mathbb{T}^2} h_1(\varphi,\theta)\overline{h_2(\varphi,\theta)}d\varphi d\theta = \sum_{(l,j)\in\mathbb{Z}^2} h_{1,l,j}\overline{h_{2,l,j}}.$$

The following notation is of constant use in this work

$$\int_{\mathbb{T}} f(x)dx \triangleq \frac{1}{2\pi}\int_0^{2\pi} f(x)dx. \tag{7.1}$$

The Sobolev spaces of regularity $s \in \mathbb{R}$ is given by

$$H^s(\mathbb{T}^2,\mathbb{C}) \triangleq \Big\{h \in L^2(\mathbb{T}^2,\mathbb{C}) \quad \text{s.t.} \quad \|h\|_{H^s} < \infty\Big\},$$

where

$$\|h\|_{H^s}^2 \triangleq \sum_{(l,j)\in\mathbb{Z}^2} \langle l,j\rangle^{2s}|h_{l,j}|^2, \qquad \langle l,j\rangle^2 \triangleq 1 + |l|^2 + |j|^2.$$

The subspace with zero space average functions is denoted

$$H_0^s(\mathbb{T}^2,\mathbb{C}) \triangleq \left\{h \in H^s(\mathbb{T}^2,\mathbb{C}) \quad \text{s.t.} \quad \forall\varphi\in\mathbb{T}, \int_{\mathbb{T}} h(\varphi,\theta)d\theta = 0\right\}.$$

Given a nonempty subset $\mathscr{O}$ of $\mathbb{R}$ and $\gamma \in (0,1]$, we define the Banach spaces

$$\mathrm{Lip}_\gamma(\mathscr{O}, H^s) \triangleq \Big\{h : \mathscr{O} \to H^s(\mathbb{T}^2,\mathbb{C}) \quad \text{s.t.} \quad \|h\|_s^{\mathrm{Lip}(\gamma)} < \infty\Big\},$$
$$\mathrm{Lip}_\gamma(\mathscr{O}, \mathbb{C}) \triangleq \Big\{h : \mathscr{O} \to \mathbb{C} \quad \text{s.t.} \quad \|h\|^{\mathrm{Lip}(\gamma)} < \infty\Big\},$$

with

$$\|h\|_s^{\mathrm{Lip}(\gamma)} \triangleq \sup_{\lambda\in\mathscr{O}} \|h(\lambda,\cdot)\|_{H^s} + \gamma \sup_{\substack{(\lambda_1,\lambda_2)\in\mathscr{O}^2\\ \lambda_1\neq\lambda_2}} \frac{\|h(\lambda_1,\cdot) - h(\lambda_2,\cdot)\|_{H^{s-1}}}{|\lambda_1-\lambda_2|},$$
$$\|h\|^{\mathrm{Lip}(\gamma)} \triangleq \sup_{\lambda\in\mathscr{O}} |h(\lambda)| + \gamma \sup_{\substack{(\lambda_1,\lambda_2)\in\mathscr{O}^2\\ \lambda_1\neq\lambda_2}} \frac{|h(\lambda_1) - h(\lambda_2)|}{|\lambda_1-\lambda_2|}.$$

We emphasize that in Section 7.6 related to Nash-Moser scheme we find it convenient to use the notation

$$\|h\|_{s,\mathscr{O}}^{\mathrm{Lip}(\gamma)} = \|h\|_s^{\mathrm{Lip}(\gamma)}. \tag{7.2}$$

In the sequel, we consider the list of numbers with the constraints

$$\gamma \in (0,1], \qquad \tau > 1, \qquad S \geqslant s \geqslant s_0 > 3. \tag{7.3}$$

Along this section, we will extensively use the following classical result related to the product law over weighted Sobolev spaces. Given $(\gamma, s_0, s)$ satisfying (7.3). Let $h_1, h_2 \in \mathrm{Lip}_\gamma(\mathscr{O}, H^s)$. Then $h_1h_2 \in \mathrm{Lip}_\gamma(\mathscr{O}, H^s)$ and

$$\|h_1h_2\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|h_1\|_{s_0}^{\mathrm{Lip}(\gamma)}\|h_2\|_s^{\mathrm{Lip}(\gamma)} + \|h_1\|_s^{\mathrm{Lip}(\gamma)}\|h_2\|_{s_0}^{\mathrm{Lip}(\gamma)}. \tag{7.4}$$

We need some anisotropic spaces to describe the spatial smoothing effects of some non-local operators. The anisotropic Sobolev spaces of regularity $s_1, s_2 \in \mathbb{R}$ is given by

$$H^{s_1,s_2}(\mathbb{T}^2,\mathbb{C}) \triangleq \Big\{h \in L^2(\mathbb{T}^2,\mathbb{C}) \quad \text{s.t.} \quad \|h\|_{H^{s_1,s_2}} < \infty\Big\},$$

where

$$\|h\|^2_{H^{s_1,s_2}} \triangleq \sum_{(l,j)\in\mathbb{Z}^2} \langle l,j\rangle^{2s_1}\langle j\rangle^{2s_2}|h_{l,j}|^2.$$

The corresponding Lipschitz norm for external parameters

$$\|h\|^{\text{Lip}(\gamma)}_{s_1,s_2} \triangleq \sup_{\lambda\in\mathscr{O}} \|h(\gamma,\cdot)\|_{H^{s_1,s_2}} + \gamma \sup_{\substack{(\lambda_1,\lambda_2)\in\mathscr{O}^2\\ \lambda_1\neq\lambda_2}} \frac{\|h(\lambda_1,\cdot)-h(\lambda_2,\cdot)\|_{H^{s_1-1,s_2}}}{|\lambda_1-\lambda_2|}.$$

To conclude this section, we will briefly recall the Hilbert transform on $\mathbb{T}$ . Let $h : \mathbb{T} \to \mathbb{R}$ be a continuous function with a zero average, we define its Hilbert transform by

$$\mathbf{H}h(\theta) \triangleq \int_{\mathbb{T}} h(\eta)\cot\left(\tfrac{\eta-\theta}{2}\right)d\eta = -\partial_\theta \int_{\mathbb{T}} h(\eta)\log\left(\sin^2\left(\tfrac{\theta-\eta}{2}\right)\right)d\eta. \tag{7.5}$$

The integrals are understood in the principal value sense. It is a classical that $\mathbf{H}$ is a Fourier multiplier with

$$\forall j \in \mathbb{Z}^*, \quad \mathbf{H}\mathbf{e}_j(\theta) = \mathrm{i}\,\mathrm{sign}(j)\mathbf{e}_j(\theta).$$

## 7.2 Asymptotic structure of the linearized operator

Our aim here is to analyze the asymptotic behavior in $\varepsilon$ of the differential $d_r\mathbf{G}(r)$ at a small state $r$ for the functional $\mathbf{G}$ introduced in (3.14). The main result is stated in Proposition 7.2 below. As a preliminary step and in view of (3.14)-(3.13), we need to perform the $\varepsilon$-expansion, up to order 2, of the kernel

$$K(p+\varepsilon z, p+\varepsilon\xi)$$

using its representation via the conformal mapping (2.5).

**Lemma 7.1.** *Let $V$ be an open set such that $\overline{V} \subset \mathbf{D}$. Denote also $\mathbb{D}_2$ the disc of radius 2 centered in the origin. There exist $\varepsilon_0 > 0$ and $K_3 \in C^\infty_b(V\times\mathbb{D}_2^2)$ such that for any $\varepsilon \in (0,\varepsilon_0)$ and $(p,z,\xi) \in V\times\mathbb{D}_2^2$, we have*

$$K(p+\varepsilon z, p+\varepsilon\xi) = \mathcal{R}_{\mathbf{D}}(p) + \varepsilon K_1(p,z,\xi) + \varepsilon^2 K_2(p,z,\xi) + \varepsilon^3 K_3(p,z,\xi), \tag{7.6}$$

*with*

$$\begin{aligned}
K_1(p,z,\xi) &\triangleq \mathrm{Re}\Big\{\partial_z\mathcal{R}_{\mathbf{D}}(p)\,(z+\xi)\Big\},\\
K_2(p,z,\xi) &\triangleq \mathrm{Re}\Big\{\Big(\tfrac{1}{6}S(\Phi)(p) + \tfrac{1}{2}\big(\partial_z\mathcal{R}_{\mathbf{D}}(p)\big)^2\Big)(z^2+\xi^2) + \tfrac{1}{6}S(\Phi)(p)z\xi + \tfrac{1}{r^2_{\mathbf{D}}(p)}z\overline{\xi}\Big\},
\end{aligned}$$

*where we have used the notations* (2.8), (2.11) *and* (2.16).

*Proof.* Throughout the proof, we shall make use of the following formulae

$$\begin{aligned}
\partial_\varepsilon\big(\log|1+g(\varepsilon)|\big) &= \mathrm{Re}\left\{\tfrac{\partial_\varepsilon g(\varepsilon)}{1+g(\varepsilon)}\right\}, &(7.7)\\
\partial^2_\varepsilon\big(\log|1+g(\varepsilon)|\big) &= \mathrm{Re}\left\{\tfrac{(1+g(\varepsilon))\partial^2_\varepsilon g(\varepsilon)-(\partial_\varepsilon g(\varepsilon))^2}{(1+g(\varepsilon))^2}\right\}. &(7.8)
\end{aligned}$$

In general, we can write

$$\Phi(z) = bz + f(z), \qquad b>0, \qquad f \text{ analytic in } \mathbf{D}.$$

As $\Phi$ is univalent, then we infer that for any compact subset $\mathbf{K} \subset \mathbf{D}$,

$$\max_{\zeta \in \mathbf{K}} |f'(\zeta)| < b.$$

Therefore, applying (2.5) we deduce that

$$\begin{aligned} K(z,w) &= -\log|1-\Phi(z)\overline{\Phi(w)}| + \log(b) + \log\left|1+\tfrac{1}{b}\tfrac{f(z)-f(w)}{z-w}\right| \\ &\triangleq -\log|1+g_1(z,w)| + \log(b) + \log|1+g_2(z,w)|. \end{aligned} \tag{7.9}$$

It follows that

$$K(p+\varepsilon z, p+\varepsilon\xi) = -\log|1+g_1(p+\varepsilon z, p+\varepsilon\xi)| + \log(b) + \log|1+g_2(p+\varepsilon z, p+\varepsilon\xi)|. \tag{7.10}$$

Define

$$\begin{aligned} g_1(\varepsilon) &\triangleq g_1(p+\varepsilon z, p+\varepsilon\xi) \\ &= -\Phi\big(p+\varepsilon z\big)\overline{\Phi\big(p+\varepsilon\xi\big)}. \end{aligned}$$

Then straightforward calculations based on Taylor expansion of the holomorphic function $\Phi$ yield for small $\varepsilon$

$$\begin{aligned} g_1(\varepsilon) =& -|\Phi(p)|^2 - \varepsilon\big(\Phi(p)\overline{\Phi'(p)}\,\overline{\xi} + \overline{\Phi(p)}\Phi'(p)z\big) \\ & -\tfrac{\varepsilon^2}{2}\Big(\overline{\Phi(p)}\Phi''(p)z^2 + \Phi(p)\overline{\Phi''(p)}\,\overline{\xi}^2 + 2|\Phi'(p)|^2 z\overline{\xi}\Big) + O(\varepsilon^3). \end{aligned}$$

This implies

$$\begin{aligned} g_1(0) &= -|\Phi(p)|^2, \\ \partial_\varepsilon g_1(0) &= -\big(\Phi(p)\overline{\Phi'(p)}\overline{\xi} + \overline{\Phi(p)}\Phi'(p)z\big), \\ \partial_\varepsilon^2 g_1(0) &= -\Big(\overline{\Phi(p)}\Phi''(p)z^2 + \Phi(p)\overline{\Phi''(p)}\overline{\xi}^2 + 2|\Phi'(p)|^2 z\overline{\xi}\Big). \end{aligned}$$

Combining these identities with (7.7) and (7.8) allow to get for small $\varepsilon$

$$\begin{aligned} \log|1+g_1(\varepsilon)| =& \log\big(1-|\Phi(p)|^2\big) - \varepsilon\mathrm{Re}\left\{\tfrac{\Phi(p)\overline{\Phi'(p)}\overline{\xi}+\overline{\Phi(p)}\Phi'(p)z}{1-|\Phi(p)|^2}\right\} \\ & -\tfrac{\varepsilon^2}{2}\mathrm{Re}\left\{\tfrac{\overline{\Phi(p)}\Phi''(p)z^2+\Phi(p)\overline{\Phi''(p)}\overline{\xi}^2+2|\Phi'(p)|^2 z\overline{\xi}}{1-|\Phi(p)|^2}\right\} \\ & -\tfrac{\varepsilon^2}{2}\mathrm{Re}\left\{\left(\tfrac{\Phi(p)\overline{\Phi'(p)}\overline{\xi}+\overline{\Phi(p)}\Phi'(p)z}{1-|\Phi(p)|^2}\right)^2\right\} + O(\varepsilon^3). \end{aligned} \tag{7.11}$$

Notice that $O(\varepsilon^3)$ is a smooth function On the other hand, one writes

$$\begin{aligned} g_2(\varepsilon) &\triangleq \tfrac{1}{b}\tfrac{f(p+\varepsilon z)-f(p+\varepsilon\xi)}{\varepsilon(z-\xi)} \\ &= \tfrac{1}{b}\int_0^1 f'\big(p+\varepsilon\xi+s\varepsilon(z-\xi)\big)ds. \end{aligned}$$

Therefore, for any $n \in \mathbb{N}$,

$$\partial_\varepsilon^n g_2(\varepsilon) = \frac{1}{b}\int_0^1 f^{(n+1)}\big(p+\varepsilon\xi+s\varepsilon(z-\xi)\big)\big(\xi+s(z-\xi)\big)^n ds,$$

implying in turn that

$$\partial_\varepsilon^n g_2(0) = \tfrac{f^{(n+1)}(p)}{b(n+1)}\tfrac{z^{n+1}-\xi^{n+1}}{z-\xi} = \tfrac{f^{(n+1)}(p)}{b(n+1)}\sum_{k=0}^{n} z^k\xi^{n-k}. \tag{7.12}$$

Putting together (7.7), (7.8) and (7.12), we deduce for small $\varepsilon$

$$\log|1+g_2(\varepsilon)|=\log\left|1+\tfrac{f'(p)}{b}\right|+\varepsilon\mathrm{Re}\left\{\tfrac{f''(p)(z+\xi)}{2(b+f'(p))}\right\} + \tfrac{\varepsilon^2}{6}\mathrm{Re}\left\{\tfrac{f^{(3)}(p)}{b+f'(p)}\left(z^2+\xi^2+z\xi\right)\right\} - \tfrac{\varepsilon^2}{8}\mathrm{Re}\left\{\left(\tfrac{f''(p)(z+\xi)}{b+f'(p)}\right)^2\right\}+O(\varepsilon^3).$$

Since $\Phi'(z)=b+f'(z)$, $\Phi''(z)=f''(z)$ and $\Phi^{(3)}(z)=f^{(3)}(z)$, then we end up with

$$\log|1+g_2(\varepsilon)| = -\log(b)+\log\left|\Phi'(p)\right|+\varepsilon\mathrm{Re}\left\{\tfrac{\Phi''(p)(z+\xi)}{2\Phi'(p)}\right\} + \tfrac{\varepsilon^2}{6}\mathrm{Re}\left\{\tfrac{\Phi^{(3)}(p)}{\Phi'(p)}\left(z^2+\xi^2+z\xi\right)\right\} - \tfrac{\varepsilon^2}{8}\mathrm{Re}\left\{\left(\tfrac{\Phi''(p)(z+\xi)}{\Phi'(p)}\right)^2\right\}+O(\varepsilon^3). \tag{7.13}$$

We observe that $O(\varepsilon^3)=\varepsilon^3K_3(p,z,\xi)$ with $K_3$ being a smooth function in $(p,z,\xi)\in V\times\mathbb{D}_2^2$. Inserting (7.11) and (7.13) into (7.10) and using (2.8), (2.11) and (2.16) give the desired result. □

The next target is to provide an asymptotic expansion in $\varepsilon$ of the linearized operator $\mathbf{G}$, defined in (3.14), at a small state $r$. In what follows, and throughout the remainder of this paper, we denote

$$\mathscr{O}\triangleq[\lambda_*,\lambda^*],$$

where $\lambda_*$ and $\lambda^*$ are as defined in Theorem 3.1.

**Proposition 7.2.** *There exists $\varepsilon_0\in(0,1)$ such that if $r$ is smooth with zero average in space and*

$$\varepsilon\leqslant\varepsilon_0 \quad\text{and}\quad \|r\|_{s_0+2}^{\mathrm{Lip}(\gamma)}\leqslant 1,$$

*then, the linearized operator of the map $\mathbf{G}$, at a small state $r$ in the direction $h\in L_0^2(\mathbb{T}^2;\mathbb{R})$ is given by*

$$d_r\mathbf{G}(r)[h]=\varepsilon^3\omega(\lambda)\partial_\varphi h+\varepsilon\partial_\theta\big[\mathbf{V}^\varepsilon(r)h\big]-\tfrac{\varepsilon}{2}\mathbf{H}[h]+\varepsilon^3\partial_\theta\mathbf{Q}_1[h]+\varepsilon^3\partial_\theta\mathcal{R}_1^\varepsilon[h]+\varepsilon^4\partial_\theta\mathcal{R}_2^\varepsilon[h],$$

*with the following properties.*

1. *The function $\mathbf{V}^\varepsilon(r)$ decomposes as follows*

$$\mathbf{V}^\varepsilon(r)\triangleq\tfrac12-\tfrac{\varepsilon}{2}r+\varepsilon^2\big(\tfrac12\mathtt{g}+V_1^\varepsilon(r)\big)+\varepsilon^3V_2^\varepsilon(r),$$

*with*

$$\mathtt{g}(\varphi,\theta)\triangleq\mathrm{Re}\Big\{\mathtt{w}_2\big(p(\varphi)\big)e^{2\mathrm{i}\theta}\Big\},\qquad \mathtt{w}_2(p)\triangleq\big(\partial_z\mathcal{R}_{\mathbf{D}}(p)\big)^2+\tfrac13S(\Phi)(p) \tag{7.14}$$

*and $V_1^\varepsilon(r)$, $V_2^\varepsilon(r)$ satisfy the estimates*

$$\begin{aligned}\|V_1^\varepsilon(r)\|_s^{\mathrm{Lip}(\gamma)}&\lesssim\|r\|_{s+1}^{\mathrm{Lip}(\gamma)}\|r\|_{s_0+1}^{\mathrm{Lip}(\gamma)},\\ \|V_2^\varepsilon(r)\|_s^{\mathrm{Lip}(\gamma)}&\lesssim 1+\|r\|_s^{\mathrm{Lip}(\gamma)},\\ \|\Delta_{12}V_1^\varepsilon(r)\|_s^{\mathrm{Lip}(\gamma)}&\lesssim\|\Delta_{12}r\|_{s+1}^{\mathrm{Lip}(\gamma)}+\|\Delta_{12}r\|_{s_0+1}^{\mathrm{Lip}(\gamma)}\max_{k\in\{1,2\}}\|r_k^\varepsilon(r)\|_{s+1}^{\mathrm{Lip}(\gamma)},\\ \|\Delta_{12}V_2^\varepsilon(r)\|_s^{\mathrm{Lip}(\gamma)}&\lesssim\|\Delta_{12}r\|_s^{\mathrm{Lip}(\gamma)}+\|\Delta_{12}r\|_{s_0}^{\mathrm{Lip}(\gamma)}\max_{k\in\{1,2\}}\|r_k^\varepsilon(r)\|_s^{\mathrm{Lip}(\gamma)}.\end{aligned}$$

