Steady vortex sheets in presence of surface tension

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Abstract

We prove a bifurcation result of uniformly-rotating/stationary non-trivial vortex sheets near the circular distribution for a model of two irrotational fluids with same density taking into account surface tension effects. As bifurcation parameters, we play with either the speed of rotation, the surface tension coefficient or the mean vorticity.

Contents

1 Introduction

We consider a planar Euler system for two irrotational fluids with same density (constant equal to 1) separated by an interface $\Gamma(t)$ homeomorphic to a circle and parametrized by $z(t,\cdot):\mathbb{T}\to\mathbb{R}^2$. This interface divides the plane into two open components $\Omega^{\pm}(t)$ with $\Omega^-(t)$ bounded and $\Omega^+(t)$ unbounded. The evolutionary system is thus composed of the following equations

$$\begin{cases} u_t^{\pm}+u^{\pm}\cdot\nabla u^{\pm}+\nabla p^{\pm}=0, & \text{in } \Omega^{\pm}(t),\\ \big(z_t-u^{\pm}|_{\Gamma(t)}\big)\cdot z_x^{\perp}=0, & \text{at } \Gamma(t),\\ (p^- - p^+)|_{\Gamma(t)}=\sigma\mathcal{K}(z), & \text{at } \Gamma(t),\\ u^+(t,\mathbf{x})\to 0, & \text{as } |\mathbf{x}|\to+\infty,\\ \nabla\cdot u^{\pm}=0, & \text{in } \Omega^{\pm}(t),\\ \nabla^{\perp}\cdot u^{\pm}=0, & \text{in } \Omega^{\pm}(t). \end{cases} \tag{1.1}$$

In the above set of equations, the quantities $u^{\pm},p^{\pm}$ are respectively the velocity field and pressure inside the domain $\Omega^{\pm}$. The parameter $\sigma\geqslant 0$ is the surface tension coefficient and $\mathcal{K}(z)$ is the curvature defined by

$$\mathcal{K}(z)\triangleq-\frac{z_x^{\perp}\cdot z_{xx}}{|z_x|^3}.$$

The last equation in (1.1) implies that the vorticity distribution $\boldsymbol{\omega}$ is localized on the curve $\Gamma(t)$ at time t, namely

$$\boldsymbol{\omega}(t,\mathbf{x})=\omega(t,x)\delta\big(\mathbf{x}-z(t,x)\big),\qquad \mathbf{x}\in\mathbb{R}^2,\quad x\in\mathbb{T}. \tag{1.2}$$

Such a solution is called *vortex sheet.* The velocity fields are recovered through the so-called Biot-Savart law

$$u^{\pm}(t,\mathbf{x})=\int_{\mathbb{T}}\frac{\big(\mathbf{x}-z(t,y)\big)^{\perp}}{\big|\mathbf{x}-z(t,y)\big|^2}\omega(t,y)\mathrm{d}y, \tag{1.3}$$

where we used the following convention for the integral on the torus $\mathbb{T}$

$$\int_{\mathbb{T}} f(y)\mathrm{d}y \triangleq \frac{1}{2\pi}\int_0^{2\pi} f(y)\mathrm{d}y.$$

The equations for the system (1.1) in the case in which the vorticty is a vortex sheet (i.e. (1.2)) are called *Kelvin-Helmholtz system.* From (1.3) and standard suppression of the pressure via Leray projectors it is clear that the bulk quantities $u^{\pm}$ and $p^{\pm}$ can be expressed in terms of the interface quantities ω and z, thus recasting (1.1) as a *Contour Dynamic Equation (CDE)* on $\mathbb{T}$. The resulting system, derived in Appendix A.1, writes as

$$\begin{cases} \big(z_t - \mathrm{BR}(z)\omega\big)\cdot z_x^{\perp} = 0, \\ \omega_t = \left(\omega \ \dfrac{(z_t - \mathrm{BR}(z)\omega)\cdot z_x}{|z_x|^2}\right)_x - \sigma\left(\mathcal{K}(z)\right)_x. \end{cases} \tag{1.4}$$

where the *Birkhoff-Rott integral operator* is defined by

$$\mathrm{BR}(z)\omega(t,x) \triangleq \mathrm{p.v.}\int_{\mathbb{T}} \frac{\big(z(t,x)-z(t,y)\big)^{\perp}}{|z(t,x)-z(t,y)|^2}\omega(t,y)\mathrm{d}y.$$

Let us choose a parametrization in the form

$$z(t,x) = R(t,x)e^{\mathrm{i}x}, \qquad R(t,x) \triangleq \sqrt{1+2\eta(t,x)}. \tag{1.5}$$

The choice of the parametrization (1.5), which is a graph on the unit circle, allows us to recast the system (1.4) in a more congenial form. The detailed computations are performed in Appendix A.2 and produce the system

$$\begin{cases} \eta_t = -\dfrac{1}{2}\mathscr{H}(\eta)[\omega], \\ \omega_t = -\left(\dfrac{\omega}{2}\mathscr{D}_0(\eta)[\omega]\right)_x - \sigma\big(\mathscr{K}(\eta)\big)_x, \end{cases} \tag{1.6}$$

where

$$\mathscr{H}(\eta)[\omega] \triangleq \eta_x\mathscr{D}_0(\eta)[\omega] + \mathscr{H}_0(\eta)[\omega], \tag{1.7}$$

$$\mathscr{D}_0(\eta)[\omega](x) \triangleq \mathrm{p.v.}\int_{\mathbb{T}} \frac{1-\sqrt{\frac{1+2\eta(y)}{1+2\eta(x)}}\cos(x-y)}{1+\eta(x)+\eta(y)-\sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\cos(x-y)}\omega(y)\mathrm{d}y, \tag{1.8}$$

$$\mathscr{H}_0(\eta)[\omega](x) \triangleq \mathrm{p.v.}\int_{\mathbb{T}} \frac{\sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\sin(x-y)}{1+\eta(x)+\eta(y)-\sqrt{1+2\eta(x)}\sqrt{1+2\eta(y)}\cos(x-y)}\omega(y)\mathrm{d}y, \tag{1.9}$$

$$\mathscr{K}(\eta) \triangleq \frac{\eta_{xx} - 2\left(\frac{\eta_x}{R}\right)^2}{\left(R^2+\left(\frac{\eta_x}{R}\right)^2\right)^{\frac{3}{2}}} - \left(R^2+\left(\frac{\eta_x}{R}\right)^2\right)^{-\frac{1}{2}}. \tag{1.10}$$

Using the divergence free property of the flow, Stokes' Theorem, the second equation in (1.1) and the first equation in (A.9), we get

$$0 = \int_{\Omega^-(t)} \nabla\cdot u^-(t,\mathbf{x})\,d\mathbf{x} = \int_0^{2\pi} u^-\big(t,z(t,x)\big)\cdot z_x^{\perp}(t,x)\,dx = \int_0^{2\pi} z_t(t,x)\cdot z_x^{\perp}(t,x)\,dx = -\frac{d}{dt}\int_0^{2\pi}\eta(t,x)\,dx.$$

Therefore, the space average of η is preserved and we impose

$$\int_{\mathbb{T}} \eta(x)dx = 0.$$

The second equation in (1.6) implies the conservation of the mean vorticity and in the sequel, we denote

$$\gamma \triangleq \int_{\mathbb{T}} \omega(x)dx.$$

Let us introduce the velocity potential ψ through the relation

$$\omega = \gamma + \psi_x, \qquad \text{i.e.} \qquad \psi \triangleq \partial_x^{-1}(\omega-\gamma). \tag{1.11}$$

In the new variables (η,ψ), the system (1.6) becomes

$$\begin{cases}\eta_t = -\dfrac{1}{2}\mathscr{H}(\eta)[\psi_x] - \dfrac{\gamma}{2}\mathscr{H}(\eta)[1],\\ \psi_t = -\dfrac{\psi_x+\gamma}{2}\mathscr{D}_0(\eta)[\psi_x+\gamma] - \sigma\mathscr{K}(\eta).\end{cases} \tag{1.12}$$

Let us emphazise that to obtain the second equation in (1.12), we have integrated in space the second equation in (1.6). Therefore, the second equation is well-defined up to a time-dependent additive constant.

The system (1.12) can be reformulated in a Hamiltonian form, akin to the approach taken by [6] for perturbations of the flat interface. The Hamiltonian structure is an important ingredient in the study of small divisors problems for PDEs to construct for instance quasi-periodic solutions [4, 9–11, 14, 30, 31, 34, 37, 39, 46] and a fundamental tool to study long time stability in presence of resonances [7, 8, 12, 13, 53]. Since it falls outside the scope of this manuscript and does not play an important role in our bifurcation analysis, we omit the derivation of the Hamiltonian formulation and we leave it to our future investigations.

1.1 Main results

We look for uniformly rotating vortex sheets, that is traveling wave solutions of the system (1.12), namely

$$(\eta,\psi)(t,x) = (\check{\eta},\check{\psi})(x-ct), \qquad c\in\mathbb{R}, \qquad \check{\eta},\check{\psi}\in L^2(\mathbb{T}).$$

When $c=0$, the corresponding solutions are stationary. The equations (1.12) rewrite in this context (for simplicity of notations, we still denote (η,ψ) instead of $(\check{\eta},\check{\psi})$)

$$\begin{cases} c\,\eta_x + \frac{1}{2}\mathscr{H}(\eta)[\psi_x] + \frac{\gamma}{2}\mathscr{H}(\eta)[1] = 0,\\ c\,\psi_x + \frac{\psi_x+\gamma}{2}\mathscr{D}_0(\eta)[\psi_x+\gamma] + \sigma\mathscr{K}(\eta) = 0.\end{cases} \tag{1.13}$$

Given $\mathbf{m}\in\mathbb{N}^*$, we say that a uniformly rotating vortex sheet is $\mathbf{m}$-fold if the functions $\check{\eta}$ and $\check{\psi}$ are $\frac{2\pi}{\mathbf{m}}$ periodic.

