

# Singular Contour Dynamics and Paradifferential Reduction

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## Abstract

We develop a direct paradifferential approach to nonlinear graph integral operators arising in contour dynamics. The main methodological result is an abstract structure-preserving parilinearization theorem, valid in arbitrary dimension, for a class of homogeneous kernels whose nonlinear dependence on the graph is expressed through normalized finite differences. The theorem covers both critical principal-value kernels and subcritical locally integrable kernels, gives explicit formulas for the principal symbols, and separates the finite-order symbolic contribution generated by the homogeneous singularity from order-zero far-field terms and arbitrarily smoothing remainders. A structural feature of the reduction is the cancellation of the intermediate symbolic order; at the critical endpoint the construction also preserves the exact skew-adjoint structure of the original singular integral.

As a main application, we consider the three-dimensional two-phase free-boundary Euler equations with constant interfacial background vorticity. We first derive an autonomous contour-dynamics equation in canonical surface variables and then apply the abstract theorem directly to the singular integral operators generated by the Birkhoff–Rott formulation. This yields the complete paradifferential structure of the system without taking the Dirichlet–Neumann reduction as a starting point. After introducing an Alinhac good unknown and diagonalizing the resulting system, we obtain local well-posedness for small Sobolev perturbations in the stable Kelvin–Rayleigh–Taylor regime, uniformly on compact subsets of the stable parameter set.

*Keywords:* Paradifferential calculus, Parilinearization, Singular integral operators, Contour dynamics, Free boundary problems, Water waves, Vortex sheets, Kelvin–Helmholtz instability.

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## 1 Presentation of the problem and main contributions

The analysis of fluid interfaces governed by contour-dynamics formulations leads naturally to nonlinear, nonlocal evolution equations driven by singular integral operators. Such equations arise in a broad range of models in fluid dynamics, including vortex sheets, water waves, surface quasi-geostrophic sharp fronts, porous-media interfaces, and related free-boundary problems. Their analysis has traditionally proceeded along two complementary directions: through the geometric and singular-integral structure of the underlying contour equations, and through paradifferential reductions designed to capture their quasilinear character. To the best of our knowledge, however, a general framework directly connecting nonlinear singular contour-dynamics operators with their paradifferential counterparts, while retaining the structural properties of the original formulation, has not been developed.

The purpose of this work is to provide such a framework. We introduce a direct paralinearization strategy for singular contour-dynamics operators, yielding a robust procedure for reducing nonlinear, nonlocal

interface equations to paradifferential systems while preserving their essential structural features. The strategy is formulated at the level of an abstract class of homogeneous kernels, in arbitrary dimension, whose nonlinear dependence on the interface enters through normalized finite differences of the graph function; it covers both critical principal-value kernels and subcritical locally integrable ones, produces explicit principal symbols, and isolates the finite-order symbolic contribution of the singularity from far-field terms and smoothing remainders. In particular, this approach makes it possible to work directly with the underlying singular integral operators, without first reformulating the problem in terms of a Dirichlet–Neumann operator, as is customary in the analysis of water-wave equations.

Because the reduction is carried out on the kernel itself rather than on a specific model, it applies uniformly to a number of contour-dynamics operators that have so far been treated by model-dependent techniques. In the corresponding small-interface regimes, the singular parts of the circular Kelvin–Helmholtz operators, the local parts of the operators governing  $\alpha$ -SQG sharp fronts for  $\alpha \in (1, 2)$ , and the compensated critical kernels of the three-dimensional Muskat equation all fall within the abstract class, with periodic coefficients, non-integer orders, and compensated finite-difference structures handled by the extensions of the abstract structure-preserving parilinearization theorem. These examples, collected in Remark 1.7, indicate that the framework is not tied to any particular free-boundary problem but reflects a common paradifferential structure shared across contour dynamics.

As a flagship application, we consider the motion of two immiscible, incompressible, inviscid fluids separated by a smooth interface and allowing for interfacial vorticity. This setting contains both the one-phase water-wave regime and the Kelvin–Helmholtz regime as distinguished limiting configurations of the density parameters. While substantial progress has been achieved for the water-wave problem [6, 21, 79, 106, 107], and the two-dimensional Kelvin–Helmholtz problem has a long and well-developed theory [33, 94, 95], the three-dimensional two-phase system with general density ratio and interfacial vorticity remains largely open. In this setting, the main difficulty stems from the singular and nonlocal velocity field induced on the interface by the Birkhoff–Rott integral. We address this problem within the general framework developed here: we obtain a direct parilinearization of the governing singular integral operators, identify the full quasilinear structure of the system, and use this reduction to establish local well-posedness in Sobolev spaces.

**Notation 1.1.** Throughout the manuscript we adopt the following notational conventions to distinguish scalar, vector, and matrix quantities:

- scalar quantities (functions, operators, symbols, etc.) are denoted by lower-case letters (e.g.,  $a, b, c, \dots$ );
- two-coordinate vector quantities (functions, operators, symbols, etc.) are denoted by upper-case letters (e.g.,  $A, B, C, \dots$ ) and shall be written in components as  $A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ;
- three-coordinate vector quantities (functions, operators, symbols, etc.) are denoted by lower-case bold letters (e.g.,  $\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots$ ), such vectors shall be decomposed as  $\mathbf{a} = \begin{bmatrix} A \\ a_3 \end{bmatrix}$  relatively to the first two and last component;
- two-by-two matrix quantities (functions, operators, symbols, etc.) are denoted by upper-case bold letters (e.g.,  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ );
- two-by-three matrix quantities (functions, operators, symbols, etc.) are denoted by Fraktur upper-case letters (e.g.,  $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ ).

The principal exceptions to this rule are the spatial variable  $x \in \mathbb{R}^2$  and the frequency variable  $\xi \in \mathbb{R}^2$ , which we keep in lower case to avoid excessively heavy notation. This convention leads to some nonstandard choices, such as denoting the Dirichlet–Neumann operator by  $\mathcal{G}$  instead of the usual  $G$ . We believe, however, that the improved clarity afforded by this unified notation system justifies these departures from standard conventions.