2. *The operator* $\mathbf{H}$ *is the Hilbert transform on the torus and the other non-local terms are integral operators in the form*

$$\mathbf{Q}_1[h](\varphi,\theta) \triangleq \int_{\mathbb{T}} h(\eta)\mathtt{Q}\big(p(\varphi),\theta,\eta\big)d\eta, \tag{7.15}$$

$$\mathcal{R}_j^{\varepsilon}[h](\varphi,\theta) \triangleq \int_{\mathbb{T}} h(\eta)\mathtt{K}_j^{\varepsilon}(r)(\varphi,\theta,\eta)d\eta, \quad j\in\{1,2\}, \tag{7.16}$$

*with, see* (2.8) *and* (2.16) *for the definitions of the functions below,*

$$\mathtt{Q}(p,\theta,\eta) \triangleq \frac{\cos(\theta-\eta)}{r_{\mathbf{D}}^2(p)}\cos(\theta-\eta)+\frac{1}{6}\mathrm{Re}\Big\{e^{\mathrm{i}(\theta+\eta)}S(\Phi)(p)\Big\} \tag{7.17}$$

*and* $\mathtt{K}_1^{\varepsilon}(r)$, $\mathtt{K}_2^{\varepsilon}(r)$ *satisfy the following estimates*

$$\|\mathtt{K}_1^{\varepsilon}(r)\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|r\|_{s+1}^{\mathrm{Lip}(\gamma)}\|r\|_{s_0+1}^{\mathrm{Lip}(\gamma)}, \tag{7.18}$$

$$\|\mathtt{K}_2^{\varepsilon}(r)\|_s^{\mathrm{Lip}(\gamma)} \lesssim 1+\|r\|_s^{\mathrm{Lip}(\gamma)}. \tag{7.19}$$

*Proof.* Throughout the proof, we may disregard the dependence on the variable $\varphi$, specifying it only when it is relevant. Differentiating (3.14) with respect to $r$ in the direction $h$ gives

$$\begin{aligned} d_r\mathbf{G}(r)[h](\theta) &= \varepsilon^3\omega(\lambda)\,\partial_\varphi h(\theta)-\tfrac{\varepsilon^2}{2}\partial_\theta\mathrm{Re}\Big\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{\mathrm{i}\theta}\Big\}\tfrac{h(\theta)}{R(\theta)}\\ &\quad+\partial_\theta\Big[d_r\Big(\Psi_1\big(r,z(\theta)\big)\Big)[h](\theta)+d_r\Big(\Psi_2\big(r,z(\theta)\big)\Big)[h](\theta)\Big]. \end{aligned} \tag{7.20}$$

The linearized of $\Psi_1$ has been computed in [59, Prop. 3.1] and is given by

$$\partial_\theta d_r\Big(\Psi_1\big(r,z(\theta)\big)\Big)[h](\theta)=\varepsilon\partial_\theta\Big[\big(\tfrac12-\tfrac{\varepsilon}{2}r(\theta)+\varepsilon^2V_1^{\varepsilon}(r)(\theta)\big)h(\theta)\Big]-\tfrac{\varepsilon}{2}\mathbf{H}[h](\theta)+\varepsilon^3\partial_\theta\mathcal{R}_1^{\varepsilon}[h](\theta),$$

where $V_1^{\varepsilon}$ satisfies the desired estimates and $\mathcal{R}_1^{\varepsilon}$ is an integral operator with the description (7.16) and (7.18). Linearizing (3.13), one readily finds

$$\begin{aligned} d_r\Big(\Psi_2\big(r,z(\theta)\big)\Big)[h](\theta) &= \varepsilon\int_{\mathbb{T}} h(\eta)K\big(\varepsilon z(\theta)+p,\varepsilon z(\eta)+p\big)d\eta\\ &\quad+\int_{\mathbb{T}}\int_0^{R(\eta)} d_r\Big(K\big(\varepsilon z(\theta)+p,\varepsilon l e^{\mathrm{i}\eta}+p\big)\Big)[h](\theta)ldld\eta\\ &\triangleq J_1[h](\theta)+J_2[h](\theta). \end{aligned} \tag{7.21}$$

We shall start with expanding the nonlocal term $J_1[h]$ using the structure of the kernel (7.6),

$$\partial_\theta J_1[h](\theta)=\varepsilon\partial_\theta\sum_{j=1}^{3}\varepsilon^j\int_{\mathbb{T}} h(\eta)K_j\big(p,z(\theta),z(\eta)\big)d\eta.$$

We have used that the test function $h$ is of zero spatial average. From $K_1$, detailed in Lemma 7.1, we obtain

$$\partial_\theta\int_{\mathbb{T}} h(\eta)K_1\big(p,z(\theta),z(\eta)\big)d\eta=\partial_\theta\mathrm{Re}\left\{\partial_z\mathcal{R}_{\mathbf{D}}(p)\int_{\mathbb{T}} h(\eta)\big(z(\theta)+z(\eta)\big)d\eta\right\}.$$

Using one time more that $h$ has a zero spatial average, we deduce that

$$\partial_\theta\int_{\mathbb{T}} h(\eta)K_1\big(p,z(\theta),z(\eta)\big)d\eta=0.$$

Regarding the second term, in view of Lemma 7.1 and using once again that $h$ has a zero average, we obtain

$$\partial_\theta \int_{\mathbb{T}} h(\eta) K_2\big(p, z(\theta), z(\eta)\big) d\eta = \partial_\theta \mathrm{Re}\left\{\tfrac{1}{6} S(\Phi)(p) z(\theta) \int_{\mathbb{T}} h(\eta) z(\eta) d\eta\right\} + \frac{1}{r_{\mathbf{D}}^2(p)} \partial_\theta \mathrm{Re}\left\{z(\theta) \int_{\mathbb{T}} h(\eta)\overline{z(\eta)} d\eta\right\}.$$

Applying the parametrization (3.12) we infer

$$\partial_\theta \int_{\mathbb{T}} h(\eta) K_2\big(p, z(\theta), z(\eta)\big) d\eta = \partial_\theta \mathrm{Re}\left\{\tfrac{1}{6} S(\Phi)(p) \int_{\mathbb{T}} h(\eta) e^{\mathrm{i}(\theta+\eta)} d\eta\right\} + \frac{1}{r_{\mathbf{D}}^2(p)} \partial_\theta \mathrm{Re}\left\{\int_{\mathbb{T}} h(\eta) e^{\mathrm{i}(\theta-\eta)} d\eta\right\} + \varepsilon \partial_\theta \mathcal{R}_{1,2}^{\varepsilon}[h],$$

with $\mathcal{R}_{1,2}^{\varepsilon}$ a kernel operator, whose kernel satisfies the estimate (7.19). Gathering the preceding identities gives

$$\partial_\theta J_1[h] = \varepsilon^3 \partial_\theta \mathbf{Q}_1[h] + \varepsilon^4 \partial_\theta \mathcal{R}_2^{\varepsilon}[h],$$

where $\mathbf{Q}_1$ is a nonlocal operator localizing on the modes $\pm 1$ and defined by

$$\mathbf{Q}_1[h](\theta) \triangleq \int_{\mathbb{T}} h(\eta) \mathtt{Q}(p,\theta,\eta) d\eta, \tag{7.22}$$

where the kernel takes the form,

$$\mathtt{Q}(p,\theta,\eta) \triangleq \frac{\cos(\theta-\eta)}{r_{\mathbf{D}}^2(p)} + \frac{1}{6}\mathrm{Re}\left\{e^{\mathrm{i}(\theta+\eta)} S(\Phi)(p)\right\}.$$

The remainder operator $\mathcal{R}_2^{\varepsilon}$ has the form

$$\mathcal{R}_2^{\varepsilon}[h] \triangleq \mathcal{R}_{1,2}^{\varepsilon}[h] + \int_{\mathbb{T}} h(\eta) K_3\big(p, z(\theta), z(\eta)\big) d\eta.$$

From Lemma 7.1, we infer that $K_3 \in C_b^\infty(V \times \mathbb{D}_2^2)$. Therefore $\mathcal{R}_2^{\varepsilon}$ is an integral operator whose kernel obeys to the estimate of (7.19).

Next, let us move to the estimate of the local term in (7.21). Performing Taylor formula with the integral contribution $\int_0^R ..$ allows to get

$$\begin{aligned} J_2[h](\theta) = & \int_{\mathbb{T}} \int_0^1 d_r\Big(K\big(\varepsilon z(\theta) + p, \varepsilon l e^{\mathrm{i}\eta} + p\big)\Big)[h](\theta) l dl d\eta \\ & + \varepsilon \int_{\mathbb{T}} \int_0^1 \tfrac{r(\eta)}{\sqrt{1+2\varepsilon\tau r(\eta)}} d_r\Big(K\big(\tau\varepsilon z(\theta) + p, \tau\varepsilon e^{\mathrm{i}\eta} + p\big)\Big)[h](\theta) d\eta d\tau, \end{aligned} \tag{7.23}$$

where $\mathbf{L}_\varepsilon[r]$ is a non-local term with smooth kernel (depending on $K$) at least quadratic in $r$. By using the decomposition (7.6), we obtain

$$\begin{aligned} & d_r\Big(K\big(\varepsilon z(\theta) + p, \varepsilon l e^{\mathrm{i}\eta} + p\big)\Big)[h](\theta) = \varepsilon^2 \tfrac{h(\theta)}{R(\theta)} \mathrm{Re}\left\{\partial_z \mathcal{R}_{\mathbf{D}}(p) e^{\mathrm{i}\theta}\right\} \\ & + \varepsilon^3 h(\theta) \mathrm{Re}\left\{\mathtt{w}_2(p) e^{2\mathrm{i}\theta}\right\} + \varepsilon^3 \tfrac{h(\theta) l}{6R(\theta)} \mathrm{Re}\left\{S(\Phi)(p) e^{\mathrm{i}(\theta+\eta)}\right\} + \varepsilon^3 \tfrac{h(\theta) l}{r_{\mathbf{D}}^2(p) R(\theta)} \mathrm{Re}\left\{e^{\mathrm{i}(\theta-\eta)}\right\} \\ & + \varepsilon^4 \tfrac{h(\theta)}{R(\theta)} \mathrm{Re}\Big\{\partial_z K_3(p, z(\theta), l e^{\mathrm{i}\eta})\, e^{\mathrm{i}\theta}\Big\}, \end{aligned} \tag{7.24}$$

with $\mathtt{w}_2$ as in (7.14). Hence, the zero space average condition for $r$ implies

$$\varepsilon \int_{\mathbb{T}} \int_0^1 \tfrac{r(\eta)}{\sqrt{1+2\varepsilon\tau r(\eta)}} d_r\Big(K\big(\tau\varepsilon z(\theta) + p, \tau\varepsilon e^{\mathrm{i}\eta} + p\big)\Big)[h](\theta) d\eta d\tau = \varepsilon^4 V_{2,1}(r) h(\theta)$$

and we can check in a straightforward manner that $V_{2,1}^{\varepsilon}$ satisfies the estimates of $V_2^{\varepsilon}$ in Proposition 7.2. Coming back to (7.23) and using (7.24) we find

$$\begin{aligned}&\int_{\mathbb{T}}\int_0^1 d_r\Big(K\big(\varepsilon z(\theta)+p(\varphi),\varepsilon l e^{\mathrm{i}\eta}+p(\varphi)\big)\Big)[h](\theta)ldld\eta\\&=\tfrac{\varepsilon^2}{2}\tfrac{h(\theta)}{R(\theta)}\mathrm{Re}\left\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{\mathrm{i}\theta}\right\}+\tfrac{\varepsilon^3}{2}\mathrm{Re}\left\{\mathtt{w}_2\big(p(\varphi)\big)e^{2\mathrm{i}\theta}\right\}h(\theta)+\varepsilon^4V_{2,2}(r)h(\theta).\end{aligned}$$

Notice that $V_{2,2}(r)$ satisfies the same estimates of $V_2^{\varepsilon}$ in Proposition 7.2. Putting the preceding identities in (7.20) yields to the asymptotic expansion of the linearized operator as described in Proposition 7.2. □

### 7.3 Construction of an approximate solution

In the following lemma, we provide the leading term in the asymptotic expansion of $\mathbf{G}(0)$ as described by (3.14). This result will be pertinent later when developing an appropriate scheme to obtain a precise approximation, as discussed in Lemma 7.5.

**Lemma 7.3.** *There exists $\varepsilon_0\in(0,1)$ such that for all $\varepsilon\in(0,\varepsilon_0)$, one has*

$$\forall(\varphi,\theta)\in\mathbb{T}^2,\quad \mathbf{G}(0)(\varphi,\theta)=-\tfrac{\varepsilon^2}{2}\mathrm{Im}\left\{\mathtt{w}_2\big(p(\varphi)\big)e^{2\mathrm{i}\theta}\right\}+O(\varepsilon^3),$$

*where $\mathtt{w}_2$ is defined in (7.14). In addition, the function $\mathbf{G}(0)$ does not contain the spatial modes $\pm1$, that is,*

$$\int_{\mathbb{T}}\mathbf{G}(0)(\varphi,\theta)e^{\pm\mathrm{i}\theta}d\theta=0.$$

*Proof.* To simplify the notation, we shall remove the dependence in $\varphi$. Substituting $r=0$ into (3.14) gives

$$\mathbf{G}(0)(\theta)=\partial_\theta\Big[-\tfrac{\varepsilon}{2}\mathrm{Re}\Big\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{\mathrm{i}\theta}\Big\}+\Psi_1\big(0,e^{\mathrm{i}\theta}\big)+\Psi_2\big(0,e^{\mathrm{i}\theta}\big)\Big].$$

In view of (3.13), we have

$$\partial_\theta\big[\Psi_1\big(0,e^{\mathrm{i}\theta}\big)\big]=\partial_\theta\left[\int_{\mathbb{T}}\int_0^1\log\big(\big|1-le^{\mathrm{i}(\eta-\theta)}\big|\big)ldld\eta\right]=0.$$

As for the $\Psi_2$ term, one gets by virtue of (3.13)

$$\Psi_2\big(0,e^{\mathrm{i}\theta}\big)=\int_{\mathbb{T}}\int_0^1 K\big(p+\varepsilon e^{\mathrm{i}\theta},p+\varepsilon l e^{\mathrm{i}\eta}\big)ldld\eta.$$

We apply Lemma 7.1 with $z=e^{\mathrm{i}\theta}$ and $\xi=le^{\mathrm{i}\eta}$ leading in view of (7.14) to

$$\begin{aligned}\partial_\theta\Big(\Psi_2\big(0,e^{\mathrm{i}\theta}\big)\Big)&=\tfrac{\varepsilon}{2}\partial_\theta\mathrm{Re}\left\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{\mathrm{i}\theta}\right\}+\tfrac{\varepsilon^2}{2}\mathrm{Re}\left\{\mathrm{i}e^{2\mathrm{i}\theta}\mathtt{w}_2(p)\right\}+O(\varepsilon^3)\\&=\tfrac{\varepsilon}{2}\partial_\theta\mathrm{Re}\left\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{\mathrm{i}\theta}\right\}-\tfrac{\varepsilon^2}{2}\mathrm{Im}\left\{e^{2\mathrm{i}\theta}\mathtt{w}_2(p)\right\}+O(\varepsilon^3).\end{aligned}$$

Combining the foregoing identities gives the suitable expansion for $\mathbf{G}(0)$. The final goal is to prove the absence of modes $\pm1$ in the whole quantity $\mathbf{G}(0)$. This pertains to show that the Fourier expansion of $\Psi_2\big(0,e^{\mathrm{i}\theta}\big)-\frac{\varepsilon}{2}\mathrm{Re}\left\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{\mathrm{i}\theta}\right\}$ does not contain the modes $\pm1$. Applying (7.9) we infer

$$K(p+\varepsilon z,p+\varepsilon\xi)=\mathrm{Re}\left\{H(p+\varepsilon z,p+\varepsilon\xi)\right\},$$

where

$$H(z_1,z_2)\triangleq\log(b)+\mathrm{Log}\left(1+\frac{1}{b}\frac{f(z_1)-f(z_2)}{z_1-z_2}\right)-\mathrm{Log}\big(1-\Phi(z_2)\overline{\Phi(z_1)}\big)$$

and Log is the principal value of the complex logarithm function. Applying this with $z = e^{i\theta}$ and $\xi = le^{i\eta}$, we obtain

$$\Psi_2\big(0, e^{i\theta}\big) = \mathrm{Re}\left\{\int_{\mathbb{T}}\int_0^1 H\big(p+\varepsilon e^{i\theta}, p+\varepsilon l e^{i\eta}\big) l dl d\eta\right\}.$$

Noting that the function $\xi \in \mathbb{D}_2 \mapsto H(p+\varepsilon e^{i\theta}, p+\varepsilon\xi)$ is holomorphic for small $\varepsilon$ allows to get the power series expansion

$$H\big(p+\varepsilon e^{i\theta}, p+\varepsilon\xi\big) = \sum_{n=0}^{\infty} c_n(\varepsilon, p, \theta)\xi^n.$$

Therefore

$$\begin{aligned}\Psi_2\big(0, e^{i\theta}\big) &= \mathrm{Re}\left\{\int_{\mathbb{T}}\int_0^1 \sum_{n=0}^{\infty} c_n(\varepsilon, p, \theta) e^{in\eta} l^{n+1} dl d\eta\right\}\\ &= \mathrm{Re}\left\{\tfrac{1}{2}c_0(\varepsilon, p, \theta)\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}\Psi_2\big(0, e^{i\theta}\big) &= \mathrm{Re}\left\{\tfrac{1}{2}H\big(p+\varepsilon e^{i\theta}, p\big)\right\}\\ &= \mathrm{Re}\left\{\tfrac{1}{2}H_1\big(p+\varepsilon e^{i\theta}, p\big)\right\},\end{aligned}$$

with

$$H_1(z_1, z_2) \triangleq \log(b) + \mathrm{Log}\left(1 + \frac{1}{b}\frac{f(z_1)-f(z_2)}{z_1 - z_2}\right) - \mathrm{Log}\big(1 - \Phi(z_1)\overline{\Phi(z_2)}\big).$$

As the function $\xi \in \mathbb{D}_2 \mapsto H_1(p+\varepsilon\xi, p)$ is holomorphic for small $\varepsilon$, then

$$H_1(p+\varepsilon e^{i\theta}, p) = \sum_{n=0}^{\infty} \tfrac{\partial_{z_1}^n H_1(p,p)}{n!}\varepsilon^n e^{in\theta},$$

which implies that

$$\Psi_2\big(0, e^{i\theta}\big) = \tfrac{1}{2}H_1(p,p) + \tfrac{\varepsilon}{2}\mathrm{Re}\left\{\partial_{z_1}H_1(p,p)e^{i\theta}\right\} + \tfrac{1}{2}\mathrm{Re}\sum_{n=2}^{\infty}\tfrac{\partial_{z_1}^n H_1(p,p)}{n!}\varepsilon^n e^{in\theta}.$$

One can check that

$$H_1(p,p) = \mathcal{R}_{\mathbf{D}}(p) \quad \text{and} \quad \partial_{z_1}H_1(p,p) = \partial_z\mathcal{R}_{\mathbf{D}}(p).$$

Consequently,

$$\Psi_2\big(0, e^{i\theta}\big) - \tfrac{\varepsilon}{2}\mathrm{Re}\left\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{i\theta}\right\} = \tfrac{1}{2}\mathcal{R}_{\mathbf{D}}(p) + \tfrac{1}{2}\mathrm{Re}\sum_{n\geqslant 2}^{\infty}\tfrac{\partial_{z_1}^n H_1(p,p)}{n!}\varepsilon^n e^{in\theta}.$$

This shows that the Fourier expansion of $\Psi_2\big(0, e^{i\theta}\big) - \tfrac{\varepsilon}{2}\mathrm{Re}\left\{\partial_z\mathcal{R}_{\mathbf{D}}(p)e^{i\theta}\right\}$ does not contain the modes $\pm 1$. This ends the proof of the lemma.