The couple $(\eta,\psi)=(0,0)$ is a trivial solution of (1.12) for any values of the parameters c,σ,γ. This solution corresponds to the family of circular stationary vortex sheets given by

$$z(x) = e^{\mathrm{i}x}, \qquad \omega\equiv\gamma, \qquad u^-(\mathbf{x})\equiv 0, \qquad u^+(\mathbf{x}) = \gamma\frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}\cdot$$

We refer the reader to Lemma 2.1 for more details. Our main result, stated below, gives the existence of non-trivial solutions which are small amplitude perturbations of this stationary state.

Theorem 1.1. ***(Local bifurcation of vortex sheets from the circular distribution)***

(i) *Let $\sigma>0$, $\gamma\in\mathbb{R}$ and $\mathbf{m}\in\mathbb{N}^*$. Assume that*

$$\frac{\gamma^2-\sigma}{\sigma\mathbf{m}^2}\notin\mathbb{N}^* \tag{1.14}$$

supplemented by one of the following conditions

(a) $4\sigma(2-\sqrt{3}) < \gamma^2 < 4\sigma(2+\sqrt{3})$.

(b) $\gamma^2\in[0,\infty)\setminus\big[4\sigma(2-\sqrt{3}),4\sigma(2+\sqrt{3})\big]$ *and* $\mathbf{m}\in\mathbb{R}\setminus[m_-(\sigma,\gamma),m_+(\sigma,\gamma)]$, *with*

$$m_\pm(\sigma,\gamma) = \frac{\gamma^2}{4\sigma}\pm\frac{1}{4\sigma}\sqrt{(\gamma^2-8\sigma)^2-48\sigma^2}\cdot$$

Then, there exist two branches of $\mathbf{m}$-fold uniformly rotating vortex-sheets bifurcating from the circular distribution at angular speed

$$c_{\mathbf{m}}^\pm(\sigma,\gamma) = -\frac{\gamma}{2}\pm\frac{1}{2}\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\cdot$$

(ii) *Let $(c,\gamma)\in\mathbb{R}^2\setminus\{(0,0)\}$. Fix $\mathbf{m}\in\mathbb{N}\setminus\{0,1\}$ such that*

$$\mathbf{m} > \frac{2\gamma^2}{(2c+\gamma)^2+\gamma^2}\cdot$$

Assume in addition that

$$\frac{(2\mathbf{m}-1)\gamma^2-(2c+\gamma)^2}{\mathbf{m}(2c+\gamma)^2+(\mathbf{m}-2)\gamma^2}\notin\mathbb{N}^*. \tag{1.15}$$

Then, there exists one branch of $\mathbf{m}$*-fold uniformly rotating vortex-sheets with speed* c *bifurcating from the circular distribution for*

$$\sigma_{\mathbf{m}}(c,\gamma)=\frac{\mathbf{m}\,(2c+\gamma)^2+(\mathbf{m}-2)\gamma^2}{2(\mathbf{m}^2-1)}\cdot$$

(iii) Let $\sigma>0$ *and* $\mathbf{m}\in\mathbb{N}\setminus\{0,1\}$. *Then, there exist two branches of* $\mathbf{m}$*-fold stationary vortex-sheets bifurcating from the circular distribution for*

$$\gamma_{\mathbf{m}}^{\pm}(\sigma)=\pm\sqrt{\sigma(\mathbf{m}+1)}\cdot$$

Remark 1.1. *Let us make some remarks.*

1. *The condition* (1.14) *is satisfied in particular for*

$$\frac{\gamma^2}{\sigma}\in\mathbb{R}\setminus\mathbb{Q}.$$

2. *Notice that formally* $\sigma_{\mathbf{m}}(0,\gamma)=\frac{\gamma^2}{\mathbf{m}+1}$, *which is in accordance with the point* (*iii*). *Nevertheless, the condition* (1.15) *is not compatible with* $c=0$ *but is satisfied in particular for*

$$\left(\frac{2c}{\gamma}+1\right)^2\in\mathbb{R}\setminus\mathbb{Q}.$$

The Theorem 1.1 is obtained by means of local bifurcation theory, more precisely by applying Crandall-Rabinowitz Theorem A.1. For this aim, we see solutions to (1.13) as zeros of a nonlinear functional $\mathcal{F}$ (defined by (2.1)-(2.2)):

$$\mathcal{F}(c,\sigma,\gamma,\eta,\psi)=0.$$

We prove in Lemma 2.1 that $(\eta,\psi)=(0,0)$ is indeed a solution. Then, in Proposition 2.1, we study the regularity of $\mathcal{F}$ and we give in Proposition 2.2 the expression of its differential $\mathcal{L}_{c,\sigma,\gamma}$ at the trivial solution $(\eta,\psi)=(0,0)$. We dispose of three parameters to bifurcate: the speed of rotation c, the surface tension coefficient σ and the mean vorticity γ. The corresponding analysis are led in Subsections 2.2, 2.3 and 2.4, respectively. Let us emphasize that the restrictions (1.14) and (1.15) are made for avoiding spectral collisions and get a one dimensional kernel for $\mathcal{L}_{c,\sigma,\gamma}$. The transversality conditions are obtained via a duality argument as developed in [32, 41, 55, 56].

1.2 Known results

The study of the motion of two immiscible fluids separated by a sharp interface has been a subject of study since the mid-19th century, when von Helmholtz derived the evolution equations for such phenomenon in 1858 in [40]. When there is no surface tension and no gravity the problem is generally ill-posed in Sobolev spaces, even at linear level (cf. [52, Section 9.3]), this phenomenon, known as *Kelvin-Helmholtz instability*, lead to the formation of vortices or waves at the interface or within the shear layer. It is named after Lord Kelvin (William Thomson) and Hermann von Helmholtz, who independently investigated the instability in the late 19th century.

When the fluid interface is a perturbation of the flat rest state and there are no surface tension effects ($\sigma=0$) stability in the analytic framework was proved in [48, 58, 59]. Adding the effects of the surface tension to (1.1) ($\sigma>0$) induces stabilizing effects and the resulting equations are well-posed, linearly [49], and non-linearly [1–3, 19, 22, 57], even in presence of two fluids with different densities. A more comprehensive stability criterion has been investigated by D. Lannes in [50] in the context of two fluids with nonzero density, when gravitational effects are taken in consideration. We highlight that when we have a shear flow the presence of the surface tension is crucial in order to avoid the growth of high-modes, that is characteristically induced by the Kelvin-Helmholtz instability, otherwise the equations are ill-posed [16, 51, 60], in sharp contrast with what happens in the context of the Water-Waves equations. We mention as well the foundational work of Delort [28] which proved the existence of global weak solutions for measured-valued vorticities that include the vortex sheet problem.

The stability results discussed are constrained by an existence time proportional to the capillarity coefficient σ, leaving the long-term stability of (1.12) largely unexplored. When $\sigma=0$, circles and lines are non-trivial steady solutions, as are segments of length $2a$ with strength $\omega(x) = \Omega\sqrt{a^2 - x^2}$, rotating uniformly at angular velocity Ω. This category also includes the family of solutions identified by B. Protas and T. Sakajo [54] and the solutions [17, 18] of Cao-Qin-Zou. When $\Gamma(t)$ is a closed curve and $\sigma = 0$, recent findings [36] have demonstrated that steady solutions bifurcate from degenerate eigenvalues. In [35], the same authors proved rigidity results, namely the non-existence of non-trivial solutions for a certain range of angular speed. Over the past decades, bifurcation techniques have been succesfully implemented to obtain traveling periodic solutions for instance, for the 3D liquid drop model with capillarity [5] or in the vortex patch class where a broad spectrum of steady solutions for CDEs were discovered, starting with Burbea's seminal work [15] and extended in [20, 21, 26, 27, 33, 38, 42–45]. The previous list is non-exhaustive since restricted to the Eulerian model. Let us mention the recent generalization to a large class of models [47]. This manuscript adds to this literature by presenting an existence result for the Kelvin-Helmholtz equations near a circular vortex with surface tension effects. The combined contributions of our study and [36] establish, for the first time, the presence of closed curves as global solutions to Kelvin-Helmholtz equations.

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2 Proof of the results

This section is devoted to the proof of Theorem 1.1. We first introduce the functional of interest and give the expression of its differential at the circular distribution. Then, we study the bifurcations with respect to the various parameters of the problem.

2.1 Functional of interest and its linear operator

Let us introduce the functional

$$\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2),$$

where

$$\mathcal{F}_1(c, \sigma, \gamma, \eta, \psi) \triangleq c\,\eta_x + \frac{1}{2}\mathscr{H}(\eta)[\psi_x] + \frac{\gamma}{2}\mathscr{H}(\eta)[1], \tag{2.1}$$

$$\mathcal{F}_2(c, \sigma, \gamma, \eta, \psi) \triangleq c\,\psi_x + \frac{\psi_x + \gamma}{2}\mathscr{D}_0(\eta)[\psi_x + \gamma] + \sigma\mathscr{K}(\eta). \tag{2.2}$$

Then, clearly solutions of (1.13) are zeros of the functional $\mathcal{F}$ and we have the following result stating that the circular distribution is indeed a trivial solution for any values of the parameters.