We emphasise that this convention is restricted to the unknowns and operators that compose the system under study, and shall be especially useful and handy in Section 5; standard mathematical objects such as

Banach and Hilbert spaces, norms, inner products, and other common objects of functional analysis retain their usual notation.

We denote generic elements in the plane and space as  $x := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$  and  $\mathbf{x} := \begin{bmatrix} x \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ . The corresponding gradient operators in  $\mathbb{R}^2$  and  $\mathbb{R}^3$  are defined as  $\nabla := \begin{bmatrix} \partial_{x_1} \\ \partial_{x_2} \end{bmatrix}$  and  $\nabla_{\mathbf{x}} := \begin{bmatrix} \nabla \\ \partial_{x_3} \end{bmatrix}$ . We define

$$\nabla^\perp := J\nabla, \quad J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Given a finite time  $T > 0$ , we shall work on the finite time interval  $I := (-T, T)$ . The interface separating both fluids is denoted  $\Gamma(t)$  and is parametrized by  $\boldsymbol{\gamma}(t, x)$ , namely

$$\Gamma(t) := \{\boldsymbol{\gamma}(t, x) \in \mathbb{R}^3 \mid x \in \mathbb{R}^2\}.$$

In what follows, we shall make the assumption that the interface  $\Gamma(t)$  is a graph on  $\mathbb{R}^2$  parametrized as

$$(t, x) \mapsto \boldsymbol{\gamma}(t, x) = (x, \eta(t, x)) \in \mathcal{C}^1(I \times \mathbb{R}^2; \mathbb{R}^3). \quad (1.1)$$

The elevation function  $\eta : I \times \mathbb{R}^2 \rightarrow \mathbb{R}$  is one of the main quantities of interest in our study. It separates  $\mathbb{R}^3$  into two open and simply-connected sets

$$\begin{aligned} \Omega^+(t) &:= \{(t, \mathbf{x}) = (t, x, x_3) \in I \times \mathbb{R}^3 \mid x_3 > \eta(x)\}, \\ \Omega^-(t) &:= \{(t, \mathbf{x}) = (t, x, x_3) \in I \times \mathbb{R}^3 \mid x_3 < \eta(x)\}. \end{aligned}$$

We assume that the domains  $\Omega^\pm(t)$  are filled with inviscid, immiscible fluids, with constant densities

$$\rho^- \geq \rho^+ \geq 0.$$

The associated velocity and pressure fields are denoted  $\mathbf{u}^\pm$  and  $p^\pm$ , respectively. The fluid is considered to be incompressible in both phases, i.e.

$$\operatorname{div}_{\mathbf{x}} \mathbf{u}^\pm := \nabla_{\mathbf{x}} \cdot \mathbf{u}^\pm = 0, \quad \text{in } I \times \Omega^\pm(t). \quad (1.2)$$

Here and in the sequel, the notation  $u \cdot v := u^\top v$  stands for the usual Euclidean scalar product. The evolution of the fluids is described by the incompressible non-homogeneous Euler equations

$$\rho^\pm (\partial_t \mathbf{u}^\pm + (\mathbf{u}^\pm \cdot \nabla_{\mathbf{x}}) \mathbf{u}^\pm) = -\nabla_{\mathbf{x}} p^\pm - g \rho^\pm \mathbf{e}_3, \quad \text{in } I \times \Omega^\pm(t). \quad (1.3)$$

where  $g \in \mathbb{R}$  is the gravitational constant and  $\mathbf{e}_3 := \nabla_{\mathbf{x}} x_3$ . Since the fluids are immiscible, the interface  $\Gamma(t)$  is transported by the normal velocity component. Specifically, we define the velocity field  $\mathbf{u}^\pm$  by continuous extension to the interface, acknowledging that while the normal component is continuous, the tangential component may exhibit a jump across  $\Gamma(t)$ . Thus, the interface evolves according to the transport equation:

$$\left( \partial_t \boldsymbol{\gamma} - \mathbf{u}^\pm|_{\Gamma(t)} \right) \cdot \mathbf{v} = 0, \quad \mathbf{v} = \mathbf{v}(\boldsymbol{\gamma}) := \partial_{x_1} \boldsymbol{\gamma} \times \partial_{x_2} \boldsymbol{\gamma} = \begin{bmatrix} -\nabla \eta \\ 1 \end{bmatrix}. \quad (1.4)$$

Given a vector field  $\omega^\pm$  defined on  $\Omega^\pm(t)$  with trace  $\omega^\pm$  on  $\Gamma(t)$ , we denote

$$\llbracket \omega^\pm \rrbracket := \omega^- - \omega^+, \quad \langle\langle \omega^\pm \rangle\rangle := \frac{\omega^- + \omega^+}{2}.$$

Moreover, we define the *Atwood number*

$$\mathfrak{a} := \mathfrak{a}(\rho^\pm) := \frac{\llbracket \rho^\pm \rrbracket}{\langle\langle \rho^\pm \rangle\rangle} = 2 \frac{\rho^- - \rho^+}{\rho^- + \rho^+} \in [0, 2]. \quad (1.5)$$

The boundary condition for the pressure is given by the equation

$$\llbracket p^\pm|_{\Gamma(t)} \rrbracket = \sigma \kappa(\boldsymbol{\gamma}), \quad (1.6)$$

where  $\sigma \geq 0$  is the surface tension coefficient and  $\kappa(\boldsymbol{\gamma})$  is the mean curvature of the surface  $\boldsymbol{\gamma}$  which, under the assumption Eq. (1.1), becomes

$$\kappa(\boldsymbol{\gamma}) \Big|_{\boldsymbol{\gamma}(x)=(x,\eta(x))} = -\nabla_x \cdot \mathcal{K}(\nabla \eta), \quad \mathcal{K}(\nabla \eta) := \frac{\nabla_x \eta}{\sqrt{1 + |\nabla_x \eta|^2}}. \quad (1.7)$$