□

We have already seen in Lemma 7.3 that the function $\mathbf{G}(0)$ does not contain the modes $\pm 1$. Furthermore, Proposition 7.2 ensures that $\mathsf{g}$ is localized on the mode 2. Leveraging these insights and adopting a similar approach as developed in [59, Lemma 3.3, Proposition 3.4] we derive a suitable approximate solution $r_\varepsilon$. Upon rescaling the functional, the behavior of the linearized operator is quite similar to the original one. Our result reads as follows.

**Lemma 7.4.** *There exists $r_\varepsilon \in C^\infty(\mathbb{T}^2)$, satisfying*

$$\|r_\varepsilon + \mathtt{g}\|_s^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon,$$

*such that the nonlinear functional*

$$\mathbf{F}(\rho) \triangleq \tfrac{1}{\varepsilon^{2+\mu}}\mathbf{G}(\varepsilon r_\varepsilon + \varepsilon^{1+\mu}\rho), \qquad \mu \in (0,1), \tag{7.25}$$

*satisfies the following estimates*

$$\|\mathbf{F}(0)\|_s^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^{3-\mu}.$$

*Proof.* The starting point is to write a Taylor expansion of the functional $\mathbf{G}$ around 0,

$$\mathbf{G}(r) = \mathbf{G}(0) + \partial_r\mathbf{G}(0)[r] + \tfrac{1}{2}\partial_r^2\mathbf{G}(0)[r,r] + \tfrac{1}{2}\int_0^1 (1-\tau)^2\partial_r^3\mathbf{G}(\tau r)[r,r,r]d\tau.$$

Then, we look for an approximate solution in the form $r = \varepsilon r_0 + \varepsilon^2 r_1$. Hence,

$$\begin{aligned}\mathbf{G}(\varepsilon r_0 + \varepsilon^2 r_1) &= \mathbf{G}(0) + \varepsilon\partial_r\mathbf{G}(0)[r_0] + \tfrac{\varepsilon^2}{2}\partial_r^2\mathbf{G}(0)[r_0,r_0] + \varepsilon^2\partial_r\mathbf{G}(0)[r_1]\\ &+ \varepsilon^3\partial_r^2\mathbf{G}(0)[r_0,r_1] + \tfrac{\varepsilon^4}{2}\partial_r^2\mathbf{G}(0)[r_1,r_1] + \tfrac{1}{2}\int_0^1 (1-\tau)^2\partial_r^3\mathbf{G}(\tau r)[r,r,r]d\tau.\end{aligned} \tag{7.26}$$

In view of Proposition 7.2 one has

$$\begin{aligned}&\mathbf{G}(0) + \varepsilon\partial_r\mathbf{G}(0)[r_0] + \tfrac{\varepsilon^2}{2}\partial_r^2\mathbf{G}(0)[r_0,r_0] = \mathbf{G}(0) + \varepsilon^4\omega(\lambda)\partial_\varphi r_0\\ &+ \varepsilon^2\partial_\theta\Big[\big(\tfrac{1}{2} + \tfrac{\varepsilon^2}{2}\mathtt{g} + \varepsilon^3 V_2^\varepsilon(0)\big)r_0\Big] - \tfrac{\varepsilon^2}{2}\mathbf{H}[h] + \varepsilon^4\partial_\theta\mathbf{Q}_1[r_0] - \tfrac{\varepsilon^4}{4}\partial_\theta\big(r_0^2\big) + \varepsilon^5\partial_\theta\mathcal{E}^\varepsilon(r_0),\end{aligned} \tag{7.27}$$

with

$$\|\partial_\theta\mathcal{E}^\varepsilon(r_0)\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|r_0\|_{s+2}^{\mathrm{Lip}(\gamma)}.$$

At the main order we shall solve the equation

$$\tfrac{\varepsilon^2}{2}\big[\partial_\theta - \mathbf{H}\big]r_0 + \mathbf{G}(0) = 0.$$

According to Lemma 7.3, the source term is of size $O(\varepsilon^2)$ and does not contain the modes 0 and $\pm1$. Therefore, we can solve the previous equation

$$\begin{aligned}r_0(\varphi,\theta) &= -2\varepsilon^{-2}\big[\partial_\theta - \mathbf{H}\big]^{-1}\mathbf{G}(0)\\ &= -\mathrm{Re}\Big\{\mathtt{w}_2\big(p(\varphi)\big)e^{2i\theta}\Big\} + \varepsilon\underline{r}_0(\varphi,\theta)\\ &= -\mathtt{g}(\varphi,\theta) + \varepsilon\underline{r}_0(\varphi,\theta),\end{aligned} \tag{7.28}$$

where $\mathtt{w}_2$ and $\mathtt{g}$ are defined in (7.14) and $\underline{r}_0 \in C^\infty(\mathbb{T}^2)$. Moreover, $\underline{r}_0$ does not contain the modes 0 and $\pm1$ which implies according to (7.15) and (7.17) that

$$\partial_\theta\mathbf{Q}_1[r_0] = 0.$$

Gathering the identities above, we conclude that

$$\mathbf{G}(0) + \varepsilon\partial_r\mathbf{G}(0)[r_0] + \tfrac{\varepsilon^2}{2}\partial_r^2\mathbf{G}(0)[r_0,r_0] = \varepsilon^4\mathbf{B}_1 + \varepsilon^5\mathbf{B}_2, \tag{7.29}$$

where $\mathbf{B}_1$ is given by

$$\mathbf{B}_1 \triangleq -\omega(\lambda)\partial_\varphi\mathtt{g} - \tfrac{3}{4}\partial_\theta\big(\mathtt{g}^2\big)$$

and $\mathbf{B}_2$ satisfies

$$\|\mathbf{B}_2\|_s^{\mathrm{Lip}(\gamma)} \lesssim 1.$$

Thus, $\mathbf{B}_1$ contains only even modes and the equation

$$\tfrac{1}{2}\big[\partial_\theta - \mathbf{H}\big]r_1 + \varepsilon\mathbf{B}_1 = 0 \tag{7.30}$$

admits a solution that satisfies the estimates

$$\|r_1\|_s^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon.$$

Since $r_1$ does not contain the modes $\pm 1$ then

$$\partial_\theta \mathbf{Q}_1[r_1] = 0.$$

Therefore, by (7.28)

$$\begin{aligned} r_\varepsilon &\triangleq r_0 + \varepsilon r_1 \\ &= -\mathtt{g} + \varepsilon\big(\underline{r}_0 + r_1\big) \end{aligned} \tag{7.31}$$

and inserting (7.27), (7.30) into (7.26) we obtain

$$\|\mathbf{G}(\varepsilon r_\varepsilon)\|_s^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^5.$$

This concludes the proof of Lemma 7.4. □

The next objective is to describe the asymptotic structure of the linearized operator associated with the rescaled function $F$ introduced in (7.25).

**Lemma 7.5.** *There exists $\varepsilon_0 \in (0,1)$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and any smooth function $\rho$ with zero space average and satisfying*

$$\|\rho\|_{s_0+2}^{\mathrm{Lip}(\gamma)} \leqslant 1,$$

*the following holds true. The linearized operator $\mathcal{L}_0 \triangleq d_\rho \mathbf{F}(\rho)$, see (7.25), has the form*

$$\mathcal{L}_0 h = \varepsilon^2 \omega(\lambda)\partial_\varphi h + \partial_\theta\big[\mathbf{V}_1^\varepsilon(\rho) h\big] - \tfrac{1}{2}\mathbf{H}[h] + \varepsilon^2 \partial_\theta \mathbf{Q}_1[h] + \varepsilon^3 \partial_\theta \mathcal{R}_0^\varepsilon(\rho)[h], \tag{7.32}$$

*where the function $\mathbf{V}_1^\varepsilon(\rho)$ is given by*

$$\mathbf{V}_1^\varepsilon(\rho) \triangleq \tfrac{1}{2} + \varepsilon^2 \mathtt{g} - \tfrac{\varepsilon^{2+\mu}}{2}\rho + \varepsilon^3 V^\varepsilon(\rho). \tag{7.33}$$

*The function $\mathtt{g}$ is defined in Proposition 7.2 and $V^\varepsilon(\rho)$ enjoys the estimates: for any $s \geqslant s_0$*

$$\begin{aligned} \|V^\varepsilon(\rho)\|_s^{\mathrm{Lip}(\gamma)} &\lesssim 1 + \varepsilon^\mu \|\rho\|_{s+1}^{\mathrm{Lip}(\gamma)}, \\ \|\Delta_{12} V^\varepsilon(\rho)\|_s^{\mathrm{Lip}(\gamma)} &\lesssim \varepsilon^\mu \|\Delta_{12}\rho\|_{s+1}^{\mathrm{Lip}(\gamma)} + \varepsilon^\mu \|\Delta_{12}\rho\|_{s_0+1}^{\mathrm{Lip}(\gamma)} \max_{\ell\in\{1,2\}} \|\rho_\ell\|_{s+1}^{\mathrm{Lip}(\gamma)}. \end{aligned} \tag{7.34}$$

*The operator $\mathbf{Q}_1$ is given in Proposition 7.2 the operator $\mathcal{R}_0^\varepsilon(\rho)$ is an integral operator of the form*

$$\mathcal{R}_0^\varepsilon(\rho)[h](\varphi,\theta) \triangleq \int_{\mathbb{T}} h(\varphi,\eta)\mathcal{K}_0^\varepsilon(\rho)(\varphi,\theta,\eta)d\eta,$$

*where the kernel $\mathcal{K}_0^\varepsilon(\rho)$ satisfies: for any $s \geqslant s_0$*

$$\|\mathcal{K}_0^\varepsilon(\rho)\|_s^{\mathrm{Lip}(\gamma)} \lesssim 1 + \varepsilon^\mu \|\rho\|_{s+1}^{\mathrm{Lip}(\gamma)}. \tag{7.35}$$

*Moreover, for any $s \geqslant s_0$, one has*

$$\begin{aligned} \big\|d_\rho \mathbf{F}(\rho)[h]\big\|_s^{\mathrm{Lip}(\gamma)} &\lesssim \|h\|_{s+2}^{\mathrm{Lip}(\gamma)} + \|\rho\|_{s+2}^{\mathrm{Lip}(\gamma)}\|h\|_{s_0+2}^{\mathrm{Lip}(\gamma)}, \\ \big\|d_\rho^2 \mathbf{F}(\rho)[h,h]\big\|_s^{\mathrm{Lip}(\gamma)} &\lesssim \varepsilon^2 \|h\|_{s+2}^{\mathrm{Lip}(\gamma)}\|h\|_{s_0+2}^{\mathrm{Lip}(\gamma)} + \varepsilon^2 \|\rho\|_{s+2}^{\mathrm{Lip}(\gamma)}\big(\|h\|_{s_0+2}^{\mathrm{Lip}(\gamma)}\big)^2. \end{aligned} \tag{7.36}$$

*Proof.* Linearizing the functional $\mathbf{F}$ defined by (7.25) an using Proposition 7.2 we obtain

$$\begin{aligned} \partial_\rho \mathbf{F}(\rho)[h] &= \tfrac{1}{\varepsilon}(\partial_r \mathbf{G})(\varepsilon r_\varepsilon + \varepsilon^{1+\mu}\rho)[h] \\ &= \varepsilon^2 \omega_0 \partial_\varphi h + \partial_\theta\Big[\Big(\tfrac{1}{2} - \tfrac{\varepsilon^2}{2} r_\varepsilon - \tfrac{\varepsilon^{2+\mu}}{2}\rho(\varphi,\theta) + \tfrac{\varepsilon^2}{2}\mathtt{g} \\ &\quad + \varepsilon^2 V_1^\varepsilon(\varepsilon r_\varepsilon + \varepsilon^{1+\mu}\rho) + \varepsilon^3 V_2^\varepsilon(\varepsilon r_\varepsilon + \varepsilon^{1+\mu}\rho)\Big) h\Big] - \tfrac{1}{2}\mathbf{H}[h] \\ &\quad + \varepsilon^2 \mathbf{Q}_1[h] + \varepsilon^2 \partial_\theta \mathcal{R}_1^\varepsilon(\varepsilon r_\varepsilon + \varepsilon^{1+\mu}\rho)[h] + \varepsilon^3 \partial_\theta \mathcal{R}_2^\varepsilon(\varepsilon r_\varepsilon + \varepsilon^{1+\mu}\rho)[h]. \end{aligned}$$

By (7.31) we infer

$$-r_\varepsilon(\varphi,\theta)+\mathtt{g}(\varphi,\theta)\triangleq 2\mathtt{g}(\varphi,\theta)+2\varepsilon r_2(\varphi,\theta),\qquad \text{with}\qquad \|r_2\|_s^{\mathrm{Lip}(\gamma)}\lesssim 1. \tag{7.37}$$

Setting

$$\begin{aligned} V^\varepsilon(\rho)&\triangleq r_2+\tfrac{1}{\varepsilon}V_1^\varepsilon(\varepsilon r_\varepsilon+\varepsilon^{1+\mu}\rho)+V_2^\varepsilon(\varepsilon r_\varepsilon+\varepsilon^{1+\mu}\rho),\\ \mathcal{R}_0^\varepsilon(\rho)[h]&\triangleq \tfrac{1}{\varepsilon}\mathcal{R}_1^\varepsilon(\varepsilon r_\varepsilon+\varepsilon^{1+\mu}\rho)[h]+\mathcal{R}_2^\varepsilon(\varepsilon r_\varepsilon+\varepsilon^{1+\mu}\rho)[h],\end{aligned}$$

gives (7.32) and (7.33). The estimates (7.34), (7.35), (7.36) follow easily from Proposition 7.2. This ends the proof of Lemma 7.5. □

### 7.4 Straightening the transport part

The purpose of this section is to conjugate the linearized operator $\mathcal{L}_0$ in (7.32) through appropriate symplectic changes of variables so that in the new coordinates, the transport part has constant coefficients. Following the approach outlined in [59, Sec. 4], we begin by performing a suitable change of variable acting only on space to transform the vector field in the advection part into a constant one, up to an error term of small size in $\varepsilon$. Notably, the time degeneracy is advantageous in this step. With this new structure, one may apply the standard KAM reduction method in the spirit of [8, 58].

**Proposition 7.6.** *Let $(\gamma,s_0,S)$ as in (7.3). There exists $\varepsilon_0>0$ such that if*

$$\varepsilon\leqslant\varepsilon_0,\qquad \|\rho\|_{s_0+2}^{\mathrm{Lip}(\gamma)}\leqslant 1, \tag{7.38}$$

*there exist $\beta_1$ taking the form*

$$\beta_1(\lambda,\varphi,\theta)=\varepsilon b_1(\lambda,\varphi)+\varepsilon^2 b_2(\lambda,\varphi,\theta)+\varepsilon^{2+\mu}\mathfrak{r}(\lambda,\varphi,\theta),$$

*with*

$$b_2(\lambda,\varphi,\theta)\triangleq \mathrm{Im}\Big\{\mathtt{w}_2\big(p_\lambda(\varphi)\big)e^{\mathrm{i}2\theta}\Big\}, \tag{7.39}$$

*where $\mathtt{w}_2(p)$ is defined in (7.14) and an invertible symplectic change of variables*

$$\mathscr{B}_1 h(\lambda,\varphi,\theta)\triangleq\big(1+\partial_\theta\beta_1(\lambda,\varphi,\theta)\big)h\big(\lambda,\varphi,\theta+\beta_1(\lambda,\varphi,\theta)\big) \tag{7.40}$$

*such that*

$$\mathscr{B}_1^{-1}\Big(\varepsilon^2\omega(\lambda)\partial_\varphi+\partial_\theta\big[\mathbf{V}_1^\varepsilon(\rho)\,\cdot\,\big]\Big)\mathscr{B}_1=\varepsilon^2\omega(\lambda)\partial_\varphi+\partial_\theta\big[\big(\mathtt{c}_0+\varepsilon^6\mathbf{V}_2^\varepsilon(\rho)\big)\,\cdot\,\big]. \tag{7.41}$$

*Furthermore, we have*

1. *The space-time independent function $\lambda\mapsto\mathtt{c}_0(\lambda)$ writes*

$$\mathtt{c}_0\triangleq\tfrac12+\varepsilon^3\mathtt{c}_1,\qquad \|\mathtt{c}_1\|^{\mathrm{Lip}(\gamma)}\lesssim 1. \tag{7.42}$$

2. *The functions $\beta_1$, $\mathfrak{r}$, $b_1$ and $\mathbf{V}_2^\varepsilon(\rho)$ satisfy the following estimates for all $s\in[s_0,S]$,*

$$\begin{aligned}\|\beta_1\|_s^{\mathrm{Lip}(\gamma)}&\lesssim\varepsilon\left(1+\|\rho\|_{s+2}^{\mathrm{Lip}(\gamma)}\right),\\ \|b_1\|_s^{\mathrm{Lip}(\gamma)}+\|\mathfrak{r}\|_s^{\mathrm{Lip}(\gamma)}&\lesssim 1+\|\rho\|_{s+2}^{\mathrm{Lip}(\gamma)},\\ \|\mathbf{V}_2^\varepsilon(\rho)\|_s^{\mathrm{Lip}(\gamma)}&\lesssim 1+\|\rho\|_{s+3}^{\mathrm{Lip}(\gamma)}.\end{aligned} \tag{7.43}$$

3. *The changes of variables $\mathscr{B}_1^{\pm1}$ satisfy the following estimates for all $s\in[s_0,S]$,*

$$\|\mathscr{B}_1^{\pm1}h\|_s^{\mathrm{Lip}(\gamma)}\lesssim\|h\|_s^{\mathrm{Lip}(\gamma)}+\varepsilon\|\rho\|_{s+3}^{\mathrm{Lip}(\gamma)}\|h\|_{s_0}^{\mathrm{Lip}(\gamma)}. \tag{7.44}$$

*4. Given two states $\rho_1$ and $\rho_2$ with the smallness property* (7.38)*, then*

$$\|\Delta_{12}\mathtt{c}_0\|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^3\|\Delta_{12}\rho\|^{\mathrm{Lip}(\gamma)}_{s_0+2}, \tag{7.45}$$
$$\|\Delta_{12}\mathbf{V}^\varepsilon_2\|^{\mathrm{Lip}(\gamma)}_s \lesssim \|\Delta_{12}\rho\|^{\mathrm{Lip}(\gamma)}_{s+3} + \|\Delta_{12}\rho\|^{\mathrm{Lip}(\gamma)}_{s_0+3}\max_{k\in\{1,2\}}\|\rho_k\|^{\mathrm{Lip}(\gamma)}_{s+3}. \tag{7.46}$$

*Proof.* Let us consider a symplectic diffeomorphism of the torus in the form

$$\mathscr{B}_1 h(\varphi,\theta) \triangleq \big(1+\partial_\theta\beta_1(\varphi,\theta)\big)\mathrm{B}_1 h(\varphi,\theta), \qquad \mathrm{B}_1 h(\varphi,\theta) \triangleq h\big(\varphi,\theta+\beta_1(\varphi,\theta)\big), \tag{7.47}$$

with $\beta_1$ to be chosen later in order to reduce the size of the coefficients. Due to the symplectic nature of $\mathscr{B}_1$, we have

$$\langle h\rangle_\theta = 0 \quad\Rightarrow\quad \langle \mathscr{B}_1 h\rangle_\theta = 0.$$