Lemma 2.1. *The state $(\eta, \psi) = (0, 0)$ is a trivial solution. More precisely,*

$$\forall (c, \sigma, \gamma) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}, \quad \mathcal{F}(c, \sigma, \gamma, 0, 0) = 0.$$

This corresponds to

$$\omega \equiv \gamma, \qquad \Omega^-(t) = D(0, 1) \qquad \text{and} \qquad \begin{cases} u^-(\mathbf{x}) = 0, & \text{if } \mathbf{x} \in D(0, 1), \\ u^+(\mathbf{x}) = \gamma\frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}, & \text{if } \mathbf{x} \in \mathbb{R}^2 \setminus \overline{D(0, 1)}. \end{cases} \tag{2.3}$$

Proof. From (1.8) and (1.10), one readily gets

$$\mathscr{D}_0(0)[f](x) = \int_{\mathbb{T}} f(y)\mathrm{d}y, \qquad \mathscr{K}(0) = 1. \tag{2.4}$$

Then, using the classical relations

$$\sin(u) = 2\sin\left(\tfrac{u}{2}\right)\cos\left(\tfrac{u}{2}\right), \qquad 1 - \cos(u) = 2\sin^2\left(\tfrac{u}{2}\right), \tag{2.5}$$

we obtain from (1.7) and (1.9)

$$\mathscr{H}(0)[f](x) = \mathscr{H}_0(0)[f](x) = \text{p.v.}\int_{\mathbb{T}} \frac{\sin(x - y)}{1 - \cos(x - y)} f(y)\mathrm{d}y = \text{p.v.}\int_{\mathbb{T}} \cot\left(\frac{x - y}{2}\right) f(y)\mathrm{d}y = \mathcal{H}[f](x), \tag{2.6}$$

where $\mathcal{H}$ denotes the 2π-periodic Hilbert transform. Inserting the foregoing calculations into (2.1)-(2.2) gives

$$\mathcal{F}_1(c,\sigma,\gamma,0,0)=\frac{\gamma}{2}\mathscr{H}(0)[1]=\frac{\gamma}{2}\mathcal{H}[1]=0$$

and

$$\mathcal{F}_2(c,\sigma,\gamma,0,0)=\frac{\gamma^2}{2}\mathscr{D}_0(0)[1]+\sigma\mathscr{K}(0)=\frac{\gamma^2}{2}+\sigma\sim 0.$$

Recall that the second equation is well-defined modulo constants. Now let us turn to the fluid description. We employ the usual identification $\mathbb{R}^2\simeq\mathbb{C}$. For $(\eta,\psi)=0$, (1.5) and (1.3) imply that

$$\omega\equiv\gamma\qquad\text{and}\qquad z(x)=e^{\mathrm{i}x}.$$

Note that

$$\frac{z_x(x)}{\mathrm{i}z(x)}\equiv 1. \tag{2.7}$$

We compute the complex conjugate of the velocity field for $\mathbf{x}\in\mathbb{C}\setminus\partial D(0,1)$ which, in view of (1.11), is given by the following holomorphic function

$$\overline{u^{\pm}(\mathbf{x})}=\frac{\gamma}{2\pi\mathrm{i}}\int_0^{2\pi}\frac{1}{\mathbf{x}-z(x)}dx\overset{(2.7)}{=}\frac{\gamma}{\mathrm{i}}\frac{1}{2\pi\mathrm{i}}\int_0^{2\pi}\frac{z_x(x)}{(\mathbf{x}-z(x))z(x)}dx=\frac{\gamma}{\mathrm{i}}\frac{1}{2\pi\mathrm{i}}\int_{\partial D(0,1)}\frac{dz}{(\mathbf{x}-z)z}=\begin{cases}0, & \mathbf{x}\in D(0,1),\\ \frac{\gamma}{\mathrm{i}\,\mathbf{x}}, & \mathbf{x}\notin D(0,1),\end{cases}$$

where, to obtain the last equality, we used the residue Theorem. Taking the complex conjugate, we get

$$u^-(\mathbf{x})=0,\qquad u^+(\mathbf{x})=\gamma\frac{\mathrm{i}}{\overline{\mathbf{x}}}=\gamma\frac{\mathrm{i}\mathbf{x}}{|\mathbf{x}|^2}=\gamma\frac{\mathbf{x}^\perp}{|\mathbf{x}|^2}.$$

This ends the proof of Lemma 2.1. □

We consider the following $\mathbf{m}$-fold ($\mathbf{m}\in\mathbb{N}^*$) Sobolev spaces with regularity index $s\geqslant 0$,

$$H^s_{\mathrm{even},\mathbf{m}}\triangleq\left\{f(x)=\sum_{n=1}^{\infty}a_n\cos(n\mathbf{m}x),\quad\sum_{n=1}^{\infty}\langle n\rangle^{2s}|a_n|^2<\infty,\quad a_n\in\mathbb{R}\right\},$$
$$H^s_{\mathrm{odd},\mathbf{m}}\triangleq\left\{f(x)=\sum_{n=1}^{\infty}b_n\sin(n\mathbf{m}x),\quad\sum_{n=1}^{\infty}\langle n\rangle^{2s}|b_n|^2<\infty,\quad b_n\in\mathbb{R}\right\},$$

where we have used the classical notation $\langle n\rangle\triangleq\max(1,n)$. Then we set

$$X^s_{\mathbf{m}}\triangleq H^{s+\frac14}_{\mathrm{even},\mathbf{m}}\times H^{s-\frac14}_{\mathrm{odd},\mathbf{m}},\qquad Y^s_{\mathbf{m}}\triangleq H^{s-\frac54}_{\mathrm{odd},\mathbf{m}}\times H^{s-\frac74}_{\mathrm{even},\mathbf{m}}$$

and for $r>0$,

$$\mathcal{B}^s_{\mathbf{m}}(r)\triangleq\Big\{(\eta,\psi)\in X^s_{\mathbf{m}}\quad\text{s.t.}\quad\|\eta\|_{H^{s+\frac14}}+\|\psi\|_{H^{s-\frac14}}<r\Big\}.$$

In the next proposition, we state the regularity of the functional $\mathcal{F}$ with respect to these function spaces.

Proposition 2.1. *Let $\mathbf{m}\in\mathbb{N}^*$ there exists $s_0>0$ such that for any $s\geqslant s_0$ there exists $r=r(s)>0$ such that the functional*

$$\mathcal{F}:\mathbb{R}\times(0,\infty)\times\mathbb{R}\times\mathcal{B}^s_{\mathbf{m}}(r)\longrightarrow Y^s_{\mathbf{m}}$$

defined in (2.1)-(2.2) *is well defined and of class C^1.*

Proof. Assume η and ψ respectively even and odd (in space). Then η_x and ψ_x are respectively odd and even. Now by performing a change of variables $y\mapsto -y$ we can easily see from (1.8) and (1.9) that

$$f\text{ even}\quad\Rightarrow\quad\Big(\mathscr{D}_0(\eta)[f]\text{ even}\quad\text{and}\quad\mathscr{H}_0(\eta)[f]\text{ odd}\Big).$$

Coming back to the expression (1.7), we see that

$$f\text{ even}\quad\Rightarrow\quad\mathscr{H}(\eta)[f]\text{ odd}.$$

With this in hand, we deduce from (2.1)-(2.2) that

$$\mathcal{F}_1(c,\sigma,\gamma,\eta,\psi)\text{ is odd}\qquad\text{and}\qquad\mathcal{F}_2(c,\sigma,\gamma,\eta,\psi)\text{ is even}.$$

The $\mathbf{m}$-fold preserving property follows similarly by using the change of variables $y \mapsto y + \frac{2\pi}{\mathbf{m}}$.

Since the parity and $\mathbf{m}$-fold symmetry is preserved as explained above, using (1.7), (2.1) and (2.2) the result follows if we can prove that given $(\eta,\psi) \in \mathcal{B}^s_{\mathbf{m}}(r)$ then $\mathscr{H}_0(\eta)[\psi_x], \mathscr{D}_0(\eta)[\psi_x]$ are in $H^{s-\frac{5}{4}}$, the C^1 differentiability shall stem since the integrand functions that define the operators $\mathscr{H}_0$ and $\mathscr{D}_0$ are analytic in η and ψ. We prove the statement for $\mathscr{H}_0(\eta)[\psi_x]$ being the one for $\mathscr{D}_0(\eta)[\psi_x]$ similar. Notice that

$$\mathscr{H}_0\left(\eta\right)\left[\psi_x\right] = |D|\psi + 2\,\text{p.v.}\int_{\mathbb{T}} \left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right) - 1\right)\frac{\psi_x\left(x-z\right)}{2\tan\left(z/2\right)}\mathrm{d}z,$$

where

$$\Delta_z\eta \triangleq \frac{\eta(x)-\eta(x-z)}{2\sin\left(z/2\right)}$$

and

$$\mathsf{K}_z\left(\mathsf{X}\right) \triangleq \frac{\sqrt{1-4\mathsf{X}\ \sin\left(z/2\right)}}{2\left(1-2\mathsf{X}\ \sin\left(z/2\right)-\sqrt{1-4\mathsf{X}\ \sin\left(z/2\right)}\cos z\right)}\ \left(2\sin\left(z/2\right)\right)^2\ . \tag{2.8}$$

Notice that $(z,\mathsf{X}) \mapsto \mathsf{K}_z\left(\mathsf{X}\right)$ is analytic in $(-\pi,\pi)\times\left(-\frac{1}{4},\frac{1}{4}\right)$. In particular we can Taylor-expand in z the application $z \mapsto \mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right) - 1$ and we obtain that

$$\mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right) - 1 = \mathsf{K}^0\left(\frac{\eta_x}{1+2\eta}\right) + R^1\left(\eta;x,z\right),$$

where

$$\begin{aligned} \mathsf{K}^0\left(\mathsf{X}\right) &\triangleq -\frac{\mathsf{X}^2}{1+\mathsf{X}^2}, \\ R^1\left(\eta;x,z\right) &\triangleq \left(\mathsf{K}_z\left(\frac{\Delta_z\eta}{1+2\eta}\right)-1\right) - \mathsf{K}^0\left(\frac{\eta_x}{1+2\eta}\right). \end{aligned} \tag{2.9}$$