We complement the system identified by Eqs. (1.2) to (1.4) and (1.7) by fixing a constant background sheet strength  $\bar{\Omega} \in \mathbb{R}^2$  and working in the centered Galilean frame in which the common horizontal velocity at infinity vanishes, i.e., we impose

$$\mathbf{u}^\pm(t, x, z) \xrightarrow{|z| \rightarrow \infty} \pm \frac{1}{2} \begin{bmatrix} \bar{\Omega}^\perp \\ 0 \end{bmatrix}, \quad (1.8)$$

and assume that the perturbations of these constant far-field states have the Sobolev regularity specified below. We also assume that

$$\nabla_x \times \mathbf{u}^\pm = 0 \quad \text{in} \quad \Omega^- \cup \Omega^+.$$

The system that we finally obtain, and that we shall study along the manuscript, is the following

$$\begin{cases} \rho^\pm (\partial_t \mathbf{u}^\pm + (\mathbf{u}^\pm \cdot \nabla_x) \mathbf{u}^\pm) = -\nabla_x p^\pm - g \rho^\pm \mathbf{e}_3, & \text{in } I \times \Omega^\pm(t), \\ (\partial_t \boldsymbol{\gamma} - \mathbf{u}^\pm|_{\Gamma(t)}) \cdot \boldsymbol{\nu} = 0, & \text{in } I \times \Gamma(t), \\ \llbracket p^\pm|_{\Gamma(t)} \rrbracket = \sigma \kappa(\boldsymbol{\gamma}), & \text{in } I \times \Gamma(t), \\ \nabla_x \cdot \mathbf{u}^\pm = 0, & \text{in } I \times \Omega^\pm(t), \\ \nabla_x \times \mathbf{u}^\pm = 0, & \text{in } I \times \Omega^\pm(t). \end{cases} \quad (1.9)$$

It is well known that the irrotationality assumption in Eq. (1.9) implies that there exists  $\phi^\pm : I \times \Omega^\pm(t) \rightarrow \mathbb{R}$  such that

$$\nabla_x \phi^\pm = \mathbf{u}^\pm, \quad \text{in } I \times \Omega^\pm(t). \quad (1.10)$$

**Remark 1.2.** Let us mention that the two limiting cases  $\alpha = 0$  and  $\alpha = 2$  correspond to the Kelvin-Helmholtz and deep gravity-capillary 3D water-waves systems, respectively.  $\blacklozenge$

## 1.1 Functional setting

Given a function  $u \in L^1(\mathbb{R}^2; \mathbb{C})$ , we denote with

$$\hat{u}(\xi) := \frac{1}{2\pi} \int_{\mathbb{R}^2} u(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^2,$$

so that, if  $\hat{u} \in L^1(\mathbb{R}^2; \mathbb{C})$ , we can recover  $u$  from  $\hat{u}$  via the classical inversion formula

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{u}(\xi) e^{ix \cdot \xi} d\xi.$$

It is immediate that  $u$  is real valued if and only if  $\hat{u}(\xi) = \overline{\hat{u}(-\xi)}$ . We define, for any  $s \in \mathbb{R}$  the non-homogeneous Sobolev space  $H^s := H^s(\mathbb{R}^2; \mathbb{C})$  as the closure of  $C_c^\infty(\mathbb{R}^2; \mathbb{C})$  functions w.r.t. the norm

$$\|u\|_{H^s} := \sqrt{\int_{\mathbb{R}^2} \langle \xi \rangle^{2s} |\hat{u}(\xi)|^2 d\xi}, \quad \langle \xi \rangle := \sqrt{1 + |\xi|^2};$$

We use the notation

$$u \otimes v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \otimes \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := uv^\top = \begin{bmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{bmatrix}.$$

**Notation 1.3.** Let  $T, r > 0$  and  $s \in \mathbb{R}$ .

1. we denote with  $I := [-T, T]$ ;

2. for  $u \in H^s$ , we set

$$\|u\|_s := \|u\|_{H^s};$$

3. for  $u \in \mathcal{C}(I; H^s)$  and  $\vec{U}_p := (u_1, \dots, u_p)$ , we set

$$\|u\|_{s,I} := \sup_{t \in I} \|u(t)\|_s, \quad \|\vec{U}_p\|_{s,I} := \max_{1 \leq j \leq p} \|u_j\|_{s,I};$$

4. we denote with

$$B_s(I; r) := B_X(0, r), \quad X = \mathcal{C}(I; H^s(\mathbb{R}^2; \mathbb{C})).$$

5. we denote with

$$B_{s,\mathbb{R}}(I; r) := B_X(0, r), \quad X = \mathcal{C}(I; H^s(\mathbb{R}^2; \mathbb{R})).$$

All symbol, kernel, and smoothing-operator classes below are scalar. Their statements and estimates apply componentwise to finite-dimensional vectors and matrices. A separately estimated argument written in brackets does not define a new class: it denotes the evaluation of a continuous linear map with values in the corresponding Fréchet space, according to Notation 5.1. Such an argument is not included among the small backgrounds and is not required to be small. Time is suppressed in pointwise formulas, and all estimates are uniform in  $t \in I$ .

## 1.2 Main results

We organize the main results around the three steps that connect the free-boundary Euler system to the Cauchy theory: the autonomous canonical contour-dynamics formulation; an abstract parilinearization theorem for graph integral operators, together with its application to the canonical CDE; and the local well-posedness theorem.

### 1.2.1 The contour-dynamics formulation

The first step is to eliminate the bulk variables and close the evolution on the interface. At the level of contour dynamics, one could already obtain a closed system by working with the elevation  $\eta$  and the surface vorticity  $\Omega$ . We do not use these variables, however, since the natural Hamiltonian coordinates for the two-fluid problem are the elevation and the density-weighted trace potential  $\psi$  defined by

$$\psi := \frac{\llbracket \rho^\pm \psi^\pm \rrbracket}{\llbracket \rho^\pm \rrbracket}, \quad \psi^\pm := \phi^\pm|_{\Gamma(t)}. \quad (1.11)$$

Passing from  $\Omega$  to  $\psi$  therefore does more than close the equations in scalar surface variables: it puts the contour dynamics directly in the canonical coordinates which will later be compatible with the Hamiltonian symmetrization and the energy estimates.