Applying Lemma A.1-2, we obtain

$$\mathscr{B}_1^{-1}\Big(\varepsilon^2\omega(\lambda)\partial_\varphi + \partial_\theta\big[\mathbf{V}^\varepsilon_1(\rho)\,\cdot\,\big]\Big)\mathscr{B}_1 = \varepsilon^2\omega(\lambda)\partial_\varphi + \partial_y\Big[\mathrm{B}_1^{-1}\big(\widehat{\mathbf{V}}^\varepsilon_1(\rho)\big)\,\cdot\,\Big],$$

with, using (7.33),

$$\begin{aligned}\widehat{\mathbf{V}}^\varepsilon_1(\rho) &\triangleq \varepsilon^2\omega(\lambda)\partial_\varphi\beta_1 + \mathbf{V}^\varepsilon_1(\rho)\big(1+\partial_\theta\beta_1\big)\\ &= \varepsilon^2\omega(\lambda)\partial_\varphi\beta_1 + \Big(\tfrac12 + \varepsilon^2\mathtt{g} - \tfrac{\varepsilon^{2+\mu}}{2}\rho + \varepsilon^3 V^\varepsilon(\rho)\Big)\big(1+\partial_\theta\beta_1\big).\end{aligned}$$

Select $\beta_1$ in the following form

$$\beta_1(\lambda,\varphi,\theta) = \beta_{1,1}(\lambda,\varphi) + \varepsilon^2\beta_{1,2}(\lambda,\varphi,\theta) + \varepsilon^4\beta_{1,3}(\lambda,\varphi,\theta), \tag{7.48}$$

such that for any $n\in\{1,2,3\}$, the function $\beta_{1,n}$ satisfies the following constraints

$$\begin{cases}\tfrac12\,\partial_\theta\beta_{1,2} &= \mathtt{F}_1 - \langle\mathtt{F}_1\rangle_\theta, \qquad \mathtt{F}_1 \triangleq -\mathtt{g} + \tfrac12\varepsilon^\mu\rho - \varepsilon V^\varepsilon(\rho),\\ \tfrac12\,\partial_\theta\beta_{1,3} &= \mathtt{F}_2 - \langle\mathtt{F}_2\rangle_\theta, \qquad \mathtt{F}_2 \triangleq -\omega(\lambda)\partial_\varphi\beta_{1,2} - \big(\mathtt{g} - \tfrac12\varepsilon^\mu\rho + \varepsilon V^\varepsilon(\rho)\big)\partial_\theta\beta_{1,2},\end{cases} \tag{7.49}$$

and

$$\omega(\lambda)\partial_\varphi\beta_{1,1} = -\langle\mathtt{F}_1\rangle_\theta - \varepsilon^2\langle\mathtt{F}_2\rangle_\theta + \langle\mathtt{F}_1\rangle_{\theta,\varphi} + \varepsilon^2\langle\mathtt{F}_2\rangle_{\theta,\varphi}. \tag{7.50}$$

With these choices, one can check that

$$\widehat{\mathbf{V}}^\varepsilon_1(\rho) = \tfrac12 - \varepsilon^2\langle\mathtt{F}_1\rangle_{\theta,\varphi} - \varepsilon^4\langle\mathtt{F}_2\rangle_{\theta,\varphi} + \varepsilon^6\widetilde{\mathbf{V}}^\varepsilon_1(\rho),$$

with

$$\widetilde{\mathbf{V}}^\varepsilon_1(\rho) \triangleq \omega(\lambda)\partial_\varphi\beta_{1,3} + \big(\mathtt{g} - \tfrac12\varepsilon^\mu\rho + \varepsilon V^\varepsilon(\rho)\big)\partial_\theta\beta_{1,3}. \tag{7.51}$$

Using in particular (7.14), we have

$$\langle\rho\rangle_\theta = 0, \qquad \langle\mathtt{g}\rangle_\theta = 0. \tag{7.52}$$

Therefore,

$$\langle\mathtt{F}_1\rangle_\theta = -\varepsilon\langle V^\varepsilon(\rho)\rangle_\theta \tag{7.53}$$

and

$$\widehat{\mathbf{V}}^\varepsilon_1(\rho) = \tfrac12 + \varepsilon^3\langle V^\varepsilon(\rho)\rangle_{\theta,\varphi} - \varepsilon^4\langle F_2\rangle_{\theta,\varphi} + \varepsilon^6\widetilde{\mathbf{V}}^\varepsilon_1(\rho).$$

By defining

$$\mathbf{V}^\varepsilon_2(\rho) \triangleq \mathrm{B}_1^{-1}\widetilde{\mathbf{V}}^\varepsilon_1(\rho), \qquad \mathtt{c}_0 \triangleq \tfrac12 + \varepsilon^3\mathtt{c}_1, \qquad \mathtt{c}_1 \triangleq \big(\langle V^\varepsilon(\rho)\rangle_{\theta,\varphi} - \varepsilon\langle F_2\rangle_{\theta,\varphi}\big), \tag{7.54}$$

we end up with the identity (7.41). The first equation in (7.49) writes

$$\tfrac{1}{2}\,\partial_\theta\beta_{1,2}(\varphi,\theta)=-\mathtt{g}(\varphi,\theta)+\tfrac{1}{2}\varepsilon^{\mu}\rho-\varepsilon V^{\varepsilon}(\rho)+\varepsilon\langle V^{\varepsilon}(\rho)\rangle_{\theta}.$$

The right hand-side has zero space average according to (7.52). Consequently, we can solve this equation via Fourier expansion to get from (7.14) that

$$\beta_{1,2}=b_2+\varepsilon^{\mu}\mathfrak{r}_2,\tag{7.55}$$

with $b_2$ as in (7.39) and $\mathfrak{r}_2$ satisfying, in view of (7.34), the following esitmate

$$\|\mathfrak{r}_2\|_{s}^{\mathrm{Lip}(\gamma)}\lesssim 1+\varepsilon^{\mu}\|\rho\|_{s+1}^{\mathrm{Lip}(\gamma)}.$$

As consequence

$$\|\beta_{1,2}\|_{s}^{\mathrm{Lip}(\gamma)}\lesssim 1+\|\rho\|_{s+1}^{\mathrm{Lip}(\gamma)}.\tag{7.56}$$

One can proceed in a similar way with the second equation in (7.49) and get

$$\|\beta_{1,3}\|_{s}^{\mathrm{Lip}(\gamma)}\lesssim 1+\|\rho\|_{s+2}^{\mathrm{Lip}(\gamma)}.\tag{7.57}$$

Now, from (7.53), the equation (7.50) becomes

$$\omega(\lambda)\partial_\varphi\beta_{1,1}=\varepsilon\big[\langle V^{\varepsilon}(\rho)\rangle_{\theta}-\langle V^{\varepsilon}(\rho)\rangle_{\theta,\varphi}+\varepsilon\langle \mathtt{F}_2\rangle_{\theta,\varphi}-\varepsilon\langle \mathtt{F}_2\rangle_{\theta}\big].$$

The $\varphi$-average in the right hand-side vanishes so we can invert and find from (7.34),

$$\beta_{1,1}=\varepsilon b_1,\qquad \|b_1\|_{s}^{\mathrm{Lip}(\gamma)}\lesssim 1+\varepsilon^{\mu}\|\rho\|_{s}^{\mathrm{Lip}(\gamma)}.\tag{7.58}$$

Putting together the identities (7.48),(7.55) and (7.58) leads to

$$\beta_1(\varphi,\theta)=\varepsilon b_1(\varphi)+\varepsilon^{2}b_2(\varphi,\theta)+\varepsilon^{2+\mu}\mathfrak{r}(\varphi,\theta),\qquad\text{with}\qquad \mathfrak{r}(\varphi,\theta)\triangleq\mathfrak{r}_2(\varphi,\theta)+\varepsilon^{2-\mu}\beta_{1,3}(\varphi,\theta).$$

Moreover, from the last identity and the foregoing estimates, we obtain

$$\|\beta_1\|_{s}^{\mathrm{Lip}(\gamma)}+\|\mathfrak{r}\|_{s}^{\mathrm{Lip}(\gamma)}\lesssim\varepsilon\big(1+\|\rho\|_{s+2}^{\mathrm{Lip}(\gamma)}\big).$$

This estimate together with Lemma A.2 concludes (7.44). Now, from (7.49), (7.54), (7.56) and (7.34), we infer

$$\|\mathtt{c}_1\|^{\mathrm{Lip}(\gamma)}\lesssim 1.$$

In addition, by (7.51), (7.34), (7.57), and the product law (7.4), we find

$$\|\widetilde{\mathbf{V}}_1^{\varepsilon}(\rho)\|_{s}^{\mathrm{Lip}(\gamma)}\lesssim 1+\|\rho\|_{s+3}^{\mathrm{Lip}(\gamma)}.\tag{7.59}$$

Applying (A.4) together with (7.59) and (7.38), we get

$$\begin{aligned}\|\mathbf{V}_2^{\varepsilon}(\rho)\|_{s}^{\mathrm{Lip}(\gamma)}&=\|\mathrm{B}_1^{-1}\widetilde{\mathbf{V}}_1^{\varepsilon}(\rho)\|_{s}^{\mathrm{Lip}(\gamma)}\\&\leqslant\|\widetilde{\mathbf{V}}_1^{\varepsilon}(\rho)\|_{s}^{\mathrm{Lip}(\gamma)}\big(1+C\|\beta_1\|_{s_0}^{\mathrm{Lip}(\gamma)}\big)+C\|\beta_1\|_{s}^{\mathrm{Lip}(\gamma)}\|\widetilde{\mathbf{V}}_1^{\varepsilon}(\rho)\|_{s_0}^{\mathrm{Lip}(\gamma)}\Big)\\&\lesssim 1+\|\rho\|_{s+3}^{\mathrm{Lip}(\gamma)}.\end{aligned}$$

Similarly, from Lemma A.2, the second estimate in (7.34), (7.38) we conclude (7.45). Finally, by (7.42) and (7.34), we find (7.46). This ends the proof of Proposition 7.6. □

Now, we shall state the final reduction step based on KAM tools. The proof is quite similar to that of [59, Prop. 4.2] and we recommend the reader this references for more details. To state the result, we need to use the following sequence $(N_n)_{n\in\mathbb{N}\cup\{-1\}}$ given by

$$N_{-1}\triangleq 1,\qquad\forall n\in\mathbb{N},\quad N_n\triangleq N_0^{\left(\frac{3}{2}\right)^{n}},\tag{7.60}$$

for some $N_0\geqslant 2$.

**Proposition 7.7.** *Let* $(\gamma, s_0, S)$ *as in* (7.3) *and*

$$\mu_2 \geqslant 4\tau + 3. \tag{7.61}$$

*There exists* $\varepsilon_0 > 0$ *such that if*

$$\gamma\varepsilon^{-2} \leqslant 1, \qquad N_0^{\mu_2}\varepsilon^6\gamma^{-1} \leqslant \varepsilon_0, \qquad \|\rho\|_{\frac{3}{2}\mu_2+2s_0+2\tau+4}^{\mathrm{Lip}(\gamma)} \leqslant 1, \tag{7.62}$$

*then, we can find* $\mathtt{c} \triangleq \mathtt{c}(\lambda,\rho) \in \mathrm{Lip}_\gamma(\mathscr{O},\mathbb{R})$ *and* $\beta_2 \in \mathrm{Lip}_\gamma(\mathscr{O}, H^S)$ *with the following properties:*

1. *The function* $\lambda \in \mathscr{O} \mapsto \mathtt{c}(\lambda,\rho)$ *satisfies the following estimate,*

$$\|\mathtt{c} - \mathtt{c}_0\|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^6,$$

*where* $\mathtt{c}_0$ *is defined in* (7.42).

2. *Consider the transformation* $\mathscr{B}_2$ *given by*

$$\mathscr{B}_2 h(\lambda,\varphi,\theta) \triangleq \big(1 + \partial_\theta\beta_2(\lambda,\varphi,\theta)\big)h\big(\lambda,\varphi,\theta+\beta_2(\lambda,\varphi,\theta)\big). \tag{7.63}$$

*It is invertible with inverse taking the form*

$$\mathscr{B}_2^{-1} h(\lambda,\varphi,\theta) \triangleq \big(1 + \partial_\theta\widehat{\beta}_2(\lambda,\varphi,\theta)\big)h\big(\lambda,\varphi,\theta+\widehat{\beta}_2(\lambda,\varphi,\theta)\big).$$

*Moreover, for any* $s \in [s_0, S]$,

$$\|\mathscr{B}_2^{\pm 1} h\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|h\|_s^{\mathrm{Lip}(\gamma)} + \varepsilon^6\gamma^{-1}\|\rho\|_{s+2\tau+5}^{\mathrm{Lip}(\gamma)}\|h\|_{s_0}^{\mathrm{Lip}(\gamma)}$$

*and*

$$\|\widehat{\beta}_2\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|\beta_2\|_s^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^6\gamma^{-1}\left(1 + \|\rho\|_{s+2\tau+4}^{\mathrm{Lip}(\gamma)}\right). \tag{7.64}$$

3. *For any* $n \in \mathbb{N}$, *if* $\lambda$ *belongs to the Cantor set*

$$\mathscr{O}_n^1(\rho) \triangleq \bigcap_{\substack{(l,j)\in\mathbb{Z}^2\\ 1\leqslant|j|\leqslant N_n}} \Big\{\lambda \in \mathscr{O} \quad \text{s.t.} \quad \big|\varepsilon^2\omega(\lambda)l + j\mathtt{c}(\lambda,\rho)\big| \geqslant \gamma|j|^{-\tau}\Big\}, \tag{7.65}$$

*then the following identity holds*

$$\mathscr{B}_2^{-1}\Big(\varepsilon^2\omega(\lambda)\partial_\varphi + \partial_\theta\big[\big(\mathtt{c}_0 + \varepsilon^6\mathbf{V}_2^\varepsilon(\rho)\big)\cdot\big]\Big)\mathscr{B}_2 = \varepsilon^2\omega(\lambda)\partial_\varphi + \mathtt{c}(\lambda,\rho)\partial_\theta + \mathtt{E}_n, \tag{7.66}$$

*with* $\mathtt{E}_n$ *a linear operator satisfying*

$$\|\mathtt{E}_n h\|_{s_0}^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^6 N_0^{\mu_2} N_{n+1}^{-\mu_2}\|h\|_{s_0+2}^{\mathrm{Lip}(\gamma)}. \tag{7.67}$$

4. *Given two small states* $\rho_1$ *and* $\rho_2$ *both satisfying the smallness properties* (7.62), *we have*

$$\|\Delta_{12}\mathtt{c}\|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^3\|\Delta_{12}\rho\|_{2s_0+2\tau+3}^{\mathrm{Lip}(\gamma)}. \tag{7.68}$$

Now we can state the main result of this subsection which addresses the conjugation of the linearized operator $\mathcal{L}_0$, as described by (7.25), using the changes of coordinates discussed before.

**Proposition 7.8.** *Given the conditions* (7.3), (7.61) *and* (7.62). *Let* $\mathscr{B}_1, \mathscr{B}_2$ *as in* (7.40) *and* (7.63). *Then, the following properties hold true.*

1. *The operator*

$$\mathscr{B} \triangleq \mathscr{B}_1\mathscr{B}_2$$

*writes*

$$\mathscr{B}h(\lambda,\varphi,\theta) = \big(1+\partial_\theta\beta(\lambda,\varphi,\theta)\big)h\big(\lambda,\varphi,\theta+\beta(\lambda,\varphi,\theta)\big), \tag{7.69}$$

*with*

$$\beta(\lambda,\varphi,\theta) \triangleq \beta_1(\lambda,\varphi,\theta) + \beta_2\big(\lambda,\varphi,\theta+\beta_1(\lambda,\varphi,\theta)\big)$$

*satisfying the estimate*

$$\|\beta\|_s^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon\big(1+\|\rho\|_{s+2\tau+4}^{\mathrm{Lip}(\gamma)}\big).$$

*Moreover, $\mathscr{B}$ is invertible and satisfies the estimate, for any $s \in [s_0, S]$*

$$\|\mathscr{B}^{\pm1}h\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|h\|_s^{\mathrm{Lip}(\gamma)} + \|\rho\|_{s+2\tau+5}^{\mathrm{Lip}(\gamma)}\|h\|_{s_0}^{\mathrm{Lip}(\gamma)}.$$

2. *For any $\lambda$ in the Cantor set $\mathscr{O}_n^1(\rho)$, defined in* (7.65), *one has*

$$\mathcal{L}_1 \triangleq \mathscr{B}^{-1}\mathcal{L}_0\mathscr{B} = \varepsilon^2\omega(\lambda)\partial_\varphi + \mathtt{c}(\lambda,\rho)\partial_\theta - \tfrac{1}{2}\mathbf{H} + \varepsilon^2\partial_\theta\mathfrak{Q}_1 + \partial_\theta\mathfrak{R}_1^\varepsilon + \mathtt{E}_n,$$

*where the linear operator $\mathtt{E}_n$ is the same as in Proposition* 7.7 *and*

$$\mathfrak{Q}_1[h](\varphi,\theta) \triangleq \int_{\mathbb{T}} h(\varphi,\eta)\mathtt{Q}_1\big(p(\varphi),\theta,\eta\big)d\eta,$$
$$\mathtt{Q}_1(p,\theta,\eta) \triangleq \tfrac{\cos(\theta-\eta)}{r_{\mathbf{D}}^2(p)} - \tfrac{1}{2}\mathrm{Re}\Big\{\big(\partial_z\mathcal{R}_{\mathbf{D}}(p)\big)^2 e^{\mathrm{i}(\theta+\eta)}\Big\}.$$

*In addition, the operator $\partial_\theta\mathfrak{R}_1^\varepsilon$ satisfies the estimates: for any $s \in [s_0, S]$, $N \geqslant 0$,*

$$\|\partial_\theta\mathfrak{R}_1^\varepsilon h\|_{s,N}^{\mathrm{Lip}(\gamma)} \lesssim \big(\varepsilon^{2+\mu}+\varepsilon^6\gamma^{-1}\big)\Big(\|h\|_s^{\mathrm{Lip}(\gamma)}\big(1+\|\rho\|_{s_0+2\tau+5+N}^{\mathrm{Lip}(\gamma)}\big) + \|h\|_{s_0}^{\mathrm{Lip}(\gamma)}\|\rho\|_{s+2\tau+5+N}^{\mathrm{Lip}(\gamma)}\Big).$$

*Proof.* **1**. The structure (7.69) follows from Lemma A.1-1 together with (7.40) and (7.63). Then, making use of Lemma A.2-1, (7.43) and (7.64), we get the desired estimates.
**2**. According to (7.32), (7.41) and (7.66), we get that for any $\lambda \in \mathscr{O}_n^1(\rho)$

$$\mathscr{B}^{-1}\mathcal{L}_0\mathscr{B} = \varepsilon^2\omega(\lambda)\partial_\varphi + \mathtt{c}(\lambda,\rho)\partial_\theta + \mathtt{E}_n - \tfrac{1}{2}\mathscr{B}^{-1}\mathbf{H}\mathscr{B} + \varepsilon^2\mathscr{B}^{-1}\partial_\theta\mathbf{Q}_1\mathscr{B} + \varepsilon^3\mathscr{B}^{-1}\partial_\theta\mathcal{R}_0^\varepsilon(\rho)\mathscr{B}. \tag{7.70}$$