We thus have that

$$\mathscr{H}_0\left(\eta\right)\left[\psi_x\right] = \left(1+\mathsf{K}^0\left(\frac{\eta_x}{1+2\eta}\right)\right)|D|\psi + \underbrace{2\,\text{p.v.}\int_{\mathbb{T}} R^1\left(\eta;x,z\right)\frac{\psi_x\left(x-z\right)}{2\tan\left(z/2\right)}\mathrm{d}z}_{\triangleq R_{\mathscr{H}_0}(\eta)[\psi_x]}. \tag{2.10}$$

Applying standard Moser tame estimates and composition theorems to (2.9) it is immediate that there exists $s_0>0$ so that for any $s \geqslant s_0$ there exists a $r=r\left(s\right)>0$ so that

$$(\eta,\psi) \mapsto \left(1+\mathsf{K}^0\left(\frac{\eta_x}{1+2\eta}\right)\right)|D|\psi \in C^1\left(\mathcal{B}^s_{\mathbf{m}}(r); H^{s-\frac{5}{4}}\right), \tag{2.11}$$

thus we can focus our attention on the remainder term $R_{\mathscr{H}_0}\left(\eta\right)\left[\psi_x\right]$ in (2.10). We have that, if $s>7/4$

$$\left\|R_{\mathscr{H}_0}\left(\eta\right)\left[\psi_x\right]\right\|_{H^{s-\frac{5}{4}}} \lesssim \left\|\frac{R^1\left(\eta;x,z\right)}{2\tan\left(z/2\right)}\right\|_{L^\infty_z H^{s-\frac{5}{4}}_x} \left\|\psi\right\|_{H^{s-\frac{1}{4}}} \tag{2.12}$$

Since $R^1\left(\eta;x,z\right)$ is a Taylor-1 remainder its explicit expression is given by

$$R^1\left(\eta;x,z\right) = z\int_0^1 \left(\partial_z\mathsf{K}_{\vartheta z}\left(\frac{\Delta_{\vartheta z}\eta}{1+2\eta}\right) + \mathsf{K}'_{\vartheta z}\left(\frac{\Delta_{\vartheta z}\eta}{1+2\eta}\right)\eta'\left(x-\vartheta z\right)\right)d\vartheta, \tag{2.13}$$

so that from (2.13) and the fact that $\partial_z\mathsf{K}_z\left(0\right)=0$ it is clear that if $\left\|\eta\right\|_{H^{s-\frac{1}{4}}} \ll 1$ then

$$\sup_{z\in\mathbb{T}}\left\|\frac{R^1\left(\eta;\bullet,z\right)}{2\tan\left(z/2\right)}\right\|_{H^{s-\frac{5}{4}}} \lesssim \left\|\eta\right\|_{H^{s-\frac{1}{4}}}, \tag{2.14}$$

so that (2.14) and (2.12) prove that (after a relabeling of s_0, s and r, if needed)

$$(\eta,\psi) \mapsto R_{\mathscr{H}_0}\left(\eta\right)\left[\psi_x\right] \in C^0\left(\mathcal{B}^s_{\mathbf{m}}(r); H^{s-\frac{5}{4}}\right). \tag{2.15}$$

The fact that the application (after a relabeling of s_0, s and r, if needed)

$$(\eta,\psi) \mapsto R_{\mathscr{H}_0}\left(\eta\right)\left[\psi_x\right] \in C^1\left(\mathcal{B}^s_{\mathbf{m}}(r); H^{s-\frac{5}{4}}\right). \tag{2.16}$$

can be proved by computations similar to the ones performed to prove (2.15), and are omitted for the sake of brevity. Combining (2.11), (2.16) and (2.10), we obtain that (after a relabeling of s_0, s and r, if needed)

$$(\eta,\psi)\mapsto \mathscr{H}_0(\eta)[\psi_x]\in C^1\left(\mathcal{B}^s_{\mathbf{m}}(r);H^{s-\frac{5}{4}}\right). \tag{2.17}$$

Computations similar to the ones used to prove (2.17) show us that

$$(\eta,\psi)\mapsto \mathscr{D}_0(\eta)[\psi_x]\in C^1\left(\mathcal{B}^s_{\mathbf{m}}(r);H^{s-\frac{5}{4}}\right). \tag{2.18}$$

Putting together (2.18) and (2.17) proves, combined with standard product estimates, that, fixed c,γ and σ

$$(\eta,\psi)\mapsto \big(\mathcal{F}_1(c,\sigma,\gamma,\eta,\psi),\mathcal{F}_2(c,\sigma,\gamma,\eta,\psi)-\sigma\mathscr{K}(\eta)\big)\in C^1\left(\mathcal{B}^s_{\mathbf{m}}(r);H^{s-\frac{5}{4}}_{\mathrm{odd},\mathbf{m}}\times H^{s-\frac{5}{4}}_{\mathrm{even},\mathbf{m}}\right). \tag{2.19}$$

In addition, it is clear from (1.10) and composition estimates that for $\|\eta\|_{H^{s+\frac{1}{4}}}\ll 1$,

$$\|\mathscr{K}(\eta)\|_{H^{s-\frac{7}{4}}}<\infty. \tag{2.20}$$

From (2.19) and (2.20), we deduce that

$$(\eta,\psi)\mapsto \big(\mathcal{F}_1(c,\sigma,\gamma,\eta,\psi),\mathcal{F}_2(c,\sigma,\gamma,\eta,\psi)\big)\in C^1\left(\mathcal{B}^s_{\mathbf{m}}(r);H^{s-\frac{5}{4}}_{\mathrm{odd},\mathbf{m}}\times H^{s-\frac{7}{4}}_{\mathrm{even},\mathbf{m}}\right).$$

Thus, since the differentiability in c,γ and σ is immediate from (2.1)-(2.2), we conclude the desired result. □

Our next goal is to linearize the system (1.13) at the trivial solution $(\eta,\psi)=(0,0)$. The corresponding operator has a good Fourier multiplier structure and enjoys Fredholmness property with respect to the function spaces introduced above. More precisely, we have the following proposition.

Proposition 2.2. *Let $(c,\sigma,\gamma)\in\mathbb{R}\times(0,\infty)\times\mathbb{R}$. We denote*

$$\mathcal{L}_{c,\sigma,\gamma}\triangleq d_{\eta,\psi}\mathcal{F}(c,\sigma,\gamma,0,0).$$

(i) The operator $\mathcal{L}_{c,\sigma,\gamma}$ writes

$$\mathcal{L}_{c,\sigma,\gamma}=\begin{pmatrix}\left(c+\frac{\gamma}{2}\right)\partial_x & \frac{1}{2}|D|\\ \sigma-\gamma^2+\frac{\gamma^2}{2}|D|-\sigma|D|^2 & \left(c+\frac{\gamma}{2}\right)\partial_x\end{pmatrix}. \tag{2.21}$$

In particular, it is a Fourier multiplier. Its action on $(\hat{\eta},\hat{\psi})$, admitting the Fourier expansions

$$\hat{\eta}(x)=\sum_{n=1}^{\infty}a_n\cos(n\mathbf{m}x),\qquad \hat{\psi}(x)=\sum_{n=1}^{\infty}b_n\sin(n\mathbf{m}x),\qquad a_n,b_n\in\mathbb{R},$$

is given by

$$\mathcal{L}_{c,\sigma,\gamma}\begin{pmatrix}\hat{\eta}\\ \hat{\psi}\end{pmatrix}(x)=\sum_{n=1}^{\infty}\begin{pmatrix}\sin(n\mathbf{m}x) & 0\\ 0 & \cos(n\mathbf{m}x)\end{pmatrix}M_{n\mathbf{m}}(c,\sigma,\gamma)\begin{pmatrix}a_n\\ b_n\end{pmatrix},$$

with

$$M_n(c,\sigma,\gamma)\triangleq\begin{pmatrix}-\left(c+\frac{\gamma}{2}\right)n & \frac{n}{2}\\ \sigma-\gamma^2+\frac{\gamma^2}{2}n-\sigma n^2 & \left(c+\frac{\gamma}{2}\right)n\end{pmatrix}. \tag{2.22}$$

(ii) The operator $\mathcal{L}_{c,\sigma,\gamma}:X^s_{\mathbf{m}}\longrightarrow Y^s_{\mathbf{m}}$ is Fredholm with zero index.

Proof. (*i*) First observe that from the expression (1.10), one readily gets

$$d_\eta\mathscr{K}(0)[\hat{\eta}]=\hat{\eta}+\hat{\eta}_{xx}=(\mathrm{Id}-|D|^2)\eta. \tag{2.23}$$

Then, differentiating in (1.9), we infer

$$d_\eta\mathscr{H}_0(0)[\hat{\eta}][f](x)=0. \tag{2.24}$$

Now, differentiating in (1.8) and using one more time (2.5) yields

$$\begin{aligned}d_\eta\mathscr{D}_0(0)[\hat{\eta}][f](x)&=-\mathrm{p.v.}\int_{\mathbb{T}}\frac{\big(\hat{\eta}(y)-\hat{\eta}(x)\big)\cos(x-y)}{1-\cos(x-y)}f(y)\mathrm{d}y-\int_{\mathbb{T}}\big(\hat{\eta}(x)+\hat{\eta}(y)\big)f(y)\mathrm{d}y\\&=\frac{1}{2}\mathrm{p.v.}\int_{\mathbb{T}}\frac{\hat{\eta}(x)-\hat{\eta}(y)}{\sin^2\left(\frac{x-y}{2}\right)}f(y)\mathrm{d}y-2\hat{\eta}(x)\int_{\mathbb{T}}f(y)\mathrm{d}y.\end{aligned} \tag{2.25}$$