Since the prescribed background  $\bar{\Omega} \in \mathbb{R}^2$  is represented by an affine, and therefore non-Sobolev, potential, we further separate it by setting

$$\bar{\psi}(x) := x^\perp \cdot \bar{\Omega}, \quad \nabla^\perp \bar{\psi} = \bar{\Omega}, \quad \tilde{\psi} := \psi - \bar{\psi}. \quad (1.12)$$

The Bernoulli gauge is fixed as in Eq. (2.21), so that the flat configuration  $(\eta, \tilde{\psi}) = (0, 0)$  is stationary. The resulting reduction is the following.

**Theorem 1.4** (The vortex-sheet equations as an autonomous canonical Contour Dynamic Equation (CDE)).  
Let  $\sigma, \rho^\pm \geq 0$  with  $\langle \rho^\pm \rangle > 0$ , let  $\mathfrak{g} \in \mathbb{R}$ , and let  $\bar{\Omega} \in \mathbb{R}^2$ . There exist  $s_0 > 0$  and  $r_0 > 0$  such that, in the class of graph configurations for which  $\eta$  and  $\psi$  are defined by Eqs. (1.1) and (1.11),  $\tilde{\psi}$  is given by Eq. (1.12), and  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  the free-boundary Euler system Eqs. (1.8) and (1.9) is equivalent to

$$\begin{cases} \eta_t = g^\star(\eta) [\bar{\Omega} [\bar{\Omega}](\eta)] + g(\mathfrak{a}|\eta) [\tilde{\psi}], \\ \tilde{\psi}_t = \tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}). \end{cases} \quad (1.13a)$$

$$(1.13b)$$

The objects appearing in Eq. (1.13) are defined, in their order of appearance, as follows.

- The operator  $g^\star(\eta)$  is the normal-velocity operator in vorticity variables defined by

$$g^\star(\eta) \Omega := \mathbf{v}^\star[\Omega](\eta) \cdot \begin{bmatrix} \nabla \eta \\ -1 \end{bmatrix}, \quad \mathbf{v}^\star[\Omega](\eta) := \begin{bmatrix} V^\star[\Omega](\eta) \\ \nu_3^\star[\Omega](\eta) \end{bmatrix} = \begin{bmatrix} (\mathcal{B}_0(\eta) [\nabla \eta \cdot \Omega] - q(\eta) \Omega)^\perp \\ -\mathcal{B}_0(\eta)^\perp \cdot \Omega \end{bmatrix}, \quad (1.14)$$

where the singular integral operators  $\mathcal{B}_0$  and  $q$  are given by

$$\mathcal{B}_0(\eta) [\omega](x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{z}{(|z|^2 + (\delta_z \eta(x))^2)^{\frac{3}{2}}} \omega(x-z) dz, \quad (1.15a)$$

$$q(\eta) [\omega](x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\delta_z \eta(x)}{(|z|^2 + (\delta_z \eta(x))^2)^{\frac{3}{2}}} \omega(x-z) dz, \quad (1.15b)$$

with  $\delta_z \eta(x) := \eta(x) - \eta(x-z)$ .

- The canonical background vorticity is

$$\bar{\Omega} [\bar{\Omega}](\eta) := S(\mathfrak{a}|\eta) \tilde{\psi},$$

where the potential-to-vorticity map  $S(\mathfrak{a}|\eta)$  is defined by

$$\begin{aligned} S(\mathfrak{a}|\eta) \psi &:= \nabla^\perp (1 + \mathfrak{a} \tau_0(\eta))^{-1} \psi, \\ \tau_0(\eta) \psi &:= \Delta^{-1} \nabla^\perp \cdot (\mathcal{B}_0(\eta) [\nabla \eta \cdot \nabla^\perp \psi] - q(\eta) \nabla^\perp \psi + (\mathcal{B}_0(\eta) \cdot \nabla \psi) \nabla^\perp \eta), \end{aligned} \quad (1.16)$$

where the operators  $\mathcal{B}_0$  and  $q$  are given in Eqs. (1.15a) and (1.15b).

- The generalized Dirichlet–Neumann operator in canonical variables is

$$g(\mathfrak{a}|\eta) := g^\star(\eta) \circ S(\mathfrak{a}|\eta),$$

with  $g^\star$  and  $S$  as in Eqs. (1.14) and (1.16).

- The function  $\tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi})$  in Eq. (1.13b) is the shifted Bernoulli function defined by

$$\tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}) := \pi^\star(\rho^\pm|\eta, \bar{\Omega} [\bar{\Omega}](\eta) + S(\mathfrak{a}|\eta) \tilde{\psi}) - c_{\bar{\Omega}},$$

where

$$\begin{aligned} \pi^\star(\rho^\pm|\eta, \Omega) &:= \frac{\sigma}{\langle \rho^\pm \rangle} \nabla \cdot \mathcal{K}(\nabla \eta) - \mathfrak{g} \mathfrak{a} \eta - \Omega^\perp \cdot V^\star[\Omega](\eta) \\ &\quad - \frac{\mathfrak{a}}{4} \left[ 2 |V^\star[\Omega](\eta)|^2 - 2 |\nu_3^\star[\Omega](\eta)|^2 + 4 \nu_3^\star[\Omega](\eta) V^\star[\Omega](\eta) \cdot \nabla \eta + \frac{1}{2} \frac{|\Omega|^2 + (\Omega \cdot \nabla \eta)^2}{1 + |\nabla \eta|^2} \right], \end{aligned}$$

where  $\mathcal{K}$ ,  $V^\star$  and  $\nu_3^\star$  are given in Eqs. (1.7) and (1.14). The Bernoulli gauge

$$c_{\bar{\Omega}} := -\frac{\mathfrak{a}}{8} |\bar{\Omega}|^2$$

gives  $\tilde{\pi}(\rho^\pm, \bar{\Omega}|0, 0) = 0$ .