In the sequel, for convenience, we will omit the dependence on $\lambda$ since it does not play any role here. Denoting

$$\mathbf{H}_0 \triangleq \mathscr{B}_1^{-1}\mathbf{H}\mathscr{B}_1 - \mathbf{H},$$

we infer

$$\begin{aligned}\mathscr{B}^{-1}\mathbf{H}\mathscr{B} - \mathbf{H} &= \mathscr{B}_2^{-1}\mathscr{B}_1^{-1}\mathbf{H}\mathscr{B}_1\mathscr{B}_2 - \mathbf{H}\\ &= \mathscr{B}_2^{-1}\big[\mathscr{B}_1^{-1}\mathbf{H}\mathscr{B}_1 - \mathbf{H}\big]\mathscr{B}_2 + \mathscr{B}_2^{-1}\mathbf{H}\mathscr{B}_2 - \mathbf{H}\\ &= \mathbf{H}_0 + (\mathscr{B}_2^{-1}\mathbf{H}_0\mathscr{B}_2 - \mathbf{H}_0) + \mathscr{B}_2^{-1}\mathbf{H}\mathscr{B}_2 - \mathbf{H}.\end{aligned} \tag{7.71}$$

Note that the inverse diffeomorphism $\mathscr{B}_1^{-1}$ admits the form

$$\mathscr{B}_1^{-1}h(\varphi,y) = \big(1+\partial_y\widehat{\beta}_1(\varphi,y)\big)h\big(\varphi,y+\widehat{\beta}_1(\varphi,y)\big),$$

where, by means of [59, Lem. A.6] and Proposition 7.6

$$\widehat{\beta}_1(\varphi,\theta) = -\varepsilon\, b_1(\varphi) - \varepsilon^2 b_2\big(\varphi,\theta\big) + \varepsilon^{2+\mu}\widehat{\mathfrak{r}}\big(\varphi,\theta\big) \qquad \text{with} \qquad \|\widehat{\mathfrak{r}}\|_s^{\mathrm{Lip}(\gamma)} \lesssim 1+\|\rho\|_{s+3}^{\mathrm{Lip}(\gamma)}. \tag{7.72}$$

Thus, in view of (7.5), we get

$$\mathbf{H}_0[h](\lambda,\varphi,\theta)=\partial_\theta\int_{\mathbb{T}}\mathbb{K}_0(\lambda,\varphi,\theta,\eta)h(\lambda,\varphi,\eta)\,d\eta,$$

where

$$\begin{aligned}\mathbb{K}_0(\lambda,\varphi,\theta,\eta)&\triangleq\log\left|\frac{e^{\mathrm{i}\,(\theta+\widehat{\beta_1}(\lambda,\varphi,\theta))}-e^{\mathrm{i}\,(\eta+\widehat{\beta_1}(\lambda,\varphi,\eta))}}{e^{\mathrm{i}\,\theta}-e^{\mathrm{i}\,\eta}}\right|\\&=\varepsilon^2\mathrm{Im}\left\{\frac{e^{\mathrm{i}\,\theta}b_2(\varphi,\theta)-e^{\mathrm{i}\,\eta}b_2(\varphi,\eta)}{e^{\mathrm{i}\,\theta}-e^{\mathrm{i}\,\eta}}\right\}+\varepsilon^{2+\mu}\mathbb{K}_1(\varphi,\theta,\eta),\end{aligned}$$

with the kernel $\mathbb{K}_1$ satisfying the estimate

$$\|\mathbb{K}_1\|_s^{\mathrm{Lip}(\gamma)}\lesssim 1+\|\rho\|_{s+3}^{\mathrm{Lip}(\gamma)},\tag{7.73}$$

where we have used Taylor expansion combined with (7.72). Making appeal to the expression (7.39) and using the identity

$$\left(e^{\mathrm{i}2\theta}-e^{\mathrm{i}2\eta}\right)\mathrm{Im}\left\{\tfrac{e^{\mathrm{i}\eta}}{e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}}\right\}=\tfrac{1}{2\mathrm{i}}\left(e^{\mathrm{i}\theta}+e^{\mathrm{i}\eta}\right)^2,$$

we infer

$$\begin{aligned}\mathbb{K}_0(\varphi,\theta,\eta)&=\varepsilon^2\Big(b_2(\varphi,\theta)-b_2(\varphi,\eta)\Big)\mathrm{Im}\left\{\tfrac{e^{\mathrm{i}\eta}}{e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}}\right\}+\varepsilon^{2+\mu}\mathbb{K}_1(\varphi,\theta,\eta)\\&=\varepsilon^2\mathrm{Im}\Big\{\mathtt{w}_2\big(p(\varphi)\big)\left(e^{\mathrm{i}2\theta}-e^{\mathrm{i}2\eta}\right)\Big\}\mathrm{Im}\left\{\tfrac{e^{\mathrm{i}\eta}}{e^{\mathrm{i}\theta}-e^{\mathrm{i}\eta}}\right\}+\varepsilon^{2+\mu}\mathbb{K}_1(\varphi,\theta,\eta)\\&=-\tfrac{\varepsilon^2}{2}\mathrm{Re}\Big\{\mathtt{w}_2\big(p(\varphi)\big)\left(e^{\mathrm{i}\theta}+e^{\mathrm{i}\eta}\right)^2\Big\}+\varepsilon^{2+\mu}\mathbb{K}_1(\varphi,\theta,\eta).\end{aligned}$$

As a consequence, using in particular the fact that $h$ has zero space average, we can write

$$\mathbf{H}_0[h]=-\varepsilon^2\partial_\theta\widehat{\mathbf{Q}}_1[h]+\varepsilon^{2+\mu}\mathbf{H}_1[h],\tag{7.74}$$

where

$$\begin{aligned}\widehat{\mathbf{Q}}_1[h](\varphi,\theta)&\triangleq\int_{\mathbb{T}}\mathrm{Re}\Big\{\mathtt{w}_2\big(p(\varphi)\big)e^{\mathrm{i}(\theta+\eta)}\Big\}h(\varphi,\eta)\,d\eta,\\\mathbf{H}_1[h](\varphi,\theta)&\triangleq\partial_\theta\int_{\mathbb{T}}\mathbb{K}_1(\varphi,\theta,\eta)h(\varphi,\eta)\,d\eta.\end{aligned}\tag{7.75}$$

Besides, one has

$$\mathscr{B}^{-1}\partial_\theta\mathbf{Q}_1\mathscr{B}=\partial_\theta\mathbf{Q}_1+\mathscr{B}^{-1}\partial_\theta\mathbf{Q}_1\mathscr{B}-\partial_\theta\mathbf{Q}_1.\tag{7.76}$$

Plugging (7.71), (7.76) and (7.74) into (7.70) implies

$$\mathscr{B}^{-1}\mathscr{L}_1\mathscr{B}=\varepsilon^2\omega(\lambda)\partial_\varphi+\mathtt{c}\partial_\theta+\mathtt{E}_n^0-\tfrac{1}{2}\mathbf{H}+\varepsilon^2\partial_\theta\mathfrak{Q}_1+\partial_\theta\mathfrak{R}_1^\varepsilon,$$

where

$$\mathfrak{Q}_1\triangleq\mathbf{Q}_1+\tfrac{1}{2}\widehat{\mathbf{Q}}_1\tag{7.77}$$

and

$$\begin{aligned}\partial_\theta\mathfrak{R}_1^\varepsilon\triangleq{}&\varepsilon^3\mathscr{B}^{-1}\partial_\theta\mathcal{R}_0^\varepsilon(\rho)\mathscr{B}-\tfrac{1}{2}[\mathscr{B}_2^{-1}\mathbf{H}_0\mathscr{B}_2-\mathbf{H}_0]-\tfrac{1}{2}\big[\mathscr{B}_2^{-1}\mathbf{H}\mathscr{B}_2-\mathbf{H}\big]\\&-\varepsilon^2\big[\mathscr{B}^{-1}\partial_\theta\mathbf{Q}_1\mathscr{B}-\partial_\theta\mathbf{Q}_1\big]-\tfrac{1}{2}\varepsilon^{2+\mu}\mathbf{H}_1.\end{aligned}\tag{7.78}$$

Combining (7.77), (7.75), (7.14), (7.15) and (7.17) yields the following integral representation

$$\mathfrak{Q}_1[h](\varphi,\theta)\triangleq\int_{\mathbb{T}}h(\varphi,\eta)\mathtt{Q}_1\big(p(\varphi),\theta,\eta\big)d\eta,\qquad\mathtt{Q}_1(p,\theta,\eta)\triangleq\tfrac{\cos(\theta-\eta)}{r_{\mathbf{D}}^2(p)}-\tfrac{1}{2}\mathrm{Re}\left\{\big(\partial_z\mathcal{R}_{\mathbf{D}}(p)\big)^2e^{\mathrm{i}(\theta+\eta)}\right\}.$$

Now, by virtue of [12, Lem. 2.36], (7.64) and (7.62) we can write

$$(\mathscr{B}_2^{-1}\mathbf{H}\mathscr{B}_2 - \mathbf{H})h(\varphi,\theta) = \int_{\mathbb{T}} \mathscr{K}_1(\varphi,\theta,\eta)h(\varphi,\eta)\,d\eta,$$

with $\mathscr{K}_1$ satisfying the following estimates, for any $s \in [s_0, S]$,

$$\|\mathscr{K}_1\|_s^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon^6\gamma^{-1}\left(1 + \|\rho\|_{s+2\tau+5}^{\mathrm{Lip}(\gamma)}\right). \tag{7.79}$$

Moreover, according to Lemma A.3, (7.43), (7.64) and (7.62), we have

$$\begin{aligned}
\big(\mathscr{B}^{-1}\partial_\theta \mathbf{Q}_1\mathscr{B} - \partial_\theta\mathbf{Q}_1\big)[h](\varphi,\theta) &= \int_{\mathbb{T}} h(\varphi,\eta)\mathscr{K}_2(\varphi,\theta,\eta)d\eta,\\
\big(\mathscr{B}_2^{-1}\mathbf{H}_0\mathscr{B}_2 - \mathbf{H}_0\big)[h](\varphi,\theta) &= \int_{\mathbb{T}} h(\varphi,\eta)\mathscr{K}_3(\varphi,\theta,\eta)d\eta,\\
\mathscr{B}^{-1}\partial_\theta\mathcal{R}_0^\varepsilon(\rho)\mathscr{B}[h](\varphi,\theta) &= \int_{\mathbb{T}} h(\varphi,\eta)\mathscr{K}_4(\varphi,\theta,\eta)d\eta,
\end{aligned}$$

with

$$\begin{aligned}
\|\mathscr{K}_2\|_s^{\mathrm{Lip}(\gamma)} &\lesssim \varepsilon\left(1 + \|\rho\|_{s+2\tau+5}^{\mathrm{Lip}(\gamma)}\right),\\
\|\mathscr{K}_3\|_s^{\mathrm{Lip}(\gamma)} &\lesssim \varepsilon^6\gamma^{-1}\left(1 + \|\rho\|_{s+2\tau+5}^{\mathrm{Lip}(\gamma)}\right),\\
\|\mathscr{K}_4\|_s^{\mathrm{Lip}(\gamma)} &\lesssim 1 + \|\rho\|_{s+2\tau+5}^{\mathrm{Lip}(\gamma)}.
\end{aligned}$$

Thus, the operator $\partial_\theta\mathfrak{R}_1^\varepsilon$, introduced in (7.78), is an integral operator with the kernel

$$\mathcal{K}_1^\varepsilon \triangleq \varepsilon^3\mathscr{K}_4 - \tfrac{1}{2}\mathscr{K}_3 - \tfrac{1}{2}\mathscr{K}_1 - \varepsilon^2\mathscr{K}_2 + \varepsilon^3\mathcal{K}_0^\varepsilon - \tfrac{1}{2}\varepsilon^{2+\mu}\partial_\theta\mathbb{K}_1,$$

which satisfies the estimate

$$\|\mathcal{K}_1^\varepsilon\|_s^{\mathrm{Lip}(\gamma)} \lesssim \left(\varepsilon^{2+\mu} + \varepsilon^6\gamma^{-1}\right)\left(1 + \|\rho\|_{s+2\tau+5}^{\mathrm{Lip}(\gamma)}\right).$$

We conclude the estimate of $\mathfrak{R}_1^\varepsilon$ by applying Lemma A.3. This ends the proof of Proposition 7.8. □

### 7.5 Invertibility of the linearized operator

The main goal this section is to transform the linearized operator $\mathcal{L}_0$, as defined in (7.32), into a Fourier multiplier up to a small error. This step is required along Nash-Moser scheme and will be done in the spirit of [59, Sec. 5]. As the leading term of this operator degenerates on the modes $\pm1$, we shall first split the phase space into two parts: a subspace localized on the modes $\pm1$ and its complement. This leads to explore the invertibility of a matrix-valued operator that will be done in different steps as detailed below.

Let $\Pi_1$ stand for the orthogonal projection on the modes $\pm1$ acting on $L^2(\mathbb{T}^2,\mathbb{R})$ as follows

$$h(\varphi,\theta) = \sum_{j\in\mathbb{Z}} h_j(\varphi)e^{\mathrm{i}j\theta} \quad \Rightarrow \quad \Pi_1 h \triangleq \sum_{j=\pm1} h_j(\varphi)e^{\mathrm{i}j\theta}. \tag{7.80}$$

We also define

$$\Pi_1^\perp \triangleq \mathrm{Id} - \Pi_1.$$

As the function $h$ is real-valued then

$$\forall j \in \mathbb{Z}, \quad h_{-j}(\varphi) = \overline{h_j(\varphi)}.$$

Now, we decompose the phase space $\mathrm{Lip}_\gamma(\mathscr{O}, H_0^s)$ as follows,

$$X^s \triangleq \mathrm{Lip}_\gamma(\mathscr{O}, H_0^s) = X_1^s \overset{\perp}{\oplus} X_\perp^s, \tag{7.81}$$

with

$$X_1^s \triangleq \big\{h\in \mathrm{Lip}_\gamma(\mathscr{O},H_0^s) \quad \text{s.t.} \quad \Pi_1 h=h\big\},$$
$$X_\perp^s \triangleq \big\{h\in \mathrm{Lip}_\gamma(\mathscr{O},H_0^s) \quad \text{s.t.} \quad \Pi_1^\perp h=h\big\}.$$

Recall from Proposition 7.8 and Proposition 7.7 that the operator $\mathcal{L}_1$ decomposes as below

$$\mathcal{L}_1=\mathbb{L}_1+\partial_\theta\mathfrak{R}_1+\mathtt{E}_n,\qquad \mathbb{L}_1\triangleq \varepsilon^2\omega(\lambda)\partial_\varphi+\mathtt{c}(\lambda,\rho)\partial_\theta-\tfrac12\mathbf{H}+\varepsilon^2\partial_\theta\mathfrak{Q}_1, \tag{7.82}$$

with

$$\mathfrak{Q}_1[h](\varphi,\theta)\triangleq r_{\mathbf{D}}^{-2}\big(p(\varphi)\big)\mathrm{Re}\Big\{h_1(\varphi)e^{\mathrm{i}\theta}\Big\}-\tfrac12\mathrm{Re}\Big\{\big[\partial_z\mathcal{R}_{\mathbf{D}}\big(p(\varphi)\big)\big]^2h_{-1}(\varphi)e^{\mathrm{i}\theta}\Big\}$$

and

$$\mathtt{c}=\tfrac12+\varepsilon^3\mathtt{c}_1+(\mathtt{c}-\mathtt{c}_0)\triangleq\tfrac12+\varepsilon^3\mathtt{c}_2, \tag{7.83}$$

such that

$$\|\mathtt{c}_2\|^{\mathrm{Lip}(\gamma)}\leqslant C. \tag{7.84}$$

Upon straightforward analysis, it becomes apparent that the linear operator

$$\mathbb{L}_1+\partial_\theta\mathfrak{R}_1^\varepsilon:\mathrm{Lip}_\gamma(\mathscr{O},X^s)\to \mathrm{Lip}_\gamma(\mathscr{O},X^{s-1})$$

is well-defined and its action is equivalent to the matrix operator $\mathbb{M}:X_1^s\times X_\perp^s\to X_1^{s-1}\times X_\perp^{s-1}$ with

$$\begin{aligned}\mathbb{M}&\triangleq\begin{pmatrix}\Pi_1\mathbb{L}_1\Pi_1 & 0\\ 0 & \Pi_1^\perp\mathbb{L}_1\Pi_1^\perp\end{pmatrix}+\begin{pmatrix}\Pi_1\partial_\theta\mathfrak{R}_1^\varepsilon\Pi_1 & \Pi_1\partial_\theta\mathfrak{R}_1^\varepsilon\Pi_1^\perp\\ \Pi_1^\perp\partial_\theta\mathfrak{R}_1^\varepsilon\Pi_1 & \Pi_1^\perp\partial_\theta\mathfrak{R}_1^\varepsilon\Pi_1^\perp\end{pmatrix}\\ &\triangleq\mathbb{M}_1+\partial_\theta\mathbb{R}_1,\end{aligned} \tag{7.85}$$

where, according to Proposition 7.8 and (7.103), for any $H=(h_1,h_2)\in X_1^s\times X_\perp^s$, $s\in[s_0,S]$ and $N\geqslant 0$, one has

$$\|\partial_\theta\mathbb{R}_1H\|_{s,N}^{\mathrm{Lip}(\gamma)}\leqslant C\varepsilon^{2+\mu}\Big(\|H\|_s^{\mathrm{Lip}(\gamma)}\big(1+\|\rho\|_{s_0+N+2\tau+5}^{\mathrm{Lip}(\gamma)}\big)+\|\rho\|_{s+N+2\tau+5}^{\mathrm{Lip}(\gamma)}\|H\|_{s_0}^{\mathrm{Lip}(\gamma)}\Big). \tag{7.86}$$

Hence, we will examine the invertibility of the scalar operator $\mathbb{L}_1+\partial_\theta\mathfrak{R}_1^\varepsilon$ by analyzing the invertibility of the matrix operator $\mathbb{M}$. This analysis involves inverting its main part $\mathbb{M}_1$ alongside employing perturbative arguments. To achieve this, we will break the process down into several steps.

### 7.5.1 Degeneracy of the modes ±1 and monodromy matrix

The main objective is to demonstrate the invertibility of the operator $\Pi\mathbb{L}_1\Pi$, defined through (7.85) and (7.82). More precisely, by virtue of (7.82), (7.83) and the identity

$$(\partial_\theta-\mathbf{H})\Pi_1=0,$$

one has

$$\mathbb{L}_{1,1}\triangleq\Pi_1\mathbb{L}_1\Pi_1=\varepsilon^2\Big(\omega(\lambda)\partial_\varphi+\varepsilon\mathtt{c}_2(\lambda)\Pi_1\partial_\theta+\Pi_1\partial_\theta\mathfrak{Q}_1\Pi_1\Big).$$

Notably, this operator localizes in the Fourier spatial modes ±1 and displays a degeneracy in $\varepsilon$. To invert it, we consider an arbitrary real-valued function

$$(\varphi,\theta)\mapsto g(\varphi,\theta)=\sum_{j=\pm1}g_j(\varphi)e^{\mathrm{i}j\theta}\in X_1^{s-1}$$

and we shall solve in the real space $X_1^s$ the equation

$$\mathbb{L}_{1,1}h=g.$$

Since $g$ and $h$ are real then $g_{-n} = \overline{g_n}$ and $h_{-n} = \overline{h_n}$. Thus, using Fourier expansion, the last equation is equivalent to

$$\partial_\varphi H - \mathbf{A}(\varepsilon, \lambda, \varphi) H = \varepsilon^{-2} G, \tag{7.87}$$

where

$$\mathbf{A}(\varepsilon, \lambda, \varphi) \triangleq \begin{pmatrix} \mathtt{u}_\lambda(\varphi) - \frac{\mathrm{i}\varepsilon \mathtt{c}_2(\lambda)}{2\omega(\lambda)} & \mathtt{v}_\lambda(\varphi) \\ \overline{\mathtt{v}_\lambda(\varphi)} & \overline{\mathtt{u}_\lambda(\varphi)} + \frac{\mathrm{i}\varepsilon \mathtt{c}_2(\lambda)}{2\omega(\lambda)} \end{pmatrix}, \qquad \begin{aligned} \mathtt{u}_\lambda(\varphi) &= -\tfrac{\mathrm{i}}{2\omega(\lambda)} \tfrac{1}{r_{\mathbf{D}}^2(p)}, \\ \mathtt{v}_\lambda(\varphi) &= \tfrac{\mathrm{i}}{4\omega(\lambda)} \big(\partial_z \mathcal{R}_{\mathbf{D}}(p)\big)^2 \end{aligned} \tag{7.88}$$

and

$$G(\varphi) \triangleq \frac{1}{\omega(\lambda)} \begin{pmatrix} g_1(\varphi) \\ \overline{g_1(\varphi)} \end{pmatrix} \in \mathbb{C}^2, \qquad H(\varphi) \triangleq \begin{pmatrix} h_1(\varphi) \\ \overline{h_1(\varphi)} \end{pmatrix} \in \mathbb{C}^2.$$

We intend to prove the following result.