For later purposes, we shall study the case $f=1$. Using Fourier symbols representation, we see that

$$|D|=\partial_x\mathcal{H}=\mathcal{H}\partial_x. \tag{2.26}$$

In addition, since $\cot'(x)=-\frac{1}{\sin^2(x)}$ and $\mathcal{H}[1]=0$, we get

$$\begin{aligned}\partial_x\mathcal{H}[f](x)&=\partial_x\mathcal{H}[f](x)-f(x)\partial_x\mathcal{H}[1](x)\\&=\text{p.v.}\int_{\mathbb{T}}\partial_x\left[\cot\left(\frac{x-y}{2}\right)\right]\big(f(y)-f(x)\big)\mathrm{d}y\\&=\frac{1}{2}\text{p.v.}\int_{\mathbb{T}}\frac{f(x)-f(y)}{\sin^2\left(\frac{x-y}{2}\right)}\mathrm{d}y.\end{aligned} \tag{2.27}$$

Putting together (2.25), (2.26) and (2.27) yields

$$d_\eta\mathscr{D}_0(0)[\hat\eta][1]=|D|\hat\eta-2\hat\eta. \tag{2.28}$$

Combining (1.7) and (2.4) and (2.24), we get

$$d_\eta\mathscr{H}(0)[\hat\eta][f]=\hat\eta_x\mathscr{D}_0(0)[f]+d_\eta\mathscr{H}_0(0)[f]=\hat\eta_x\int_{\mathbb{T}}f(y)\mathrm{d}y. \tag{2.29}$$

As a consequence, differentiating (2.1) and using (2.6), (2.26) and (2.29) implies

$$\begin{aligned}d_{\eta,\psi}\mathcal{F}_1(c,\sigma,\gamma,0,0)[\hat\eta,\hat\psi]&=c\,\hat\eta_x+\frac{1}{2}\mathscr{H}(0)[\hat\psi_x]+\frac{\gamma}{2}d_\eta\mathscr{H}(0)[\hat\eta][1]\\&=c\,\hat\eta_x+\frac{1}{2}\mathcal{H}\partial_x\hat\psi+\frac{\gamma}{2}\hat\eta_x\\&=\left(c+\frac{\gamma}{2}\right)\partial_x\hat\eta+\frac{|D|}{2}\hat\psi.\end{aligned} \tag{2.30}$$

Besides, differentiating (2.2) and making appeal to (2.4), (2.23) and (2.28), we find

$$\begin{aligned}d_{\eta,\psi}\mathcal{F}_2(c,\sigma,\gamma,0,0)[\hat\eta,\hat\psi]&=c\,\hat\psi_x+\frac{\gamma}{2}\hat\psi_x\mathscr{D}_0(0)[1]+\frac{\gamma^2}{2}d_\eta\mathscr{D}_0(0)[\hat\eta][1]+\frac{\gamma}{2}\mathscr{D}_0(0)[\hat\psi_x]+\sigma d_\eta\mathscr{K}(0)[\hat\eta]\\&=\left(c+\frac{\gamma}{2}\right)\partial_x\hat\psi+(\sigma-\gamma^2)\hat\eta+\frac{\gamma^2}{2}|D|\hat\eta-\sigma|D|^2\hat\eta.\end{aligned} \tag{2.31}$$

Putting together (2.30) and (2.31), we obtain the matrix representation (2.21) for the linearized operator.

(ii) Coming back to the expression (2.21), we decompose the operator as follows

$$\mathcal{L}_{c,\sigma,\gamma}=I_\sigma+K_{c,\sigma,\gamma},$$

where

$$I_\sigma\triangleq\begin{pmatrix}0 & \frac{1}{2}|D|\\ -\sigma|D|^2 & 0\end{pmatrix},\qquad K_{c,\sigma,\gamma}\triangleq\begin{pmatrix}\left(c+\frac{\gamma}{2}\right)\partial_x & 0\\ \sigma-\gamma^2+\frac{\gamma^2}{2}|D| & \left(c+\frac{\gamma}{2}\right)\partial_x\end{pmatrix}.$$

Clearly, $I_\sigma:X^s_{\mathbf{m}}\to Y^s_{\mathbf{m}}$ is an isomorphism. In addition, $K_{c,\sigma,\gamma}:X^s_{\mathbf{m}}\to Y^{s+\frac{1}{2}}_{\mathbf{m}}$ is continuous. By Rellich-Kondrachov Theorem, we deduce that $K_{c,\sigma,\gamma}:X^s_{\mathbf{m}}\to Y^s_{\mathbf{m}}$ is a compact operator. This proves the claim by applying [23, Cor. 5.9]. □

2.2 Bifurcation from c

In this subsection, we fix $\sigma>0$ and $\gamma\in\mathbb{R}$ (some restrictions will be imposed later on) and study the bifurcation from the parameter c. Let us look for the values of the parameter c such that the matrix $M_n(c,\sigma,\gamma)$, introduced in (2.22), is singular. For this aim, we compute its determinant

$$\det\big(M_n(c,\sigma,\gamma)\big)=-\left(c+\frac{\gamma}{2}\right)^2n^2+\frac{n}{4}\left(2\sigma n^2-\gamma^2n+2(\gamma^2-\sigma)\right). \tag{2.32}$$

We study the sign on $[1,\infty)$ of the polynomial function

$$n\mapsto 2\sigma n^2-\gamma^2n+2(\gamma^2-\sigma).$$

The associated discriminant is

$$\Delta=\gamma^4-16\sigma\gamma^2+16\sigma^2=(\gamma^2-8\sigma)^2-48\sigma^2.$$

Hence,

$$2\sigma n^2-\gamma^2 n+2\gamma^2>0\quad\Leftrightarrow\quad(n,\sigma,\gamma)\in\mathcal{S}\triangleq\mathcal{S}_1\cup\mathcal{S}_2,$$

where

$$\mathcal{S}_1\triangleq\big\{(n,\sigma,\gamma)\in\mathbb{N}^*\times(0,\infty)\times\mathbb{R}\quad\text{s.t.}\quad 4\sigma(2-\sqrt{3})<\gamma^2<4\sigma(2+\sqrt{3})\big\},$$

$$\mathcal{S}_2\triangleq\left\{(n,\sigma,\gamma)\in\mathbb{N}^*\times(0,\infty)\times\mathbb{R}\quad\begin{array}{l}\text{s.t.}\quad\gamma^2\in[0,\infty)\setminus\Big[4\sigma(2-\sqrt{3}),4\sigma(2+\sqrt{3})\Big]\\ \text{and}\quad n\in\mathbb{R}\setminus[m_-(\sigma,\gamma),m_+(\sigma,\gamma)],\\ \text{with}\quad m_\pm(\sigma,\gamma)\triangleq\dfrac{\gamma^2}{4\sigma}\pm\dfrac{1}{4\sigma}\sqrt{(\gamma^2-8\sigma)^2-48\sigma^2}\end{array}\right\}.$$

Then, for $(n,\sigma,\gamma)\in\mathcal{S}$, we have

$$\det\big(M_n(c,\sigma,\gamma)\big)=0\quad\Leftrightarrow\quad c=c_n^\pm(\sigma,\gamma)\triangleq-\frac{\gamma}{2}\pm\frac{1}{2}\sqrt{2\sigma n-\gamma^2+\frac{2(\gamma^2-\sigma)}{n}}\cdot\tag{2.33}$$

Proposition 2.3. *Let $(\mathbf{m},\sigma,\gamma)\in\mathcal{S}$ satisfying the additional condition*

$$\frac{\gamma^2-\sigma}{\sigma\mathbf{m}^2}\not\in\mathbb{N}^*.\tag{2.34}$$

Then, the following properties hold true.

(i) The kernel of the operator $\mathcal{L}_{c_{\mathbf{m}}^\pm(\sigma,\gamma),\sigma,\gamma}$ is one dimensional. More precisely

$$\ker\left(\mathcal{L}_{c_{\mathbf{m}}^\pm(\sigma,\gamma),\sigma,\gamma}\right)=\mathtt{span}(x_{0,\sigma,\gamma,\mathbf{m}}^\pm),\qquad x_{0,\sigma,\gamma,\mathbf{m}}^\pm(x)\triangleq\begin{pmatrix}\cos(\mathbf{m}x)\\ \pm\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\sin(\mathbf{m}x)\end{pmatrix}.$$

(ii) The operator $\mathcal{L}_{c_{\mathbf{m}}^\pm(\sigma,\gamma),\sigma,\gamma}:X_{\mathbf{m}}^s\longrightarrow Y_{\mathbf{m}}^s$ is Fredholm with zero index.

(iii) The transversality condition holds, namely

$$\partial_c\mathcal{L}_{c,\sigma,\gamma}|_{c=c_{\mathbf{m}}^\pm(\sigma,\gamma)}\left[x_{0,\sigma,\gamma,\mathbf{m}}^\pm\right]\not\in R\left(\mathcal{L}_{c_{\mathbf{m}}^\pm(\sigma,\gamma),\sigma,\gamma}\right).\tag{2.35}$$

Proof. (i) Let us study the spectral collisions of the $\mathbf{m}$-fold spectrum. More precisely, we shall solve

$$c_{\mathbf{m}}^{\kappa_1}(\sigma,\gamma)=c_{k\mathbf{m}}^{\kappa_2}(\sigma,\gamma),\qquad(\kappa_1,\kappa_2)\in\{-,+\}^2,\qquad k\in\mathbb{N}^*.\tag{2.36}$$