### 1.2.2 Paradifferential reduction of the CDE

The second, and main technical, result of the manuscript consists in the paradifferential reduction of the CDE Eq. (1.13). This is achieved via a two-steps process:

- i we prove an abstract theorem (Theorem 1.5) that allows us to parilinearize arbitrary nonlinear convolution kernel operators in any dimension  $d \geq 1$  with suitable integrability hypothesis and;
- ii we apply this abstract result to parilinearize the terms in the CDE for the 3D vortex-sheet equation that resists classical tools of paradifferential reduction, mainly the singular integral operators  $\mathcal{B}_0$  and  $q$  in Eqs. (1.15a) and (1.15b). Next, we perform a careful analysis of every term involved in the reduction in order to obtain the final paradifferential system.

At this stage we deliberately avoid introducing the full paradifferential framework needed for a completely quantitative formulation, including the precise symbol and smoothing-operator classes and their associated tame estimates. The statement below should therefore be regarded as a concise preview of the parilinearization mechanism, whose notation and precise meaning are developed in Section 4. Throughout this preview, the Sobolev estimates are understood for every  $s \geq s_0$ , where  $s_0$  is a sufficiently large regularity threshold whose value is not made explicit. The techniques developed in the manuscript would in principle allow one to track an admissible value of  $s_0$ . We have chosen not to do so, since this would require a substantial amount of additional computations and low-order estimates without introducing any new conceptual ingredient, while considerably burdening the exposition.

**Parilinearization of abstract nonlinear convolution operators in arbitrary dimensions.** This is one of the main methodological contributions of the paper. We consider  $d \in \mathbb{N}^*$  and graph integral operators on  $\mathbb{R}^d$  of the schematic form

$$t^{(n)}(\eta)[\omega](x) = \int_{\mathbb{R}^d} \frac{h(\hat{z})}{|z|^n} k\left(\frac{\eta(x) - \eta(x-z)}{|z|}\right) \omega(x-z) dz, \quad 0 < n \leq d, \quad \hat{z} := \frac{z}{|z|}, \quad (1.17)$$

where the functions  $h$  and  $k$  are smooth (cf. Definition 5.4 for the precise regularity requirements) and the integral is understood in the principal-value sense in the critical case  $n = d$  whereas in the subcritical case the convolution kernel is locally integrable around zero. Thus the singular or weakly singular kernel depends nonlinearly on finite differences of the graph itself. This interaction between the  $z$ -singularity and the nonlinear geometry prevents a direct application of standard parilinearization techniques.

Rather than analyzing such operators on a case-by-case basis, we develop an abstract structure-preserving parilinearization theorem which generalizes the mechanisms introduced in [24, 85]. The theorem works in arbitrary dimension, covers both the critical and subcritical regimes in Eq. (1.17), and captures the structural cancellation of the intermediate symbolic order. This is one of the main technical innovations of the paper and provides the operator-level foundation for the paradifferential reduction of the vortex-sheet system. The proof is correspondingly divided into two parts: the critical case is established in Section 5.1, while the subcritical case is proved in Appendix B, here we present a statement that is common to both critical and subcritical case.

**Theorem 1.5** (Critical and subcritical parilinearization of graph kernel operators). *Let  $d \geq 1$ ,  $0 < n \leq d$ , and let  $(h, k)$  be an admissible homogeneous pair of order  $-n$  and vanishing order  $\ell \in \mathbb{N}^*$  in the sense of Definition 5.4. Set*

$$\beta := d - n \geq 0.$$

*When  $\beta > 0$ , assume in addition Eq. (B.1). There exists  $\varrho_0 \geq 0$  such that, for every  $\varrho \geq \varrho_0$ , there exist  $s_0 > d/2 + 1$ ,  $\mu \geq 0$ , and  $r_0 > 0$  such that, for every  $s \geq s_0$  and every*

$$\eta \in H^{s+1+\mu}(\mathbb{R}^d; \mathbb{R}), \quad \omega \in H^{s+\mu}(\mathbb{R}^d; \mathbb{R}), \quad \|\eta\|_{H^{s_0+1+\mu}} < r_0,$$

*the graph operator in Eq. (1.17) satisfies*

$$t^{(n)}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(A_{1,-\beta}^{h,k}[\omega](\eta; x, \xi) + \left[\begin{smallmatrix} a_{[0]}[\omega](\eta; x, \xi) \\ a_{[-\beta-2]}(\eta; x, \xi) \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} \eta \\ \omega \end{smallmatrix}\right] + R_{[\ell]}^{\text{lin}}(\eta, \omega) \cdot \left[\begin{smallmatrix} \eta \\ \omega \end{smallmatrix}\right]\right). \quad (1.18)$$

*The objects in Eq. (1.18) are as follows.*

- The principal vector symbol is

$$A_{1,-\beta}^{h,k}[\omega](\eta; x, \xi) := \begin{bmatrix} \omega(x) a_1^{h,k;\eta}(\nabla\eta(x), \xi) \\ a_{-\beta}^{h,k;\omega}(\nabla\eta(x), \xi) \end{bmatrix}.$$

Its components have the following properties.

- The transported symbol  $a_1^{h,k;\eta}(\nabla\eta, \xi) \in \Gamma_{\geq \ell-1}^1$  is given explicitly by

$$a_1^{h,k;\eta}(p, \xi) := \begin{cases} \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\vartheta \cdot \xi| h(\vartheta) k'(p \cdot \vartheta) d\sigma(\vartheta), & \beta = 0, \\ \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n+1}} k'(p \cdot \hat{z}) (1 - e^{-iz \cdot \xi}) dz, & \beta > 0, \end{cases}$$

where  $\chi$  is the cut-off function introduced in Eq. (4.7).