**Proposition 7.9.** *Assume* (3.18), *there exists* $\varepsilon_1 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_1)$, *the operator* $\mathbb{L}_{1,1} : X_1^s \to X_1^{s-1}$ *is invertible with inverse* $\mathbb{L}_{1,1}^{-1}$ *satisfying*

$$\|\mathbb{L}_{1,1}^{-1} g\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\varepsilon^{-2} \|g\|_{s-1}^{\mathrm{Lip}(\gamma)}.$$

*Proof.* Let $(\varphi, \phi) \in \mathbb{R}^2 \mapsto \mathscr{F}(\varepsilon, \lambda, \varphi, \phi)$ be the fundamental matrix defined through the $2 \times 2$ matrix ODE

$$\begin{cases} \partial_\varphi \mathscr{F}(\varepsilon, \lambda, \varphi, \phi) - \mathbf{A}(\varepsilon, \lambda, \varphi) \mathscr{F}(\varepsilon, \lambda, \varphi, \phi) = 0, \\ \mathscr{F}(\phi, \phi) = \mathrm{Id}. \end{cases}$$

Then the solution $H$ can be expressed in the form

$$H(\varphi) = \mathscr{F}(\varepsilon, \lambda, \varphi, 0) H(0) + \varepsilon^{-2} \int_0^\varphi \mathscr{F}(\varepsilon, \lambda, \varphi, \phi) G(\phi) d\phi. \tag{7.89}$$

Observe that the matrix $A$, defined by (7.88), is $2\pi-$periodic in the variable $\varphi$. Then $H$ is $2\pi-$periodic if and only if

$$H(2\pi) = H(0).$$

Equivalently,

$$\big(\mathrm{Id} - \mathscr{F}(\varepsilon, \lambda, 2\pi, 0)\big) H(0) = \varepsilon^{-2} \int_0^{2\pi} \mathscr{F}(\varepsilon, \lambda, 2\pi, \phi) G(\phi) d\phi. \tag{7.90}$$

To find a unique solution to this equation it is enough to show that the matrix $\mathrm{Id} - \mathscr{F}(\varepsilon, \lambda, 2\pi, 0)$ is invertible. To this end, we use the decomposition

$$\mathbf{A}(\varepsilon, \lambda, \varphi) = \mathbb{A}_\lambda(\varphi) + \mathbf{B}(\varepsilon, \lambda, \varphi), \qquad \mathbb{A}_\lambda(\varphi) \triangleq \mathbf{A}(0, \lambda, \varphi), \qquad \mathbf{B}(\varepsilon, \lambda, \varphi) \triangleq \mathbf{A}(\varepsilon, \lambda, \varphi) - \mathbf{A}(0, \lambda, \varphi).$$

According to (7.84), one has

$$\sup_{\varphi \in \mathbb{R}} \|\mathbf{B}(\varepsilon, \lambda, \varphi)\| \lesssim \varepsilon. \tag{7.91}$$

Now, consider the fundamental solution of the unperturbed problem

$$\begin{cases} \partial_\varphi \mathscr{M}_\lambda(\varphi) - \mathbb{A}_\lambda(\varphi) \mathscr{M}_\lambda(\varphi) = 0, \\ \mathscr{M}_\lambda(0) = \mathrm{Id}. \end{cases} \tag{7.92}$$

Then one may write

$$\mathscr{F}(\varepsilon, \lambda, \varphi, 0) = \mathscr{M}_\lambda(\varphi) + \mathscr{G}(\varepsilon, \lambda, \varphi),$$

with

$$\partial_\varphi \mathscr{G}(\varepsilon, \lambda, \varphi) - \mathbb{A}_\lambda(\varphi) \mathscr{G}(\varepsilon, \lambda, \varphi) = -\mathbf{B}(\varepsilon, \lambda, \varphi) \mathscr{F}, \quad \mathscr{G}(\varepsilon, \lambda, 0) = 0.$$

and

$$\sup_{\varphi\in[0,2\pi]}\|\mathscr{G}(\varepsilon,\lambda,\varphi)\|\leqslant C\varepsilon,$$

where we have used for the last inequality the estimate (7.91). The resolvant matrix must have the following form

$$\mathscr{M}_\lambda(\varphi)=\begin{pmatrix}a_\lambda(\varphi) & b_\lambda(\varphi)\\ \overline{b_\lambda(\varphi)} & \overline{a_\lambda(\varphi)}\end{pmatrix},\qquad a_\lambda(0)=1,\qquad b_\lambda(0)=0. \tag{7.93}$$

Remark that the property $\mathrm{Tr}(\mathbb{A}_\lambda)\equiv 0$ gives

$$|a_\lambda(\varphi)|^2-|b_\lambda(\varphi)|^2=\det\big(\mathscr{M}_\lambda(\varphi)\big)=\exp\left(\int_0^\varphi \mathrm{Tr}\big(\mathbb{A}_\lambda(u)\big)du\right)=1.$$

Hence,

$$\det(\mathscr{M}_\lambda-\mathrm{Id})=|a_\lambda-1|^2-|b_\lambda|^2=2\big(1-\mathrm{Re}(a_\lambda)\big)=2-\mathrm{Tr}\big(\mathscr{M}_\lambda(2\pi)\big).$$

In particular,

$$\det\big(\mathscr{M}_\lambda(2\pi)-\mathrm{Id}\big)\neq 0\qquad\Longleftrightarrow\qquad \mathrm{Tr}\big(\mathscr{M}_\lambda(2\pi)\big)\neq 2. \tag{7.94}$$

Thus, under the assumption (3.18), the matrix $\mathscr{M}_\lambda(2\pi)-\mathrm{Id}$ is invertible and there exist $C>0$ such that

$$\|\big(\mathscr{M}_\lambda(2\pi)-\mathrm{Id}\big)^{-1}\|^{\mathrm{Lip}(\gamma)}\leqslant C.$$

Therefore, by using perturbation arguments we conclude that the matrix $\mathscr{F}(\varepsilon,\lambda,2\pi)-\mathrm{Id}$ is invertible on $[\lambda_*,\lambda^*]$ and we have

$$\|\big(\mathscr{F}(\varepsilon,2\pi)-\mathrm{Id}\big)^{-1}\|^{\mathrm{Lip}(\gamma)}\leqslant C.$$

Consequentely, the equation (7.90) admits a unique solution satisfying

$$\|H(0)\|_s^{\mathrm{Lip}(\gamma)}\leqslant C\varepsilon^{-2}\|G\|_{L^2(\mathbb{T})}, \tag{7.95}$$

where we have used, for $\varepsilon$ small enough, the estimate

$$\sup_{\varphi,\phi\in[0,2\pi]}\|\mathscr{F}(\varepsilon,\lambda,\varphi,\phi)\|\leqslant C.$$

From the previous analysis we conclude that the equation (7.87) admits a unique solution which satisfies, in view of (7.95) and (7.89) the estimate: for all $s>1$

$$\|H\|_s^{\mathrm{Lip}(\gamma)}\leqslant C\varepsilon^{-2}\|G\|_{s-1}^{\mathrm{Lip}(\gamma)}.$$

This implies that the linear operator $\mathbb{L}_{1,1}:X_1^s\to X_1^{s-1}$ is invertible on $[\lambda_*,\lambda^*]$ and

$$\|\mathbb{L}_{1,1}^{-1}g\|_s^{\mathrm{Lip}(\gamma)}\leqslant C\varepsilon^{-2}\|g\|_{s-1}^{\mathrm{Lip}(\gamma)}.$$

This achieves the proof of the desired result. □

We shall end this section with some comments on the monodromy matrix that can be linked to Riccatti equation. As we have seen before,

$$\det(\mathscr{M}_\lambda)=1,$$

which implies that the map

$$\varphi\in\mathbb{R}\mapsto\mathscr{M}_\lambda(\varphi)\in SL(2;\mathbb{C})$$

is well defined. We will associate to each element $\mathscr{M}_\lambda(\varphi)$ the Möbius transform

$$T(\varphi):z\in\mathbb{D}\mapsto\frac{a_\lambda(\varphi)\,z+b_\lambda(\varphi)}{\overline{b_\lambda(\varphi)}\,z+\overline{a_\lambda(\varphi)}}\in\mathbb{D}.$$

Notice that $T(\varphi):\mathbb{D}\to\mathbb{D}$ is an automorphism with $T(0)=\mathrm{Id}$. Straightforward computations based on (7.92) lead to

$$\begin{cases}\partial_\varphi T=-\mathrm{i}\mu T-\overline{\mathtt{v}_\lambda}T^2+\mathtt{v}_\lambda,\\ T(0,z)=z\in\mathbb{D},\end{cases}$$

with $T^2(\varphi,z)\triangleq\big(T(\varphi,z)\big)^2$ and

$$\mu\triangleq\tfrac{e^{2\lambda}}{\omega(\lambda)},\qquad \mathtt{v}_\lambda(\varphi)\triangleq\tfrac{\mathrm{i}}{4\omega(\lambda)}\Big(\partial_z\mathcal{R}_{\mathbf{D}}\big(p(\varphi)\big)\Big)^2.$$

By setting

$$\mathtt{T}(\varphi)\triangleq e^{-i\mu\varphi}T(\varphi),\qquad \varrho(\varphi)\triangleq e^{i\mu\varphi}\mathtt{v}_\lambda(\varphi),$$

we get Riccatti equation

$$\begin{cases}\partial_\varphi\mathtt{T}=-\overline{\varrho}\;\mathtt{T}^2+\varrho,\\ \mathtt{T}(0,z)=z\in\mathbb{D}.\end{cases}$$

### 7.5.2 Invertibility of $\Pi_1^\perp\mathbb{L}_1\Pi_1^\perp$

The purpose here is to find a right inverse for the operator

$$\mathbb{L}_{1,\perp}\triangleq\Pi_1^\perp\mathbb{L}_1\Pi_1^\perp,$$

where $\mathbb{L}_1$ is defined through (7.82) and (7.80). Recall from Proposition 7.8-2 that the operator $\mathfrak{Q}_1$ localizes on the modes $\pm1$ implying that

$$\Pi_1^\perp\mathfrak{Q}_1\Pi_1^\perp=0.$$

Therefore, the linear operator

$$\mathbb{L}_{1,\perp}:X_\perp^s\to X_\perp^{s-1}$$

is well-defined and assumes the structure

$$\mathbb{L}_{1,\perp}=\varepsilon^2\omega(\lambda)\partial_\varphi+\mathcal{D}_{1,\perp},\qquad \mathcal{D}_{1,\perp}\triangleq\mathtt{c}(\lambda)\partial_\theta-\tfrac{1}{2}\mathbf{H}.\tag{7.96}$$

In view of (7.83), one has

$$\forall l\in\mathbb{Z},\quad\forall|j|\geqslant2,\quad \mathcal{D}_{1,\perp}\mathbf{e}_{l,j}=\mathrm{i}\mu_{j,2}\,\mathbf{e}_{l,j},\tag{7.97}$$

with

$$\mu_{j,2}(\lambda)=j\left(\tfrac{1}{2}+\varepsilon^3\mathtt{c}_2(\lambda)\right)-\tfrac{1}{2}\tfrac{j}{|j|}.\tag{7.98}$$

The main result of this subsection reads as follows.

**Proposition 7.10.** *Let $(\tau,\gamma,s_0,S)$ as in (7.3). There exists $\varepsilon_0>0$ small enough such that for any $\varepsilon\in(0,\varepsilon_0)$, there exists a family of linear operators $\big(\mathtt{T}_n\big)_{n\in\mathbb{N}}$ satisfying for any $N\geqslant0$ and $s\in[s_0,S]$,*

$$\sup_{n\in\mathbb{N}}\|\mathtt{T}_nh\|_{s,N}^{\mathrm{Lip}(\gamma)}\lesssim\gamma^{-1}\|h\|_{s,2\tau+N}^{\mathrm{Lip}(\gamma)}$$

*and for $\lambda$ in the Cantor set*

$$\mathscr{O}_n^2(\rho)\triangleq\bigcap_{\substack{(l,j)\in\mathbb{Z}^2\\ 2\leqslant|j|\leqslant N_n}}\Big\{\lambda\in\mathscr{O}\quad\text{s.t.}\quad|\varepsilon^2\omega(\lambda)l+\mu_{j,2}(\lambda)|\geqslant\gamma|j|^{-\tau}\Big\},\tag{7.99}$$

*we get*

$$\mathbb{L}_{1,\perp}\mathtt{T}_n=\mathrm{Id}+\mathtt{E}_n^2,$$

*where $\mathtt{E}_n^2$ satisfies for any $s\in[s_0,S]$,*

$$\|\mathtt{E}_n^2h\|_{s_0}^{\mathrm{Lip}(\gamma)}\lesssim\gamma^{-1}N_n^{s_0-s}\|h\|_{s,2\tau+1}^{\mathrm{Lip}(\gamma)}.$$

*Proof.* In view of (7.96), one may write

$$\mathbb{L}_{1,\perp} = \mathtt{L}_n + \Pi_{N_n}^{\perp}\mathcal{D}_{1,\perp}, \qquad \mathtt{L}_n \triangleq \varepsilon^2\omega(\lambda)\partial_\varphi + \Pi_{N_n}\mathcal{D}_{1,\perp}, \tag{7.100}$$

where the projectors $\Pi_{N_n}$ and $\Pi_{N_n}^{\perp}$ are defined by

$$\Pi_{N_n} h \triangleq \sum_{\substack{(l,j)\in\mathbb{Z}^2\\ \langle l,j\rangle\leqslant N_n}} h_{l,j}\mathbf{e}_{l,j} \qquad \text{and} \qquad \Pi_{N_n}^{\perp} \triangleq \mathrm{Id} - \Pi_{N_n}.$$

Therefore,

$$\mathtt{L}_n\mathbf{e}_{l,j} = \begin{cases} \mathrm{i}\big(\varepsilon^2\omega(\lambda)\,l + \mu_{j,2}\big)\mathbf{e}_{l,j}, & \text{if } 2\leqslant |j|\leqslant N_n \text{ and } l\in\mathbb{Z},\\ \mathrm{i}\,\varepsilon^2\omega(\lambda)\,l\mathbf{e}_{l,j}, & \text{if } |j|>N_n \text{ and } l\in\mathbb{Z}.\end{cases}$$

Define the operator $\mathtt{T}_n$ by

$$\mathtt{T}_n h(\varphi,\theta) \triangleq -\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^2\\ 2\leqslant|j|\leqslant N_n}} \tfrac{\chi\left((\varepsilon^2\omega(\lambda)\,l+\mu_{j,2})\gamma^{-1}|j|^{\tau}\right)}{\varepsilon^2\omega(\lambda)\,l+\mu_{j,2}} h_{l,j}\,\mathbf{e}_{l,j}(\varphi,\theta),$$

where $\chi\in\mathscr{C}^\infty(\mathbb{R},[0,1])$ is an even positive cut-off function such that

$$\chi(\xi) = \begin{cases} 0, & \text{if } |\xi|\leqslant \frac{1}{3},\\ 1, & \text{if } |\xi|\geqslant\frac{1}{2}.\end{cases}$$

In the Cantor set $\mathscr{O}_n^2(\rho)$ one has

$$\mathtt{L}_n\mathtt{T}_n = \mathrm{Id} + \Pi_{N_n}^{\perp}. \tag{7.101}$$

Arguing in a similar way to [59, Prop. 5.3] one can show that, for any $N\geqslant 0$,

$$\sup_{n\in\mathbb{N}} \|\mathtt{T}_n h\|_{s,N}^{\mathrm{Lip}(\gamma)} \leqslant C\gamma^{-1}\|h\|_{s,2\tau+N}^{\mathrm{Lip}(\gamma)}. \tag{7.102}$$

Now, from (7.100) and (7.101) we conclude that on the Cantor set $\mathscr{O}_{n,2}$ we have the identity

$$\mathbb{L}_{1,\perp}\mathtt{T}_n = \mathrm{Id} + \mathtt{E}_n^2, \qquad \mathtt{E}_n^2 \triangleq \Pi_{N_n}^{\perp} + \Pi_{N_n}^{\perp}\mathcal{D}_{1,\perp}\mathtt{T}_n.$$

Finally, the estimate on the remainder $\mathtt{E}_n^2$ follows immediately from (7.102) and (7.97). This ends the proof of the desired result. □

### 7.5.3 Invertibility of $\mathbb{M}$

The next goal is to revisit the matrix operator introduced in (7.85) and explore its invertibility. Here is our key result.