First observe that there can exist values of $k\in\mathbb{N}\setminus\{0,1\}$ such that $(k\mathbf{m},\sigma,\gamma)\not\in\mathcal{S}$. In this case, $c_{k\mathbf{m}}^{\kappa_2}(\sigma,\gamma)$ is either equal to $-\frac{\gamma}{2}$ or an element of $\mathbb{C}\setminus\mathbb{R}$. In both cases the equation (2.36) is not satisfied. So we can restrict the discussion to the case where $(k\mathbf{m},\sigma,\gamma)\in\mathcal{S}$. Coming back to the expression of $c_n^\pm(\sigma,\gamma)$ in (2.33), the equation (2.36) implies

$$\sigma\mathbf{m}+\frac{\gamma^2-\sigma}{\mathbf{m}}=\sigma k\mathbf{m}+\frac{\gamma^2-\sigma}{k\mathbf{m}}$$

or equivalently

$$\sigma\mathbf{m}^2(k-1)\left(k-\frac{\gamma^2-\sigma}{\sigma\mathbf{m}^2}\right)=\sigma\mathbf{m}^2k^2-\left(\sigma\mathbf{m}^2+\gamma^2-\sigma\right)k+\gamma^2-\sigma=0.$$

Therefore, there are two solutions

$$k_1=1,\qquad k_2=\frac{\gamma^2-\sigma}{\sigma\mathbf{m}^2}\cdot$$

Since $c_{\mathbf{m}}^-(\sigma,\gamma)\neq c_{\mathbf{m}}^+(\sigma,\gamma)$, the case $k=k_1=1$ corresponds to the trivial solution. Thanks to the condition (2.34), we deduce that the equation (2.36) has no non-trivial solution. As a consequence

$$\det\Big(M_{\mathbf{m}}\big(c_{\mathbf{m}}^\pm(\sigma,\gamma),\sigma,\gamma\big)\Big)=0$$

and

$$\forall n\in\mathbb{N}\setminus\{0,1\},\quad\det\Big(M_{n\mathbf{m}}\big(c_{\mathbf{m}}^\pm(\sigma,\gamma),\sigma,\gamma\big)\Big)\neq0.$$

This implies the one dimensional kernel property for the linearized operator and the generator is obtain by remarking that

$$\begin{pmatrix} 1 \\ \pm\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\end{pmatrix} \in \ker\Big(M_{\mathbf{m}}\big(c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma\big)\Big).$$

(*ii*) Follows from Proposition 2.2-(*ii*).

(*iii*) We introduce on $Y^s_{\mathbf{m}}$ the scalar product $\langle\cdot,\cdot\rangle$ defined as follows: for (f,g) and $(\widetilde{f},\widetilde{g})$ in $Y^s_{\mathbf{m}}$ admitting the Fourier representations

$$f(x)=\sum_{n=1}^{\infty}a_n\sin(n\mathbf{m}x), \qquad \widetilde{f}(x)=\sum_{n=1}^{\infty}\widetilde{a}_n\sin(n\mathbf{m}x)$$

and

$$g(x)=\sum_{n=1}^{\infty}b_n\cos(n\mathbf{m}x), \qquad \widetilde{g}(x)=\sum_{n=1}^{\infty}\widetilde{b}_n\cos(n\mathbf{m}x),$$

with $a_n,\widetilde{a}_n,b_n,\widetilde{b}_n\in\mathbb{R}$, their scalar product is given by

$$\langle (f,g),(\widetilde{f},\widetilde{g})\rangle \triangleq \sum_{n=1}^{\infty}a_n\widetilde{a}_n+b_n\widetilde{b}_n.$$

Define

$$y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}(x)\triangleq\begin{pmatrix}\mp\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\sin(\mathbf{m}x)\\ \cos(\mathbf{m}x)\end{pmatrix}.$$

Let us prove that

$$R\left(\mathcal{L}_{c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma}\right)=\mathtt{span}\left(y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right)^{\perp_{\langle\cdot,\cdot\rangle}}. \tag{2.37}$$

Take an element $y\in R\left(\mathcal{L}_{c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma}\right)$. By construction,

$$y(x)=\sum_{n=1}^{\infty}\begin{pmatrix}\sin(n\mathbf{m}x) & 0\\ 0 & \cos(n\mathbf{m}x)\end{pmatrix}M_{n\mathbf{m}}(c,\sigma,\gamma)\begin{pmatrix}a_n\\ b_n\end{pmatrix}.$$

Then,

$$\begin{aligned}\langle y,y^{\pm}_0\rangle &= \left\langle M_{\mathbf{m}}\left(c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma\right)\begin{pmatrix}a_n\\ b_n\end{pmatrix},\begin{pmatrix}\mp\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\\ 1\end{pmatrix}\right\rangle_{\mathbb{R}^2}\\ &=\left\langle \begin{pmatrix}a_n\\ b_n\end{pmatrix},M^{\top}_{\mathbf{m}}\left(c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma\right)\begin{pmatrix}\mp\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\\ 1\end{pmatrix}\right\rangle_{\mathbb{R}^2}\\ &=0.\end{aligned}$$

The last identity is obtained because by construction

$$\begin{pmatrix}\mp\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\\ 1\end{pmatrix}\in\ker\Big(M^{\top}_{\mathbf{m}}\big(c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma\big)\Big).$$

Recall that the notation $M^{\top}$ denotes the transposed of the matrix M. This proves that

$$R\left(\mathcal{L}_{c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma}\right)\subset\mathtt{span}\left(y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right)^{\perp_{\langle\cdot,\cdot\rangle}}. \tag{2.38}$$

Now, since the space $\mathtt{span}\left(y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right)$ is of finite dimension, then we can apply the orthogonal supplementary Theorem in the pre-Hilbertian space $(Y^s_{\mathbf{m}},\langle\cdot,\cdot\rangle)$ to get

$$Y^s_{\mathbf{m}}=\mathtt{span}\left(y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right)\overset{\perp}{\oplus}\mathtt{span}\left(y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right)^{\perp_{\langle\cdot,\cdot\rangle}}.$$

This proves that $\mathtt{span}\left(y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right)^{\perp_{\langle\cdot,\cdot\rangle}}$ is of codimension one in $Y^s_{\mathbf{m}}$. Besides, the points (*i*) and (*ii*) give that $R\left(\mathcal{L}_{c^{\pm}_{\mathbf{m}}(\sigma,\gamma),\sigma,\gamma}\right)$ is also of codimension one in $Y^s_{\mathbf{m}}$. Together with the inclusion (2.38), we conclude (2.37). With this in hand, we can now check the transversality condition. Notice that, from (2.21), we get

$$\partial_c\mathcal{L}_{c,\sigma,\gamma}\big|_{c=c^{\pm}_{\mathbf{m}}(\sigma,\gamma)}=\begin{pmatrix}\partial_x & 0\\ 0 & \partial_x\end{pmatrix}.$$

Thus, a straitforward computation together with the fact that $(\mathbf{m},\sigma,\gamma)\in\mathcal{S}$ give

$$\left\langle \partial_c\mathcal{L}_{c,\sigma,\gamma}|_{c=c^{\pm}_{\mathbf{m}}(\sigma,\gamma)}\left[x^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right],y^{\pm}_{0,\sigma,\gamma,\mathbf{m}}\right\rangle=\pm2\mathbf{m}\sqrt{2\sigma\mathbf{m}-\gamma^2+\frac{2(\gamma^2-\sigma)}{\mathbf{m}}}\neq0.$$

According to (2.37), this proves the transversality condition (2.35) and achieves the proof of Proposition 2.3. □

Proof of Theorem 1.1-(i). We apply Theorem A.1 together with Lemma 2.1 and Propositions 2.1 and 2.3. □

2.3 Bifurcation from σ

In this subsection, we fix $(c,\gamma)\in\mathbb{R}^2$ (some restrictions will be imposed later on) and study the bifurcation from the parameter σ. According to (2.32), we have

$$\det\big(M_1(c,\sigma,\gamma)\big)=-c(c+\gamma)\tag{2.39}$$

is independent of σ and, for any $n\geqslant2$,

$$\det\big(M_n(c,\sigma,\gamma)\big)=0\quad\Leftrightarrow\quad\sigma=\sigma_n(c,\gamma)\triangleq\frac{\Big[(2c+\gamma)^2+\gamma^2\Big]n-2\gamma^2}{2(n^2-1)}\cdot\tag{2.40}$$

In the sequel, we denote

$$\alpha(c,\gamma)\triangleq(2c+\gamma)^2+\gamma^2,\qquad\beta(\gamma)\triangleq2\gamma^2.$$

The condition $\sigma>0$ requires

$$n\alpha(c,\gamma)-\beta(\gamma)>0,\qquad\text{i.e.}\qquad n>N(c,\gamma)\triangleq\frac{2\gamma^2}{(2c+\gamma)^2+\gamma^2}\cdot$$

Proposition 2.4. *Let $(c,\gamma)\in\mathbb{R}^2$ and $\mathbf{m}\in\mathbb{N}\setminus\{0,1\}$ with $\mathbf{m}>N(c,\gamma)$. Assume in addition that*

$$\frac{(2\mathbf{m}-1)\gamma^2-(2c+\gamma)^2}{\mathbf{m}(2c+\gamma)^2+(\mathbf{m}-2)\gamma^2}\not\in\mathbb{N}^*.\tag{2.41}$$

Then, the following properties hold true.

(i) The kernel of the operator $\mathcal{L}_{c,\sigma_{\mathbf{m}}(c,\gamma),\gamma}$ is one dimensional. More precisely,

$$\ker\left(\mathcal{L}_{c,\sigma_{\mathbf{m}}(c,\gamma),\gamma}\right)=\mathtt{span}(x_{0,c,\gamma,\mathbf{m}}),\qquad x_{0,c,\gamma,\mathbf{m}}(x)\triangleq\begin{pmatrix}\cos(\mathbf{m}x)\\(2c+\gamma)\sin(\mathbf{m}x)\end{pmatrix}.$$

(ii) The operator $\mathcal{L}_{c,\sigma_{\mathbf{m}}(c,\gamma),\gamma}:X^s_{\mathbf{m}}\longrightarrow Y^s_{\mathbf{m}}$ is Fredholm with zero index.