- The quasilinear symbol satisfies

$$a_{-\beta}^{h,k;\omega}(\nabla\eta, \xi) - a_{-\beta}^{h,k;\omega}(0, \xi) \in \Gamma_{\geq \ell}^{-\beta}$$

and is given explicitly by

$$a_{-\beta}^{h,k;\omega}(p, \xi) := \begin{cases} -\frac{i\pi}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\vartheta \cdot \xi) h(\vartheta) k(p \cdot \vartheta) d\sigma(\vartheta), & \beta = 0, \\ \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} k(p \cdot \hat{z}) e^{-iz \cdot \xi} dz, & \beta > 0. \end{cases}$$

- The lower vector symbol satisfies

$$a_{[0]}[\cdot] \in \mathcal{L}(H^{s_0}; \Gamma^0), \quad a_{[-\beta-2]} \in \Gamma_{\geq \ell}^{-\beta-2}.$$

When  $\beta = 0$ , the second component can be chosen in the form

$$a_{[-2]}(\eta; x, \xi) := i a_{[-2]}^{\mathbb{R};\omega}(\eta; x, \xi), \quad a_{[-2]}^{\mathbb{R};\omega} \in \Gamma_{\geq \ell}^{-2},$$

where  $a_{[-2]}^{\mathbb{R};\omega}$  is real-valued and odd in  $\xi$ . In particular, there is no independent symbol of order  $-\beta - 1$ .

- $R_{[\ell]}^{\text{lin}}(\eta, \omega)$  is a generic real-to-real linear smoothing remainder in the sense of Notation 5.1; its first column is the evaluation of a map in  $\mathcal{L}(H^{s_0}; \mathcal{R}^{\ell})$ , while its second column belongs to  $\mathcal{R}_{\geq \ell}^{\ell}$ .

When  $\beta = 0$ , the operator is exactly skew-adjoint,

$$a_{-\beta}^{h,k;\omega} = i a_0^{h,k;\omega},$$

with  $a_0^{h,k;\omega}$  real-valued and odd in  $\xi$ , and its complete nonsmoothing quasilinear symbol acting on  $\omega$  is

$$i \left( a_0^{h,k;\omega}(\nabla\eta(x), \xi) + a_{[-2]}^{\mathbb{R};\omega}(\eta; x, \xi) \right),$$

hence is purely imaginary and odd in  $\xi$ . The two principal components satisfy Eq. (5.23). The complete critical and subcritical statements are respectively Theorems 5.7 and B.1.

**Remark 1.6** (Scope of the model class). The class in Eq. (1.17) is intended to capture the basic singular structure of the contour-dynamics operators arising in applications, rather than to provide an exhaustive description of all possible kernels and operators. We record three extensions which are not needed for the applications developed in the present paper.

- i *Non-homogeneous kernels.* Exact homogeneity of the convolution kernel is not essential. Suppose, more generally, that a kernel  $g$  admits near  $z = 0$  a finite polyhomogeneous expansion of the form

$$g(z) = \sum_{j=0}^J \frac{h_j(\hat{z})}{|z|^{n-j}} + g_{\text{rem}}(z).$$

Each homogeneous component may be treated separately by Theorem 1.5. The expansion is continued only until the localized remainder  $\chi g_{\text{rem}}$  has sufficiently many powers of  $z$  at the origin to belong to the locally integrable kernel classes of Proposition 4.35. At that point the remaining nonlinear dependence is reduced by the usual Bony decomposition and Proposition 4.35, while the part supported away from the origin contributes an order-zero paraproduct modulo arbitrarily smoothing remainders by Proposition 4.37; cf. also Lemma 5.9. Consequently, only finitely many homogeneous components of the kernel contribute to any prescribed finite portion of the paradifferential expansion.

- ii *Periodic interfaces.* The same mechanism applies on  $\mathbb{T}^d$ , after replacing the Euclidean paradifferential calculus by its periodic counterpart. The singular analysis is entirely local near the diagonal  $z = 0$  and is therefore unchanged. In fact, the global part is simpler on the torus: after localizing the singularity, the complementary geometric kernel is smooth. Its graph dependence contributes an order-zero paraproduct modulo smoothing remainders. Thus no far-field decay assumptions, and no Euclidean tail analysis at  $|z| \rightarrow +\infty$ , are required, as can be observed for instance in the works [24, 29, 85].
- iii *Higher-order contour-dynamics operators.* The restriction  $n \leq d$  in Eq. (1.17) is likewise not intrinsic to the parilinearization mechanism. Higher-order contour-dynamics equations naturally lead to compensated singular integrals of the form

$$\int_{\mathbb{R}^d} \frac{h(\hat{z})}{|z|^n} k(\Delta_z \eta(x)) \delta_z \omega(x) dz, \quad n \in (d, d+1)$$

and, for more singular kernels, to the corresponding expressions containing higher finite differences  $\delta_z^m \omega$ . The finite differences provide precisely the cancellations required to compensate singularities with  $n > d$ ; for instance, an  $m$ -th order finite difference gives  $m$  powers of  $z$  at the origin, so that the natural locally integrable range becomes  $n < d + m$ , with the corresponding parity or moment cancellation required at the endpoints.

We do not formulate a separate abstract theorem for these operators here. Nevertheless, repeating the Bony decomposition, the homogeneous kernel analysis, and the compensated kernel-to-symbol reductions developed above yields their parilinearization by the same procedure. In particular, increasing the differential order of the CDE changes the homogeneity of the resulting symbols but not the underlying parilinearization mechanism.

- iv *Finite regularity of the kernel function  $k$ .* Real-analyticity of  $k$  in Definition 5.4 is used only in the proof of Lemma 4.45, to sum the convergent series Eq. (4.72) by Cauchy-type estimates. Since Definition 4.26 only demands, at each fixed order  $N$  of  $z$ -differentiation, a tame bound Eq. (4.15) with a loss  $\mu_N^k$  allowed to grow with  $N$ , this convergent expansion may be replaced, order by order, by the Taylor formula with integral remainder

$$k(X) = \sum_{j=0}^N \frac{k^{(j)}(0)}{j!} X^j + \frac{X^{N+1}}{N!} \int_0^1 (1-t)^N k^{(N+1)}(tX) dt,$$

estimating the remainder exactly as  $k_{[2]}$  is estimated in the proof of Lemma 4.45 the same mechanism already used at order two in Eq. (5.39). Hence  $k \in \mathcal{C}^\infty((-r_k, r_k); \mathbb{R})$  suffices in place of  $k \in \mathcal{C}^\omega((-r_k, r_k); \mathbb{R})$ ; the vanishing-order hypothesis Eq. (5.14) is unaffected, since it fixes only the first  $\ell$  Taylor coefficients of  $k$ .