**Proposition 7.11.** *Under the assumptions of Proposition* 7.10 *and the smallness condition*

$$\varepsilon^{-1}\gamma + \varepsilon^{2+\mu}\gamma^{-1} \leqslant \varepsilon_0, \qquad \|\rho\|_{s_0+4\tau+5}^{\mathrm{Lip}(\gamma)} \leqslant 1, \tag{7.103}$$

*the following holds true. There exists a family of linear operators $\mathbb{P}_n$ satisfying for any $s\in[s_0,S]$,*

$$\|\mathbb{P}_n H\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\gamma^{-1}\Big(\|H\|_{s+2\tau}^{\mathrm{Lip}(\gamma)} + \|\rho\|_{s+2\tau}^{\mathrm{Lip}(\gamma)}\|H\|_{s_0+2\tau}^{\mathrm{Lip}(\gamma)}\Big), \tag{7.104}$$

*such that in the Cantor set $\mathscr{O}_n^2$, defined in Proposition* 7.10*, we have*

$$\mathbb{M}\mathbb{P}_n = \mathrm{Id}_{X_1^s\times X_\perp^s} + \mathbb{E}_n^2,$$

*with*

$$\|\mathbb{E}_n^2 H\|_{s_0}^{\mathrm{Lip}(\gamma)} \leqslant C\gamma^{-1}N_n^{s_0-s}\Big(\|H\|_{s+2\tau+1}^{\mathrm{Lip}(\gamma)} + \|\rho\|_{s+4\tau+5}^{\mathrm{Lip}(\gamma)}\|H\|_{s_0+2\tau}^{\mathrm{Lip}(\gamma)}\Big). \tag{7.105}$$

*Proof.* Consider the operator

$$\mathbb{K}_n = \begin{pmatrix} \mathbb{L}_{1,1}^{-1} & 0 \\ 0 & \mathtt{T}_n \end{pmatrix},$$

where the operator $\mathtt{T}_n$ was defined in Proposition 7.10. For all $\lambda \in \mathscr{O}_n^2$ one has the identity

$$\mathbb{M}_1 \mathbb{K}_n = \mathrm{Id}_{X_1^s \times X_\perp^s} + \begin{pmatrix} 0 & 0 \\ 0 & \mathtt{E}_n^2 \end{pmatrix}. \tag{7.106}$$

Moreover, according to Proposition 7.9, Proposition 7.10 and (7.103), for any $H = (h_1, h_2) \in X_1^s \times X_\perp^s$ we have

$$\|\mathbb{K}_n H\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\gamma^{-1} \|H\|_{s,2\tau}^{\mathrm{Lip}(\gamma)}. \tag{7.107}$$

Inserting (7.86) into (7.107) gives, according to (7.103) and $\mu \in (0,1)$,

$$\|\mathbb{K}_n \partial_\theta \mathbb{R}_1 H\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\varepsilon^{2+\mu}\gamma^{-1}\Big(\|H\|_s^{\mathrm{Lip}(\gamma)} + \|\rho\|_{s+4\tau+5}^{\mathrm{Lip}(\gamma)} \|H\|_{s_0}^{\mathrm{Lip}(\gamma)}\Big). \tag{7.108}$$

Consequently, under the condition (7.103), the operator

$$\mathrm{Id}_{X_1^{s_0} \times X_\perp^{s_0}} + \mathbb{K}_n \partial_\theta \mathbb{R}_1 : X_1^{s_0} \times X_\perp^{s_0} \to X_1^{s_0} \times X_\perp^{s_0}$$

is invertible, with

$$\|(\mathrm{Id}_{X_1^s \times X_\perp^s} + \mathbb{K}_n \partial_\theta \mathbb{R}_1)^{-1} H\|_{s_0}^{\mathrm{Lip}(\gamma)} \leqslant 2\|H\|_{s_0}^{\mathrm{Lip}(\gamma)}.$$

As to the invertibility for $s \in [s_0, S]$, one can check by induction from (7.108), $\forall m \geqslant 1$, that under the smallness condition (7.103) one has

$$\sum_{m \geqslant 0} \|(\mathbb{K}_n \partial_\theta \mathbb{R}_1)^m H\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\|H\|_s^{\mathrm{Lip}(\gamma)} + C\|\rho\|_{s+4\tau+5}^{\mathrm{Lip}(\gamma)} \|H\|_{s_0}^{\mathrm{Lip}(\gamma)}.$$

This implies that

$$\|(\mathrm{Id}_{X_1^s \times X_\perp^s} + \mathbb{K}_n \partial_\theta \mathbb{R}_1)^{-1} H\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\|H\|_s^{\mathrm{Lip}(\gamma)} + C\|\rho\|_{s+4\tau+5}^{\mathrm{Lip}(\gamma)} \|H\|_{s_0}^{\mathrm{Lip}(\gamma)}. \tag{7.109}$$

Now define the operator

$$\mathbb{P}_n = \big(\mathrm{Id}_{X_1^s \times X_\perp^s} + \mathbb{K}_n \partial_\theta \mathbb{R}_1\big)^{-1} \mathbb{K}_n. \tag{7.110}$$

From (7.110), (7.109), (7.107) and (7.103), we obtain (7.104). Moreover, in view of (7.106) and (7.110) we deduce that in the Cantor set $\mathscr{O}_n^2$ one has

$$\begin{aligned} \mathbb{M}\mathbb{P}_n &= \left(\mathbb{M}_1 + \left(\mathbb{M}_1 \mathbb{K}_n - \begin{pmatrix} 0 & 0 \\ 0 & \mathtt{E}_n^2 \end{pmatrix}\right) \partial_\theta \mathbb{R}_1\right) \big(\mathrm{Id}_{X_1^s \times X_\perp^s} + \mathbb{K}_n \partial_\theta \mathbb{R}_1\big)^{-1} \mathbb{K}_n \\ &= \mathbb{M}_1 \mathbb{K}_n - \begin{pmatrix} 0 & 0 \\ 0 & \mathtt{E}_n^2 \end{pmatrix} \partial_\theta \mathbb{R}_1 \mathbb{P}_n \\ &= \mathrm{Id}_{X_1^s \times X_\perp^s} + \mathbb{E}_n^2, \end{aligned} \tag{7.111}$$

where

$$\mathbb{E}_n^2 \triangleq \begin{pmatrix} 0 & 0 \\ 0 & \mathtt{E}_n^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & \mathtt{E}_n^2 \end{pmatrix} \partial_\theta \mathbb{R}_1 \mathbb{P}_n.$$

By virtue of Proposition 7.10, (7.86), (7.104) and (7.103) one gets (7.105). Finally, by construction, one has the algebraic properties

$$\mathbb{M} = \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_1^\perp \end{pmatrix} \mathbb{M} \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_1^\perp \end{pmatrix} \quad \text{and} \quad \mathbb{P}_n = \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_1^\perp \end{pmatrix} \mathbb{P}_n \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_1^\perp \end{pmatrix}. \tag{7.112}$$

This completes the proof of the desired result. □

### 7.5.4 Invertibility of $\mathcal{L}_1$ and $\mathcal{L}_0$

In this section we intend to explore the existence of an approximate right inverse to the operators $\mathcal{L}_1$ and $\mathcal{L}_0$ defined in (7.82) and Proposition 7.8, respectively.

**Proposition 7.12.** *Given the conditions* (7.3), (7.61). *There exists* $\epsilon_0 > 0$ *such that under the assumptions*

$$\varepsilon^{-1}\gamma + \varepsilon^{4+\mu}\gamma^{-1}N_0^{\mu_2} + \varepsilon^{2+\mu}\gamma^{-1} \leqslant \epsilon_0, \qquad \|\rho\|^{\mathrm{Lip}(\gamma)}_{2s_0+2\tau+4+\frac{3}{2}\mu_2} \leqslant 1, \tag{7.113}$$

*the following assertions holds true.*

1. *There exists a family of linear operators* $\big(\mathbf{T}_n\big)_{n\in\mathbb{N}}$ *satisfying*

$$\forall\, s \in\, [s_0, S], \quad \sup_{n\in\mathbb{N}} \|\mathbf{T}_n h\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\gamma^{-1}\Big(\|h\|^{\mathrm{Lip}(\gamma)}_{s+2\tau} + \|\rho\|^{\mathrm{Lip}(\gamma)}_{s+2\tau}\|h\|^{\mathrm{Lip}(\gamma)}_{s_0+2\tau}\Big)$$

*and such that in the Cantor set* $\mathscr{O}_n^2$, *where* $\mathscr{O}_n^2$ *is introduced in Proposition* 7.10, *we have*

$$\mathcal{L}_1\, \mathbf{T}_n = \mathrm{Id} + \mathbf{E}_n,$$

*where* $\mathbf{E}_n$ *satisfies the following estimate*

$$\begin{aligned}\forall\, s \in [s_0, S], \quad \|\mathbf{E}_n h\|_{s_0}^{\mathrm{Lip}(\gamma)} &\leqslant C\gamma^{-1}N_n^{s_0-s}\Big(\|h\|^{\mathrm{Lip}(\gamma)}_{s+2\tau+1} + \|\rho\|^{\mathrm{Lip}(\gamma)}_{s+4\tau+5}\|h\|^{\mathrm{Lip}(\gamma)}_{s_0+2\tau}\Big)\\ &\quad + \varepsilon^6\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|h\|^{\mathrm{Lip}(\gamma)}_{s_0+2\tau+2}.\end{aligned}$$

2. *Set* $\mathcal{T}_n \triangleq \mathscr{B}\mathbf{T}_n\mathscr{B}^{-1}$. *Then,*

$$\forall\, s \in\, [s_0, S], \quad \sup_{n\in\mathbb{N}} \|\mathcal{T}_n h\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\gamma^{-1}\Big(\|h\|^{\mathrm{Lip}(\gamma)}_{s+2\tau} + \|\rho\|^{\mathrm{Lip}(\gamma)}_{s+4\tau+5}\|h\|^{\mathrm{Lip}(\gamma)}_{s_0+2\tau}\Big).$$

*Moreover, in the Cantor set* $\mathscr{O}_n^1(\rho)\cap\mathscr{O}_n^2(\rho)$ *(see* (7.65) *for* $\mathscr{O}_n^1(\rho)$*) one has the identity*

$$\mathcal{L}_0\, \mathcal{T}_n = \mathrm{Id} + \mathcal{E}_n,$$

*where* $\mathcal{E}_n := \mathscr{B}\mathbf{E}_n\mathscr{B}^{-1}$ *satisfies the estimate*

$$\begin{aligned}\|\mathcal{E}_n h\|_{s_0}^{\mathrm{Lip}(\gamma)} &\leqslant C\gamma^{-1}N_n^{s_0-s}\big(\|h\|^{\mathrm{Lip}(\gamma)}_{s+2\tau+1} + \|\rho\|^{\mathrm{Lip}(\gamma)}_{s+4\tau+6}\|h\|^{\mathrm{Lip}(\gamma)}_{s_0+2\tau}\big)\\ &\quad + C\varepsilon^6\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|h\|^{\mathrm{Lip}(\gamma)}_{s_0+2\tau+2}.\end{aligned}$$

*Proof.* In view of (7.85), (7.110) and (7.111) we may write

$$\mathbb{M} = \begin{pmatrix} M_1 & M_2\\ M_3 & M_4\end{pmatrix}, \qquad \mathbb{P}_n = \begin{pmatrix} P_1 & P_2\\ P_3 & P_4\end{pmatrix} \qquad \text{and} \qquad \mathbb{E}_n^2 = \begin{pmatrix} E^2_{n,1} & E^2_{n,2}\\ E^2_{n,3} & E^2_{n,4}\end{pmatrix}.$$

It follows from (7.82) that

$$\mathcal{L}_1 = \sum_{j=1}^4 M_j + \mathtt{E}_n.$$

By denoting

$$\mathbf{T}_n = \sum_{i=1}^4 P_i \qquad \text{and} \qquad \mathbf{E}_n = \sum_{i=1}^4 E^2_{n,i} + \mathtt{E}_n\mathbf{T}_n$$

and using the algebraic structure (7.112) and the identity (7.111) we conclude that

$$\mathcal{L}_1\mathbf{T}_n = \mathrm{Id} + \mathbf{E}_n.$$

The estimates of $\mathbf{T}_n$ and $\mathbf{E}_n$ can be easily achieved from Proposition 7.11 and Proposition 7.7. The second point follows immediately from the fact that $\mathcal{L}_1 = \mathscr{B}^{-1}\mathcal{L}_0\mathscr{B}$, Proposition 7.12-1 and Proposition 7.8 -1 together with (7.113). This achieves the proof of Proposition 7.12. ☐

### 7.6 Construction of solutions and Cantor set measure

To construct solutions to the nonlinear equation

$$\mathbf{F}(\rho) \triangleq \tfrac{1}{\varepsilon^{2+\mu}}\mathbf{G}(\varepsilon r_\varepsilon + \varepsilon^{\mu+1}\rho) = 0 \tag{7.114}$$

introduced in (7.25), we apply a modified Nash-Moser scheme, in the spirit of [59]. The tame estimates on the approximate right inverse of $\mathcal{L}_0 = d_\rho \mathbf{F}(\rho)$, obtained in Proposition 7.12 are identical to those found in [59, Prop. 5.5]. Consequently, the proof of Proposition 7.13 follows the same lines of [59, Prop. 6.1]. Therefore, we shall omit the proof and only provide a complete statement of the main results. For this end, we define the finite-dimensional space

$$E_n \triangleq \Big\{h : [\lambda_*, \lambda^*] \times \mathbb{T}^2 \to \mathbb{R} \quad \text{s.t.} \quad \Pi_{N_n} h = h\Big\},$$

where $\Pi_{N_n}$ is the projector defined by

$$h(\varphi,\theta) = \sum_{\substack{l\in\mathbb{Z}\\ j\in\mathbb{Z}^*}} h_{l,j} e^{\mathrm{i}(l\varphi + j\theta)}, \qquad \Pi_{N_n} h(\varphi,\theta) \triangleq \sum_{|l|+|j|\leqslant N_n} h_{l,j} e^{\mathrm{i}(l\varphi+j\theta)},$$

and the sequence $(N_n)_n$ was defined in (7.60). Here, we will utilize the parameters introduced in (7.3), along with the following additional value.

$$\mathtt{b}_0 = 3 - \mu. \tag{7.115}$$

We shall also fix the values of $N_0$ and $\gamma$ as below

$$N_0 \triangleq \varepsilon^{-\delta}, \qquad \gamma = \varepsilon^{2+\delta}. \tag{7.116}$$

Moreover, we shall impose the following constraints required along the Nash-Moser scheme,

$$\begin{cases} 1+\tau < a_2, \\ 3s_0 + 12\tau + 15 + \frac{3}{2}a_2 < a_1, \\ \frac{2}{3}a_1 < \mu_2, \\ 0 < \delta < \min\left(\mu, \frac{1-\mu}{a_1+2}, \frac{2+\mu}{1+\mu_2}\right), \\ 12\tau + 3 + \frac{6}{\delta} < \mu_1, \\ \max\left(s_0 + 4\tau + 3 + \frac{2}{3}\mu_1 + a_1 + \frac{4}{\delta}, 3s_0 + 4\tau + 6 + 3\mu_2\right) < b_1. \end{cases} \tag{7.117}$$

We have the following result.

**Proposition 7.13** (Nash-Moser scheme)**.** *Given the conditions* (7.115)*,* (7.116) *and* (7.117)*. There exist $C_* > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0,\varepsilon_0]$ we get for all $n \in \mathbb{N}$ the following properties,*

- $(\mathcal{P}1)_n$ *There exists a Lipschitz function*

$$\begin{aligned} \rho_n : \ [\lambda_*, \lambda^*] &\to E_{n-1} \\ \lambda &\mapsto \rho_n \end{aligned}$$

  *satisfying*

$$\rho_0 = 0 \quad \textit{and} \quad \|\rho_n\|^{\mathrm{Lip}(\gamma)}_{2s_0+2\tau+3} \leqslant C_* \varepsilon^{\mathtt{b}_0}\gamma^{-1} \quad \textit{for} \quad n \geqslant 1.$$

  *By setting*

$$u_n = \rho_n - \rho_{n-1} \quad \textit{for} \quad n \geqslant 1,$$

  *we have*

$$\forall s \in [s_0, S],\ \|u_1\|^{\mathrm{Lip}(\gamma)}_s \leqslant \tfrac{1}{2} C_* \varepsilon^{\mathtt{b}_0}\gamma^{-1} \quad \textit{and} \quad \|u_k\|^{\mathrm{Lip}(\gamma)}_{2s_0+2\tau+3} \leqslant C_* \varepsilon^{\mathtt{b}_0}\gamma^{-1} N_{k-1}^{-a_2} \quad \forall\, 2 \leqslant k \leqslant n.$$

- $(\mathcal{P}2)_n$ *Set*

$$\mathcal{A}_0 = [\lambda_*, \lambda^*] \quad \text{and} \quad \mathcal{A}_{n+1} = \mathcal{A}_n \cap \mathcal{O}_n^1(\rho_n) \cap \mathcal{O}_n^2(\rho_n), \quad \forall n \in \mathbb{N}. \tag{7.118}$$

*Then we have the following estimate*

$$\|\mathbf{F}(\rho_n)\|_{s_0,\mathcal{A}_n}^{\mathrm{Lip}(\gamma)} \leqslant C_* \varepsilon^{\mathtt{b}_0} N_{n-1}^{-a_1}.$$

- $(\mathcal{P}3)_n$ *High regularity estimate:* $\|\rho_n\|_{b_1}^{\mathrm{Lip}(\gamma)} \leqslant C_* \varepsilon^{\mathtt{b}_0} \gamma^{-1} N_{n-1}^{\mu_1}$. *Here, we have used the notation* (7.2)*.*

Next, we shall study the convergence of Nash-Moser scheme, stated in Proposition 7.13, show that the limit is a solution to the problem (7.114) and establish a lower bound for the Lebesgue measure of the final Cantor.

**Corollary 7.14.** *There exists* $\lambda \in [\lambda_*, \lambda^*] \mapsto \rho_\infty$ *satisfying*

$$\|\rho_\infty\|_{2s_0+2\tau+3}^{\mathrm{Lip}(\gamma)} \leqslant C_* \varepsilon^{\mathtt{b}_0} \gamma^{-1} \quad \text{and} \quad \|\rho_\infty - \rho_m\|_{2s_0+2\tau+3}^{\mathrm{Lip}(\gamma)} \leqslant C_* \varepsilon^{\mathtt{b}_0} \gamma^{-1} N_m^{-a_2}$$

*such that*

$$\forall \lambda \in \mathtt{C}_\infty \triangleq \bigcap_{m \in \mathbb{N}} \mathcal{A}_m, \qquad \mathbf{F}\big(\rho_\infty(\lambda)\big) = 0. \tag{7.119}$$

*Proof.* In view of Proposition 7.13 one has for all $m \geqslant 1$,

$$\rho_m = \sum_{n=1}^{m} u_n.$$

Define the formal infinite sum

$$\rho_\infty \triangleq \sum_{n=1}^{\infty} u_n.$$

Then the claimed estimates follow immediately from the estimates of $(\mathcal{P}1)_m$, (7.60) and (7.116). Thus, the sequence $(\rho_m)_{m\geqslant 1}$ converges pointwise to $\rho_\infty$. In view of $(\mathcal{P}2)_m$ we obtain

$$\|\mathbf{F}(\rho_m)\|_{s_0,\mathtt{C}_\infty}^{\mathrm{Lip}(\gamma)} \leqslant C_* \varepsilon^{\mathtt{b}_0} N_{m-1}^{-a_1}.$$

Passing to the limit leads to

$$\forall \lambda \in \mathtt{C}_\infty, \quad \mathbf{F}\big(\rho_\infty(\lambda)\big) = 0.$$

This concludes the proof of Corollary 7.14. □

The following result is proved using standard argument, in similar way to [59, Lem. 6.3].