(iii) The transversality condition holds, namely

$$\partial_\sigma\mathcal{L}_{c,\sigma,\gamma}|_{\sigma=\sigma_{\mathbf{m}}(c,\gamma)}[x_{0,c,\gamma,\mathbf{m}}]\not\in R\left(\mathcal{L}_{c,\sigma_{\mathbf{m}}(c,\gamma),\gamma}\right).\tag{2.42}$$

Proof. (i) Let us study the spectral collisions. We need to solve

$$\sigma_{\mathbf{m}}(c,\gamma)=\sigma_{k\mathbf{m}}(c,\gamma),\qquad k\in\mathbb{N}^*.\tag{2.43}$$

In what follows, we simply denote $\alpha\triangleq\alpha(c,\gamma)$ and $\beta\triangleq\beta(\gamma)$. Coming back to the expression (2.40), the previous equation is equivalent to

$$(\alpha\mathbf{m}-\beta)(k^2\mathbf{m}^2-1)=(\alpha k\mathbf{m}-\beta)(\mathbf{m}^2-1),$$

or again

$$(\alpha\mathbf{m}-\beta)\mathbf{m}^2(k-1)\left(k-\frac{\beta\mathbf{m}-\alpha}{\mathbf{m}(\alpha\mathbf{m}-\beta)}\right)=(\alpha\mathbf{m}-\beta)\mathbf{m}^2k^2-\alpha\mathbf{m}(\mathbf{m}^2-1)k+\mathbf{m}(\beta\mathbf{m}-\alpha)=0.$$

Therefore, the equation (2.43) admits two solutions

$$k_1=1\text{ (trivial solution)},\qquad k_2=\frac{\beta\mathbf{m}-\alpha}{\alpha\mathbf{m}-\beta}=\frac{(2\mathbf{m}-1)\gamma^2-(2c+\gamma)^2}{\mathbf{m}(2c+\gamma)^2+(\mathbf{m}-2)\gamma^2}\cdot$$

The condition (2.41) implies that the equation (2.43) has no non-trivial solution. Then, we can conclude as in the proof of Proposition 2.3-(i).

(ii) Follows from Proposition 2.2-(ii).

(iii) Proceeding as in the proof of Proposition 2.3-(iii), we get

$$R\left(\mathcal{L}_{c,\sigma_{\mathbf{m}(c,\gamma)},\gamma}\right)=\mathtt{span}(y_{0,c,\gamma,\mathbf{m}})^{\perp_{\langle\cdot,\cdot\rangle}},\qquad y_{0,c,\gamma,\mathbf{m}}(x)\triangleq\begin{pmatrix}-(2c+\gamma)\sin(\mathbf{m}x)\\ \cos(\mathbf{m}x)\end{pmatrix}.$$

From the expression (2.21), we see that

$$\partial_\sigma\mathcal{L}_{c,\sigma,\gamma}|_{\sigma=\sigma_{\mathbf{m}}(c,\gamma)}=\begin{pmatrix}0 & 0\\ \mathrm{Id}-|D|^2 & 0\end{pmatrix}.$$

Therefore, since $\mathbf{m}\neq 1$,

$$\langle\partial_\sigma\mathcal{L}_{c,\sigma,\gamma}|_{\sigma=\sigma_{\mathbf{m}}(c,\gamma)}[x_{0,c,\gamma,\mathbf{m}}],y_{0,c,\gamma,\mathbf{m}}\rangle=1-\mathbf{m}^2\neq 0.$$

This concludes (2.42) and the proof of Proposition 2.4. □

Proof of Theorem 1.1-(ii). We apply Theorem A.1 together with Lemma 2.1 and Propositions 2.1 and 2.4. □

2.4 Bifurcation from γ for stationary vortex sheets

In this subsection, we fix $c=0$ and $\sigma>0$. Then, we study the bifurcation from the parameter γ. According to (2.39), we get

$$\det\big(M_1(0,\sigma,\gamma)\big)=0,$$

and in view of (2.32), for any $n\geqslant 2$, we have

$$\det\big(M_n(0,\sigma,\gamma)\big)=0\quad\Leftrightarrow\quad\gamma=\gamma_n^{\pm}(\sigma)\triangleq\pm\sqrt{\sigma(n+1)}.$$

Proposition 2.5. *Let $\sigma>0$ and $\mathbf{m}\in\mathbb{N}\setminus\{0,1\}$. Then, the following properties hold true.*

(i) The kernel of the operator $\mathcal{L}_{0,\sigma,\gamma_{\mathbf{m}}^{\pm}(\sigma)}$ is one dimensional. More precisely,

$$\ker\left(\mathcal{L}_{0,\sigma,\gamma_{\mathbf{m}}^{\pm}(\sigma)}\right)=\mathtt{span}(x_{0,\sigma,\mathbf{m}}^{\pm}),\qquad x_{0,\sigma,\mathbf{m}}^{\pm}(x)\triangleq\begin{pmatrix}\cos(\mathbf{m}x)\\ \gamma_{\mathbf{m}}^{\pm}(\sigma)\sin(\mathbf{m}x)\end{pmatrix}.$$

(ii) The operator $\mathcal{L}_{0,\sigma,\gamma_{\mathbf{m}}^{\pm}(\sigma)}:X_{\mathbf{m}}^s\longrightarrow Y_{\mathbf{m}}^s$ is Fredholm with zero index.

(iii) The transversality condition holds, namely

$$\partial_\gamma\mathcal{L}_{0,\sigma,\gamma}|_{\gamma=\gamma_{\mathbf{m}}^{\pm}(\sigma)}\,[x_{0,\sigma,\mathbf{m}}^{\pm}]\notin R\left(\mathcal{L}_{0,\sigma,\gamma_{\mathbf{m}}^{\pm}(\sigma)}\right).\tag{2.44}$$

Proof. (i) Observe that $\gamma_{\mathbf{m}}^{-}(\sigma)=-\gamma_{\mathbf{m}}^{+}(\sigma)\neq 0$ and that the sequence $\big(\gamma_n^{+}(\sigma)\big)_{n\geqslant 2}$ is strictly increasing. This immediatly prevents spectral collisions and allows to conclude similarly to Proposition 2.3-(i).

(ii) Follows from Proposition 2.2-(ii).

(iii) Proceeding as in the proof of Proposition 2.3-(iii), we get

$$R\left(\mathcal{L}_{c,\sigma,\gamma_{\mathbf{m}^{\pm}(\sigma)}}\right)=\mathtt{span}(y_{0,\sigma,\mathbf{m}}^{\pm})^{\perp_{\langle\cdot,\cdot\rangle}},\qquad y_{0,\sigma,\mathbf{m}}^{\pm}(x)\triangleq\begin{pmatrix}-\gamma_{\mathbf{m}}^{\pm}(\sigma)\sin(\mathbf{m}x)\\ \cos(\mathbf{m}x)\end{pmatrix}.$$

From (2.21), we have

$$\partial_\gamma\mathcal{L}_{0,\sigma,\gamma}|_{\gamma=\gamma_{\mathbf{m}}^{\pm}(\sigma)}=\begin{pmatrix}\frac{1}{2}\partial_x & 0\\ \gamma_{\mathbf{m}}^{\pm}(\sigma)(|D|-2) & \frac{1}{2}\partial_x\end{pmatrix}.$$

Therefore, since $\mathbf{m}\neq 1$ and $\gamma_{\mathbf{m}}^{\pm}(\sigma)\neq 0$,

$$\left\langle\partial_\gamma\mathcal{L}_{c,\sigma,\gamma}|_{\gamma=\gamma_{\mathbf{m}}^{\pm}(\sigma)}[x_{0,\sigma,\mathbf{m}}^{\pm}],y_{0,\sigma,\mathbf{m}}^{\pm}\right\rangle=2\gamma_{\mathbf{m}}^{\pm}(\sigma)\,(\mathbf{m}-1)\neq 0.$$

This implies (2.44) and achieves the proof of Proposition 2.5. □

Proof of Theorem 1.1-(iii). We apply Theorem A.1 together with Lemma 2.1 and Propositions 2.1 and 2.5. □

A Appendix

A.1 Derivation of (1.4) from (1.1)

From (1.3) it is immediate that we can recast (1.1) as an evolutionary equation on the interface only, since the bulk quantities $u^\pm$ and $p^\pm$ can be recovered from the interface evolution of ω and z. Such derivation is quite well-known in the literature (cf. [24, 29]) but we perform here detailed computations for the sake of clarity.

Given two functions $f^\pm : \Omega^\pm(t) \to \mathbb{R}$ we define

$$\llbracket f^\pm \rrbracket \triangleq f^- - f^+.$$

Let us now define the trace of the velocity $u^\pm$ (cf. (1.3)) as

$$v^\pm(t,x) \triangleq u^\pm\big|_{\Gamma(t)}(t,x) = u^\pm\big(t, z(t,x)\big), \qquad (t,x) \in (0,T) \times \mathbb{T}. \tag{A.1}$$

From (1.3) we can compute $v^\pm$ defined in (A.1) as

$$v^\pm(t,x) = \lim_{\epsilon \searrow 0} u^\pm\left(t, z(t,x) \pm \epsilon z_x^\perp(t,x)\right)$$

since $z(t,x) \pm \epsilon z_x^\perp(t,x) \in \Omega^\pm(t)$ for any $x \in \mathbb{T}$, so that the trace of the velocity flow $v^\pm$ can be recasted in terms of z and ω via the *Birkhoff-Rott integral operator*

$$v^\pm = \mathrm{BR}(z)\omega \mp \frac{\omega}{2} \frac{z_x}{|z_x|^2}, \qquad \mathrm{BR}(z)\omega(t,x) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} \frac{(z(t,x) - z(t,y))^\perp}{|z(t,x) - z(t,y)|^2} \omega(t,y)\,\mathrm{d}y. \tag{A.2}$$