Thus Eq. (1.17) isolates the basic homogeneous Euclidean model in which the method can be stated most transparently, while the procedure itself is stable under the extensions described above.

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**Remark 1.7** (Representative contour-dynamics applications). In the corresponding small-interface regimes, the mechanism of Theorem 1.5, together with the extensions in Remark 1.6, applies to the following contour-dynamics operators. In each case, we use the notations of the corresponding papers [15, 24, 53, 85].

- *Circular Kelvin–Helmholtz*. With  $q_{\text{KH}} := \frac{1}{2} \log(1 + 2\eta)$ , the singular parts of  $\mathcal{H}_0$  and  $(1 + 2\eta) \mathcal{D}_0 - \Pi_0$  are the periodic critical models associated respectively with

$$(h, k, \ell) = \left( \vartheta/\pi, (1 + X^2)^{-1}, 2 \right), \quad (h, k, \ell) = \left( 1/\pi, X(1 + X^2)^{-1}, 1 \right).$$

The remaining periodic and geometric corrections start at quadratic order in the chord variable and are therefore lower order [85].

- *$\alpha$ -SQG sharp fronts*. For  $\alpha \in (1, 2)$ , after the analogous logarithmic change and the extraction of one periodic chord through  $\delta_z f = 2 \sin(z/2) \Delta_z f$ , the local parts of  $\mathcal{I}$  and  $\mathcal{J}$  have

$$n = \alpha - 1, \quad \beta = 2 - \alpha, \quad \ell_{\mathcal{I}} = 1, \quad \ell_{\mathcal{J}} = 2.$$

Their  $z$ -dependent periodic coefficients are handled by items (i)–(ii) of Remark 1.6 [24].

- *Three-dimensional Muskat*. Up to the physical prefactor, the graph equation is a sum over  $j \in \{1, 2\}$  of compensated critical kernels with

$$d = n = 2, \quad h_j(\vartheta) = \vartheta_j, \quad k(X) = (1 + X^2)^{-3/2}, \quad \ell = 2, \quad \omega = \partial_j f,$$

in which  $\omega(x - z)$  is replaced by  $\delta_z \omega(x)$ . This is the lower-endpoint case of the compensated finite-difference mechanism described in item (iii) of Remark 1.6 [15, 53].

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The operators  $\mathcal{B}_0(\eta)$  and  $q(\eta)$  in Eqs. (1.15a) and (1.15b) are the  $d = 2$  critical instances, with vanishing orders  $\ell = 2$  and  $\ell = 1$ , respectively; see Lemma 5.13 and Propositions 5.14 and 5.15. They provide the singular-integral input for the application to the canonical CDE.

**Paralinearization of the 3D Vortex-Sheet CDE.** The second part consists in the application of Theorem 1.5 to the 3D Vortex-Sheet CDE, the complete statement is Theorem 5.2.

**Theorem 1.8** (Conditional paradifferential reduction). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle\langle \rho^\pm \rangle\rangle > 0$ ,  $\mathfrak{g} \in \mathbb{R}$ , and  $\bar{\Omega} \in \mathbb{R}^2$ . There exists  $\rho_0 > 0$  such that, for every  $\rho \geq \rho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  with the following property. If*

$$\eta \in B_{s_0, \mathbb{R}}(I; r_0) \quad \tilde{\psi} \in \mathcal{C}(I; H^s(\mathbb{R}^2; \mathbb{R}))$$

*solve the canonical CDE Eq. (1.13), then, setting*

$$Y := \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix},$$

*one has*

$$\begin{aligned} \partial_t Y = & \text{Op}^{\text{BW}} \left( \mathbf{Q}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x, \xi) + \mathbf{B}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x, \xi) + i(\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x) \cdot \xi) \text{Id}_{\mathbb{C}^2} \right) Y \\ & + \text{Op}^{\text{BW}} \left( \begin{bmatrix} A_{[0]}^\eta(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x, \xi) & A_{[-1]}^\psi(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x, \xi) \end{bmatrix} \right) Y + \mathbf{R}_{[\ell]}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}) Y. \end{aligned} \quad (1.19)$$

*The objects in Eq. (1.19) have the following structure, in their order of appearance:*

- *The quasilinear symbol  $\mathbf{Q} \in (\Gamma_{\geq 0}^2)^{2 \times 2}$  is defined by*

$$\mathbf{Q} := \begin{bmatrix} 0 & \lambda_1 + \mathfrak{a}\lambda_0 \\ -\mathfrak{g}\mathfrak{a} + a_2^{\pi_0, \eta} + \lambda_1 q_0^{\tilde{\pi}} & 0 \end{bmatrix}. \quad (1.20)$$

*Its constituent symbols are defined explicitly in Eqs. (5.4) to (5.7). In particular,*

$$\mathbf{Q}(\mathfrak{a}, \bar{\Omega} | 0, 0; \xi) = \begin{bmatrix} 0 & \frac{1}{2} |\xi| \\ q(\mathfrak{a}, \bar{\Omega} | \xi) & 0 \end{bmatrix},$$

*where*

$$q(\mathfrak{a}, \bar{\Omega} | \xi) := -\mathfrak{g}\mathfrak{a} - \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2 + \frac{4 - \mathfrak{a}^2}{8} \frac{(\bar{\Omega}^\perp \cdot \xi)^2}{|\xi|}, \quad \xi \in \mathbb{R}^2 \setminus \{0\}. \quad (1.21)$$

- The order-one symbol  $\mathbf{B} \in (\Gamma_{\geq 1}^1)^{2 \times 2}$  has the Alinhac block form

$$\mathbf{B} := - \begin{bmatrix} -b_{\text{can}} & 0 \\ b_{\text{can}}^2 & b_{\text{can}} \end{bmatrix} \lambda_1. \quad (1.22)$$

The coefficient  $b_{\text{can}}$  is defined in Eq. (5.10).