**Lemma 7.15.** *Under* (7.117)*, we have the inclusion*

$$\mathcal{C}_\infty^1 \cap \mathcal{C}_\infty^2 \subset \mathtt{C}_\infty, \tag{7.120}$$

*where, for* $k \in \{1, 2\}$*,*

$$\mathcal{C}_\infty^k \triangleq \bigcap_{\substack{l \in \mathbb{Z} \\ |j| \geqslant k}} \left\{ \lambda \in [\lambda_*, \lambda^*] \quad \text{s.t.} \quad |\varepsilon^2 \omega(\lambda) l + \mu_{j,k}(\lambda, \rho_\infty)| \geqslant \tfrac{2\gamma}{|j|^\tau} \right\}$$

*and*

$$\mu_{j,k}(\lambda, \rho) = j\mathtt{c}(\lambda, \rho) - \tfrac{k-1}{2} \tfrac{j}{|j|}.$$

*In addition, there exists* $\varepsilon_0 > 0$ *and* $C > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$

$$|[\lambda_*, \lambda^*] \setminus \mathtt{C}_\infty| \leqslant C \varepsilon^\delta.$$

*Proof.* In view of (7.65), (7.99), (7.118) and (7.119) we have

$$\mathtt{C}_\infty = \mathcal{O}_{\infty,1} \cap \mathcal{O}_{\infty,2},$$

with

$$\mathcal{O}_{\infty,k} = \bigcap_{\substack{(l,j,n)\in\mathbb{Z}^2\times\mathbb{N}\\ k\leqslant|j|\leqslant N_n}} \Big\{\lambda\in[\lambda_*,\lambda^*] \quad \text{s.t.} \quad |\varepsilon^2\omega(\lambda)\, l + \mu_{j,k}(\lambda,\rho_n)| \geqslant \tfrac{\gamma}{|j|^\tau}\Big\}.$$

We shall prove that for $k\in\{1,2\}$,

$$\mathcal{C}^k_\infty \subset \mathcal{O}_{\infty,k}, \tag{7.121}$$

which implies the inclusion (7.120). Let $\lambda\in\mathcal{C}^k_\infty$ and $n\in\mathbb{N}$. Then, from the triangle inequality, we obtain for all $k\leqslant|j|\leqslant N_n$,

$$\begin{aligned}|\varepsilon^2\omega(\lambda)\, l + \mu_{j,k}(\lambda,\rho_n)| &\geqslant |\varepsilon^2\omega(\lambda)\, l + \mu_{j,k}(\lambda,\rho_\infty)| - |\mu_{j,k}(\lambda,\rho_\infty) - \mu_{j,k}(\lambda,\rho_n)|\\ &\geqslant \tfrac{2\gamma}{|j|^\tau} - |\mu_{j,k}(\lambda,\rho_\infty) - \mu_{j,k}(\lambda,\rho_n)|.\end{aligned} \tag{7.122}$$

Using (7.68), (7.116) and Corollary 7.14 we get for $\varepsilon$ small enough

$$\begin{aligned}|\mu_{j,k}(\lambda,\rho_\infty) - \mu_{j,k}(\lambda,\rho_n)| &\leqslant C\varepsilon^3|j|\|\rho_\infty-\rho_n\|^{\mathrm{Lip}(\gamma)}_{2s_0+2\tau+3}\\ &\leqslant C_*\varepsilon^{3+\mathtt{b}_0}\gamma^{-1}|j|N_n^{-a_2}\\ &\leqslant C_*\gamma|j|^{-\tau}\varepsilon^{\mathtt{b}_0-1-2\delta}N_n^{1+\tau-a_2}.\end{aligned}$$

From (7.115) and the assumption $1+\tau<a_2$ and $\delta<\frac{2-\mu}{2}$ (see (7.117)), we conclude that for small $\varepsilon$ we have

$$\begin{aligned}|\mu_{j,k}(\lambda,\rho_\infty) - \mu_{j,k}(\lambda,\rho_n)| &\leqslant C_*\gamma|j|^{-\tau}\varepsilon^{2-\mu-2\delta}N_n^{1+\tau-a_2}\\ &\leqslant \gamma|j|^{-\tau}.\end{aligned}$$

Combining the last estimate with (7.122) we get, for all $n\in\mathbb{N}$, $l\in\mathbb{Z}$ and $k\leqslant|j|\leqslant N_n$ we get

$$|\varepsilon^2\omega(\lambda)\, l + \mu_{j,k}(\lambda,\rho_n)| \geqslant \tfrac{\gamma}{|j|^\tau}.$$

Hence, we deduce that $\lambda\in\mathcal{O}_{\infty,k}$, concluding the proof of (7.121). Consequently, one has

$$\big|[\lambda_*,\lambda^*]\setminus\mathtt{C}_\infty\big| \leqslant \big|[\lambda_*,\lambda^*]\setminus\mathcal{C}^1_\infty\big| + \big|[\lambda_*,\lambda^*]\setminus\mathcal{C}^2_\infty\big|.$$

Thus, the measure estimate we only need to prove that

$$\big|[\lambda_*,\lambda^*]\setminus\mathcal{C}^k_\infty\big| \leqslant C\varepsilon^\delta. \tag{7.123}$$

For this aim we write

$$\mathcal{C}^k_\infty = \bigcap_{n\in\mathbb{N}}\bigcap_{\substack{(l,j)\in\mathbb{Z}^2\\ k\leqslant|j|\leqslant N_n}} \mathcal{U}^k_{l,j},$$

with

$$\mathcal{U}^k_{l,j} \triangleq \Big\{\lambda\in[\lambda_*,\lambda^*] \quad \text{s.t.} \quad \big|\varepsilon^2\omega(\lambda)\, l + \mu_{j,k}(\lambda,\rho_\infty)\big| \geqslant 2\tfrac{\varepsilon^{2+\delta}}{|j|^\tau}\Big\},$$

having used the fact that $\gamma=\varepsilon^{2+\delta}$. Notice that

$$\bigcap_{\substack{(l,j)\in\mathbb{Z}^2\\ k\leqslant|j|}} \mathcal{U}^k_{l,j} \subset \mathcal{C}^k_\infty.$$

Moreover, by continuity and non-degeneracy assumption on the period 3.17, we infer

$$\forall\lambda\in[\lambda_*,\lambda^*], \quad 0<\overline{b}\leqslant\omega(\lambda)\leqslant\overline{a}, \qquad \overline{a}\triangleq\max_{\lambda\in[\lambda_*,\lambda^*]}\omega_0(\lambda), \qquad \overline{b}\triangleq\min_{\lambda\in[\lambda_*,\lambda^*]}\omega_0(\lambda).$$

Moreover, from (7.83), for all $0 < \varepsilon \leqslant \varepsilon_0$ one has

$$\tfrac{1}{3} \leqslant \mathtt{c}(\lambda,\rho_\infty) \leqslant \tfrac{2}{3}. \tag{7.124}$$

We distinguish different cases.

• Case $\varepsilon^2\overline{b}|l| \geqslant |j| \geqslant k$: For $k=1$, from the triangle inequality, we get

$$\begin{aligned}\big|\varepsilon^2\omega(\lambda)\ell + j\mathtt{c}(\lambda,\rho_\infty)\big| &\geqslant \varepsilon^2\overline{b}|l| - \tfrac{2}{3}|j| \\ &\geqslant \tfrac{1}{3}|j| \\ &\geqslant \tfrac{\varepsilon^{2+\delta}}{|j|^\tau}.\end{aligned}$$

Similarly, for $k=2$, we have

$$\begin{aligned}\big|\varepsilon^2\omega(\lambda)\,l + j\mathtt{c}(\lambda,\rho_\infty) - \tfrac{j}{2|j|}\big| &\geqslant \varepsilon^2\overline{b}|l| - \tfrac{2}{3}|j| - \tfrac{1}{2} \\ &\geqslant \tfrac{1}{3}|j| - \tfrac{1}{2} \geqslant \tfrac{1}{6} \\ &\geqslant \tfrac{\varepsilon^{2+\delta}}{|j|^\tau}.\end{aligned}$$

Therefore, for $k\in\{1,2\}$,

$$\mathcal{U}_{l,j}^k = [\lambda_*,\lambda^*].$$

• Case $|j| \geqslant 24\varepsilon^2\overline{a}|l|$: For $k=1$, one gets by the triangle inequality and (7.124), together with $|j|\geqslant 1$,

$$\begin{aligned}\big|\varepsilon^2\omega(\lambda)\,l + j\mathtt{c}(\lambda,\rho_\infty)\big| &\geqslant \tfrac{1}{3}|j| - \varepsilon^2\overline{a}|l| \\ &\geqslant \big(\tfrac{1}{24}|j| - \varepsilon^2\overline{a}|l|\big) + \tfrac{7}{24}|j| \\ &\geqslant \tfrac{7}{24}.\end{aligned}$$

Thus, for small $\varepsilon$ we infer

$$\big|\varepsilon^2\omega(\lambda)\,l + j\mathtt{c}(\lambda,\rho_\infty)\big| \geqslant \tfrac{\varepsilon^{2+\delta}}{|j|^\tau}.$$

For $k=2$, one obtains by the triangle inequality and (7.124), together with $|j|\geqslant 2$,

$$\begin{aligned}\big|\varepsilon^2\omega(\lambda)\,l + j\mathtt{c}(\lambda,\rho_\infty) - \tfrac{j}{2|j|}\big| &\geqslant \tfrac{1}{3}|j| - \varepsilon^2\overline{a}|l| - \tfrac{1}{2} \\ &\geqslant \big(\tfrac{1}{24}|j| - \varepsilon^2\overline{a}|l|\big) + \tfrac{7}{24}|j| - \tfrac{1}{2} \\ &\geqslant \tfrac{1}{12}.\end{aligned}$$

Then, for small $\varepsilon$ we infer

$$\big|\varepsilon^2\omega(\lambda)\,l + j\mathtt{c}(\lambda,\rho_\infty) - \tfrac{j}{2|j|}\big| \geqslant \tfrac{\varepsilon^{2+\delta}}{|j|^\tau}.$$

Hence, for $k\in\{1,2\}$,

$$\mathcal{U}_{l,j}^k = [\lambda_*,\lambda^*].$$

Consequently, one has

$$\bigcap_{\substack{|j|\leqslant 24\varepsilon^2\overline{a}|l| \\ \varepsilon^2\overline{b}|l|\leqslant|j|}} \mathcal{U}_{l,j}^k \subset \mathcal{C}_\infty^k. \tag{7.125}$$

Set

$$f_{l,j}(\lambda) \triangleq \varepsilon^2\omega(\lambda)\,l + j\mathtt{c}(\lambda,\rho_\infty) - (k-1)\tfrac{j}{2|j|}.$$

Differentiating $f_{l,j}$ with respect to $\lambda$ and using (7.83) give

$$f_{l,j}'(\lambda) = \varepsilon^2\omega'(\lambda)\,l + \varepsilon^3 j\mathtt{c}_2'(\lambda).$$

By non-degeneracy assumption and from the estimate (7.84), we conclude that

$$\forall\lambda\in[\lambda_*,\lambda^*],\quad 0<\underline{c}\leqslant|\omega'(\lambda)|\quad\text{and}\quad|\mathtt{c}_2'(\lambda)|\leqslant c_1.$$

Since $|j|\leqslant 24\varepsilon^2\overline{a}|l|$, then for all $\lambda\in[\lambda_*,\lambda^*]$, we have

$$\begin{aligned}|f_{j,l}'(\lambda)|&\geqslant \underline{c}\varepsilon^2|l|-\varepsilon^3|j|c_1.\\&\geqslant |j|\big(\tfrac{\underline{c}}{24\overline{a}}-c_1\varepsilon^3\big).\end{aligned}$$

Thus, for $\varepsilon$ small enough,

$$|f_{j,l}'(\lambda)|\geqslant \tfrac{\underline{c}}{48\overline{a}}|j|.$$

Applying Lemma 7.16, we deduce that

$$\Big|\Big\{\lambda\in[\lambda_*,\lambda^*]\quad\text{s.t.}\quad\big|\varepsilon^2\omega(\lambda)\,l+j\mathtt{c}(\lambda,\rho_\infty)-(k-1)\tfrac{j}{2|j|}\big|<2\tfrac{\varepsilon^{2+\delta}}{|j|^\tau}\Big\}\Big|\leqslant C\tfrac{\varepsilon^{2+\delta}}{|j|^{1+\tau}}.$$

Consequently, from (7.125), we get

$$\begin{aligned}\big|[\lambda_*,\lambda^*]\setminus\mathcal{C}_\infty^k\big|&\leqslant C\sum_{\varepsilon^2\overline{b}|\ell|\leqslant|j|}\tfrac{\varepsilon^{2+\delta}}{|j|^{1+\tau}}\\&\leqslant C\varepsilon^\delta\sum_{j\in\mathbb{Z}^*}\tfrac{1}{|j|^\tau}\\&\leqslant C\varepsilon^\delta,\end{aligned}$$

proving (7.123). It is important to mention that we are slightly stretching the argument here, as the function $f_{j,l}$ is Lipschitz continuous rather than $C^1$. Nevertheless, we can rigorously support the preceding argument by working with the Lipschitz norm. This leads to the proof of the desired result. □

We finally recall the following classical result concerning bi-Lipschitz functions and measure theory.

**Lemma 7.16.** *Let $(\alpha,\beta)\in(\mathbb{R}_+^*)^2$. Consider $f:[\lambda_*,\lambda^*]\to\mathbb{R}$ a bi-Lipschitz function such that*

$$\forall(\lambda_1,\lambda_2)\in[\lambda_*,\lambda^*]^2,\quad|f(\lambda_1)-f(\lambda_2)|\geqslant\beta|\lambda_1-\lambda_2|.$$

*Then, there exists $C>0$ independent of $\|f\|_{\mathrm{Lip}}$ such that*

$$\Big|\big\{\lambda\in[\lambda_*,\lambda^*]\quad\text{s.t.}\quad|f(\lambda)|\leqslant\alpha\big\}\Big|\leqslant C\tfrac{\alpha}{\beta}.$$

# A Symplectic changes of coordinates and integral operators

This section's major objective is to highlight useful findings related to some change of coordinates system. We refer the reader to the papers [8, 11, 35, 71] for the proofs. Let $\beta:\mathscr{O}\times\mathbb{T}^2\to\mathbb{R}$ be a smooth function such that $\sup_{\lambda\in\mathscr{O}}\|\beta(\lambda,\cdot,\bullet)\|_{\mathrm{Lip}}<1$ then there exists $\widehat{\beta}:\mathscr{O}\times\mathbb{T}^2\to\mathbb{R}$ smooth such that

$$y=\theta+\beta(\lambda,\varphi,\theta)\Longleftrightarrow\theta=y+\widehat{\beta}(\lambda,\varphi,y).\tag{A.1}$$

Define the operators

$$\mathscr{B}=(1+\partial_\theta\beta)\mathrm{B},\qquad \mathrm{B}h(\lambda,\varphi,\theta)=h\big(\lambda,\varphi,\theta+\beta(\lambda,\varphi,\theta)\big).\tag{A.2}$$

Then

$$\mathscr{B}^{-1}=(1+\partial_\theta\widehat{\beta})\mathrm{B}^{-1},\qquad \mathrm{B}^{-1}h(\lambda,\varphi,y)=h\big(\lambda,\varphi,y+\widehat{\beta}(\lambda,\varphi,y)\big).\tag{A.3}$$

Now, we will provide some basic algebraic properties for the aforementioned operators.

**Lemma A.1.** *The following assertions hold true.*

1. *Let $\mathscr{B}_1, \mathscr{B}_2$ be two periodic change of variables as in* (A.2)*, then*
$$\mathscr{B}_1\mathscr{B}_2 = (1+\partial_\theta\beta)\mathrm{B},$$
*with*
$$\beta(\varphi,\theta) \triangleq \beta_1(\varphi,\theta) + \beta_2\big(\varphi,\theta+\beta_1(\varphi,\theta)\big).$$
2. *The conjugation of the transport operator by $\mathscr{B}$ keeps the same structure*
$$\mathscr{B}^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\big(V(\varphi,\theta)\,\cdot\,\big)\Big)\mathscr{B} = \omega\cdot\partial_\varphi + \partial_y\big(\mathscr{V}(\varphi,y)\,\cdot\,\big),$$
*with*
$$\mathscr{V}(\varphi,y) \triangleq \mathrm{B}^{-1}\Big(\omega\cdot\partial_\varphi\beta(\varphi,\theta) + V(\varphi,\theta)\big(1+\partial_\theta\beta(\varphi,\theta)\big)\Big).$$

In what follows, and in the rest of this appendix, we assume that $(\lambda, s, s_0)$ satisfy (7.3) and we consider $\beta \in \mathrm{Lip}_\gamma(\mathscr{O}, H^s(\mathbb{T}^2))$ satisfying the smallness condition
$$\|\beta\|_{2s_0}^{\mathrm{Lip}(\gamma)} \leqslant \varepsilon_0,$$
with $\varepsilon_0$ small enough. The following result can be found in [35, 11].

**Lemma A.2.** *The following assertions hold true.*

1. *The linear operators $\mathrm{B}, \mathscr{B} : \mathrm{Lip}_\gamma(\mathscr{O}, H^s(\mathbb{T}^2)) \to \mathrm{Lip}_\gamma(\mathscr{O}, H^s(\mathbb{T}^2))$ are continuous and invertible, with*
$$\|\mathrm{B}^{\pm 1}h\|_s^{\mathrm{Lip}(\gamma)} \leqslant \|h\|_s^{\mathrm{Lip}(\gamma)}\left(1 + C\|\beta\|_{s_0}^{\mathrm{Lip}(\gamma)}\right) + C\|\beta\|_s^{\mathrm{Lip}(\gamma)}\|h\|_{s_0}^{\mathrm{Lip}(\gamma)}, \tag{A.4}$$
$$\|\mathscr{B}^{\pm 1}h\|_s^{\mathrm{Lip}(\gamma)} \leqslant \|h\|_s^{\mathrm{Lip}(\gamma)}\left(1 + C\|\beta\|_{s_0}^{\mathrm{Lip}(\gamma)}\right) + C\|\beta\|_{s+1}^{\mathrm{Lip}(\gamma)}\|h\|_{s_0}^{\mathrm{Lip}(\gamma)}. \tag{A.5}$$
2. *The functions $\beta$ and $\widehat{\beta}$ defined through* (A.1) *satisfy the estimates*
$$\|\widehat{\beta}\|_s^{\mathrm{Lip}(\gamma)} \leqslant C\|\beta\|_s^{\mathrm{Lip}(\gamma)}.$$

The next result deals with some integral operator estimates. For the proof, we refer to [71, Lem. 4.4] and [12, Lem. 2.3].

**Lemma A.3.** *Consider a smooth real-valued kernel*
$$K : (\lambda,\varphi,\theta,\eta) \mapsto K(\lambda,\varphi,\theta,\eta)$$
*and let*
$$(\mathcal{T}_K h)(\lambda,\varphi,\theta) \triangleq \int_{\mathbb{T}} K(\lambda,\varphi,\theta,\eta)h(\lambda,\varphi,\eta)d\eta, \tag{A.6}$$
*be an integral operator. Then, for any $s_1, s_2 \geqslant 0$,*
$$\|\mathcal{T}_K h\|_{s_1,s_2}^{\mathrm{Lip}(\gamma)} \lesssim \|h\|_{s_0}^{\mathrm{Lip}(\gamma)}\|K\|_{s_1+s_2}^{\mathrm{Lip}(\gamma)} + \|h\|_{s_1}^{\mathrm{Lip}(\gamma)}\|K\|_{s_0+s_2}^{\mathrm{Lip}(\gamma)}.$$
*Given the periodic change of variables $\mathscr{B}$ defined by* (A.2)*. Then, the operators $\mathscr{B}^{-1}\mathcal{T}_K\mathscr{B}$ and $\mathscr{B}^{-1}\mathcal{T}_K\mathscr{B} - \mathcal{T}_K$ are integral operators,*
$$\big(\mathscr{B}^{-1}\mathcal{T}_K\mathscr{B}\big)h(\lambda,\varphi,\theta) = \int_{\mathbb{T}} h(\lambda,\varphi,\eta)\widehat{K}(\lambda,\varphi,\theta,\eta)d\eta$$
$$\big(\mathscr{B}^{-1}\mathcal{T}_K\mathscr{B} - \mathcal{T}_K\big)h(\lambda,\varphi,\theta) = \int_{\mathbb{T}} h(\lambda,\varphi,\eta)\widetilde{K}(\lambda,\varphi,\theta,\eta)d\eta$$
*with*
$$\|\widehat{K}\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|K\|_s^{\mathrm{Lip}(\gamma)} + \|K\|_{s_0}^{\mathrm{Lip}(\gamma)}\|\beta\|_{s+1}^{\mathrm{Lip}(\gamma)},$$
$$\|\widetilde{K}\|_s^{\mathrm{Lip}(\gamma)} \lesssim \|K\|_{s+1}^{\mathrm{Lip}(\gamma)}\|\beta\|_{s_0}^{\mathrm{Lip}(\gamma)} + \|K\|_{s_0}^{\mathrm{Lip}(\gamma)}\|\beta\|_{s+1}^{\mathrm{Lip}(\gamma)}.$$

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