From (A.2) we can deduce hence a relation between the vorticity strength and the trace of the velocity valid in $(0,T) \times \mathbb{T}$ (cf. (A.1))

$$\omega = \llbracket v^\pm \rrbracket \cdot z_x. \tag{A.3}$$

Remark A.1. *Notice that from* (A.2) *it is immediate that the normal (to* $\Gamma(t)$*) component of the velocity flow is continuous through the interface.*

The relation (A.3) allows us to express ω in terms of the trace of the velocity flow, hence, to derive the evolution equation for ω by taking the tangential (to $\Gamma(t)$) trace of the first equation of (1.1) onto $\Gamma(t)$. This procedure produces the evolutionary equations for $v^\pm$ which are computed in [29, Eq. (2.2)] and are given by

$$\left(v^\pm \cdot z_x\right)_t - \left(v^\pm \cdot z_t\right)_x + \frac{1}{2}\left(\left|v^\pm\right|^2\right)_x + \left(p^\pm\big|_{\Gamma(t)}\right)_x + (z_2)_x = 0, \tag{A.4}$$

where z_2 is the second component of the parametrization vector z. Using (A.4) we can hence compute the evolution equation for ω defined as in (A.3) which is, using the continuity of the stress tensor among the surface $\Gamma(t)$, i.e. the third equation of (1.1)

$$\omega_t - \left(\llbracket v^\pm \rrbracket \cdot z_t\right)_x + \frac{1}{2}\left(\left\llbracket \left|v^\pm\right|^2 \right\rrbracket\right)_x + \sigma\big(\mathcal{K}(z)\big)_x = 0, \tag{A.5}$$

while we use (A.2) in order to obtain the identity

$$v^+ = v^- - \omega\, \frac{z_x}{|z_x|^2},$$

from which we derive

$$v^+ \cdot z_x = v^- \cdot z_x - \omega, \qquad v^+ \cdot z_t = v^- \cdot z_t - \omega \frac{z_t \cdot z_x}{|z_x|^2}, \qquad \left|v^+\right|^2 = \left|v^-\right|^2 + \frac{\omega^2}{|z_x|^2} - 2\omega\, \frac{v^- \cdot z_x}{|z_x|^2}. \tag{A.6}$$

The relations in (A.6) and (A.2) give us that

$$-\left(\llbracket v^\pm \rrbracket \cdot z_t\right)_x + \frac{1}{2}\left(\left\llbracket \left|v^\pm\right|^2 \right\rrbracket\right)_x = -\frac{1}{2}\left(\frac{\omega^2}{|z_x|^2}\right)_x - \left(\omega\, \frac{(z_t - v^-) \cdot z_x}{|z_x|^2}\right)_x = -\left(\omega\, \frac{(z_t - \mathrm{BR}(z)\omega) \cdot z_x}{|z_x|^2}\right)_x. \tag{A.7}$$

The system (1.1) can thus be recasted as an evolutionary equation on the interface only via the unknowns (ω, z), plugging (A.7) in (A.5), thus obtaining (1.4).

A.2 Derivation of (1.6) from (1.4)

Here we explain how to get the system (1.6) from (1.4). Let us consider a parametrization of $\Gamma(t)$ in the form

$$z(t,x)=R(t,x)e^{\mathrm{i}x},\qquad R(t,x)\triangleq\sqrt{1+2\eta(t,x)}.$$

Straightforward calculations show that

$$\begin{aligned}
z_t(t,x)&=\frac{\eta_t(t,x)}{R(t,x)}e^{\mathrm{i}x},\\
z_x(t,x)&=\left(\frac{\eta_x(t,x)}{R(t,x)}+\mathrm{i}R(t,x)\right)e^{\mathrm{i}x},\\
z_x^{\perp}(t,x)&=\left(\mathrm{i}\frac{\eta_x(t,x)}{R(t,x)}-R(t,x)\right)e^{\mathrm{i}x},\\
z_{xx}(t,x)&=\left(\frac{\eta_{xx}(t,x)}{R(t,x)}-\frac{\eta_x^2(t,x)}{R^3(t,x)}+2\mathrm{i}\frac{\eta_x(t,x)}{R(t,x)}-R(t,x)\right)e^{\mathrm{i}x}.
\end{aligned}\tag{A.8}$$

As a consequence,

$$\begin{aligned}
z_t\cdot z_x^{\perp}&=-\eta_t,\\
z_t\cdot z_x&=\frac{\eta_t\eta_x}{R^2},\\
z_x^{\perp}\cdot z_{xx}&=3\left(\frac{\eta_x}{R}\right)^2-\eta_{xx}+R^2,\\
|z_x|^2&=R^2+\left(\frac{\eta_x}{R}\right)^2.
\end{aligned}\tag{A.9}$$

The last two identities combined with (1.10) give immediately

$$\mathcal{K}(z)=\frac{\eta_{xx}-2\left(\frac{\eta_x}{R}\right)^2}{\left(R^2+\left(\frac{\eta_x}{R}\right)^2\right)^{\frac{3}{2}}}-\left(R^2+\left(\frac{\eta_x}{R}\right)^2\right)^{-\frac{1}{2}}=\mathscr{K}(\eta).\tag{A.10}$$

In addition,

$$\begin{aligned}
\big(z(x)-z(y)\big)\cdot z_x(x)&=\eta_x(x)\left(1-\frac{R(y)}{R(x)}\cos(x-y)\right)+R(x)R(y)\sin(x-y),\\
|z(x)-z(y)|^2&=2\big(1+\eta(x)+\eta(y)-R(x)R(y)\cos(x-y)\big).
\end{aligned}\tag{A.11}$$

Using (A.11) and the notation (1.7), we deduce that

$$\begin{aligned}
\mathrm{BR}(z)\omega\cdot z_x^{\perp}(x)&=\mathrm{p.v.}\int_{\mathbb{T}}\frac{\big(z(x)-z(y)\big)^{\perp}\cdot z_x^{\perp}(x)}{|z(x)-z(y)|^2}\omega(y)\mathrm{d}y\\
&=\mathrm{p.v.}\int_{\mathbb{T}}\frac{\big(z(x)-z(y)\big)\cdot z_x(x)}{|z(x)-z(y)|^2}\omega(y)\mathrm{d}y\\
&=\tfrac{1}{2}\mathscr{H}(\eta)[\omega](x).
\end{aligned}$$

Together with (A.9), the first equation in (1.4) becomes

$$\eta_t=-\tfrac{1}{2}\mathscr{H}(\eta)[\omega].\tag{A.12}$$

Combining (1.4), (A.8) and (A.9), we find

$$z_t\cdot z_x=-\frac{\eta_x}{R^2}\mathrm{BR}(z)\omega\cdot z_x^{\perp}.$$

Thus,

$$\begin{aligned}
\big(z_t-\mathrm{BR}(z)\omega\big)\cdot z_x&=-\mathrm{BR}(z)\omega\cdot\left(\frac{\eta_x}{R^2}z_x^{\perp}+z_x\right)\\
&=-|z_x|^2\mathrm{BR}(z)\omega\cdot\left(\frac{\mathrm{i}e^{\mathrm{i}x}}{R}\right).
\end{aligned}$$

But

$$\big(z(x)-z(y)\big)^{\perp}\cdot\left(\frac{\mathrm{i}e^{\mathrm{i}x}}{R}\right)=1-\frac{R(y)}{R(x)}\cos(x-y).$$

The last computations with (A.11) and (1.8) give

$$\omega\frac{\big(z_t-\mathrm{BR}(z)\omega\big)\cdot z_x}{|z_x|^2}(x)=-\omega(x)\mathrm{p.v.}\int_{\mathbb{T}}\frac{\big(z(x)-z(y)\big)^{\perp}\cdot\Big(\frac{\mathrm{i}e^{\mathrm{i}x}}{R}\Big)}{|z(x)-z(y)|^2}\omega(y)\mathrm{d}y$$
$$=-\tfrac{1}{2}\omega(x)\mathscr{D}_0(\eta)[\omega](x).$$

This together with (1.4) and (A.10) imply

$$\omega_t=\Big(-\tfrac{1}{2}\omega\mathscr{D}_0(\eta)[\omega]-\sigma\mathscr{K}(\eta)\Big)_x.$$

A.3 Crandall-Rabinowitz Theorem

We state here the local bifurcation result obtained in [25] and used in this study to construct our solutions.

Theorem A.1. ***(Crandall-Rabinowitz)***
Let X and Y be two Banach spaces. Let $(p_0,u_0)\in\mathbb{R}\times X$ and U be a neighborhood of (p_0,u_0) in $\mathbb{R}\times X$. Consider a C^1 function $F:U\to Y$ such that

(1) $\forall(p,u_0)\in U,\quad F(p,u_0)=0.$

(2) *The operator $d_uF(p_0,u_0)$ is a Fredholm operator with zero index and such that*

$$\ker\big(d_uF(p_0,u_0)\big)=\mathtt{span}(x_0).$$

(3) *Transversality:*

$$\partial_p d_uF(p_0,u_0)[x_0]\notin R\big(d_uF(p_0,u_0)\big).$$

If we decompose

$$X=\mathtt{span}(x_0)\oplus Z,$$

then there exist two C^1 functions

$$p:(-\epsilon,\epsilon)\to\mathbb{R}\qquad\text{and}\qquad z:(-\epsilon,\epsilon)\to Z,\qquad\text{with}\qquad\epsilon>0,$$

such that

$$p(0)=p_0,\qquad z(0)=0$$

and the set of zeros of F in U is the union of two curves

$$\big\{(p,u)\in U\quad\text{s.t.}\quad F(p,u)=0\big\}=\big\{(p,u_0)\in U\big\}\cup\mathscr{C}_{\mathrm{local}},\qquad\mathscr{C}_{\mathrm{local}}\triangleq\big\{\big(p(\mathtt{s}),u_0+\mathtt{s}x_0+\mathtt{s}z(\mathtt{s})\big),\quad|\mathtt{s}|<\epsilon\big\}.$$

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