- The transport velocity  $\mathcal{V}_{\text{can}} \in (\mathcal{F}_{\geq 0}^{\mathbb{R}})^2$  is real-valued, independent of  $\xi$ , and satisfies

$$\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) := \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp + \mathcal{V}_{\text{can}}^{\text{nl}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}), \quad \mathcal{V}_{\text{can}}^{\text{nl}} \in (\mathcal{F}_{\geq 1}^{\mathbb{R}})^2.$$

Its full formula is given in Eq. (5.12).

- The remaining paradifferential symbols satisfy

$$A_{[0]}^\eta \in (\Gamma_{\geq 1}^0)^2, \quad A_{[-1]}^\psi \in (\Gamma_{\geq 1}^{-1})^2, \quad (1.23)$$

and hence form a two-by-two block of orders

$$\begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \end{bmatrix}.$$

- The remainder

$$\mathbf{R}_{[\varrho]} \in (\mathcal{R}_{\geq 1}^\varrho)^{2 \times 2}$$

is a tame smoothing operator in the sense of Definition 4.5 and Notation 5.1.

**Remark 1.9** (Formal recovery of the one-phase water-wave structure). It is instructive to compare Theorem 1.8 with the classical paradifferential reduction of the one-phase water-wave equations. We stress that the following discussion is only a formal identification of the quasilinear structures: no convergence result as  $\mathfrak{a} \rightarrow 2$  and  $\bar{\Omega} \rightarrow 0$  is intended, and we do not attempt to identify all the bounded symbols appearing in Eq. (1.23).

Recall first that the endpoint  $\mathfrak{a} = 2$  corresponds to  $\rho^+ = 0$ . By Eq. (2.2), at this endpoint  $\psi = 2\psi^-$ . Thus, when  $\bar{\Omega} = 0$ , so that  $\tilde{\psi} = \psi$ , the standard one-phase surface potential is

$$\psi_{\text{WW}} := \frac{1}{2} \tilde{\psi}. \quad (1.24)$$

This factor 2 is entirely due to the normalization of the canonical two-phase potential introduced following [80]. In particular, quadratic expressions in the potential acquire the corresponding factor 4; this is consistent with the quadratic dependence of the kinetic Hamiltonian on the surface potential and does not represent a difference between the two dynamics.

The comparison is especially transparent at the level of the unbounded paradifferential terms, which are the terms relevant for the derivative balance in the energy estimates. By Eq. (5.4),

$$2\lambda_1(\eta; x, \xi) = \sqrt{(1 + |\nabla \eta|^2)|\xi|^2 - (\nabla \eta \cdot \xi)^2},$$

and the right-hand side is precisely the principal symbol of the one-phase Dirichlet–Neumann operator appearing in Eq. (1.38). Accordingly, after the normalization Eq. (1.24), the order-one upper-right entry of Eq. (1.20) is the usual water-wave Dirichlet–Neumann contribution. Likewise, the order-one transport and Alinhac blocks in Eqs. (1.19) and (1.22) reduce formally to the standard transport velocity and good-unknown structure encoded by  $V$ ,  $B$ , and  $U = \psi_{\text{WW}} - T_B \eta$  in Eq. (1.38). Notice in particular that letting  $\bar{\Omega} \rightarrow 0$  removes the prescribed affine background flow, but not the nonlinear water-wave transport velocity.

The lower-left entry exhibits the other essential correspondence. Since  $\langle\langle \rho^\pm \rangle\rangle = \rho^-/2$  at  $\mathfrak{a} = 2$ , Eq. (5.6) gives

$$\frac{1}{2} a_2^{\pi_0, \eta}(\rho^\pm|\eta; x, \xi) = -\frac{\sigma}{\rho^-} \frac{|\xi|^2 + |\nabla^\perp \eta \cdot \xi|^2}{(1 + |\nabla \eta|^2)^{\frac{3}{2}}}, \quad (1.25)$$

which is the usual order-two symbol of the linearized mean-curvature operator, with the physical one-phase surface-tension coefficient  $\sigma/\rho^-$ . After dividing the second equation by the factor 2 coming from  $\tilde{\psi} = 2\psi_{\text{WW}}$ , the gravity contribution  $-2g$  similarly reduces to the standard  $-g$  term.

Most importantly, the additional order-one contribution specific to the two-phase problem disappears at the one-phase endpoint. Indeed, Eq. (5.7) gives the nonlinear identity

$$q^{\tilde{\pi}_0}(2, 0 | \eta, 2\psi_{\text{WW}}; x, \xi) = 0,$$

because  $q_0^{\tilde{\pi}}$  contains the factor  $4 - \alpha^2$ . Thus the order-one lower-left term  $\lambda_1 q_0^{\tilde{\pi}}$  in Eq. (1.20) vanishes identically. At the flat level the same mechanism is visible directly in Eq. (1.21), since

$$\frac{4 - \alpha^2}{8} \frac{(\bar{\Omega}^\perp \cdot \xi)^2}{|\xi|} \longrightarrow 0 \quad \text{as } (\alpha, \bar{\Omega}) \longrightarrow (2, 0).$$

This is precisely the Kelvin–Helmholtz order-one contribution which has no counterpart in the one-phase water-wave equations.

Consequently, at the level of all positive-order terms, Eq. (1.19) formally reduces at  $(\alpha, \bar{\Omega}) = (2, 0)$ , after the normalization  $\tilde{\psi} = 2\psi_{\text{WW}}$ , to the standard nonlinear gravity–capillary water-wave paradifferential structure recalled in Eq. (1.39); see [6, 8, 25]. Moreover, because the two-phase order-one Kelvin–Helmholtz contribution has vanished before any use of surface tension, one may formally set  $\sigma = 0$  in this endpoint: the order-two curvature term in Eq. (1.25) then disappears and one recovers the corresponding unbounded quasilinear structure of the pure-gravity water-wave system of [8]. The bounded order-zero terms, including the nonlinear Taylor coefficient in the pure-gravity formulation, are not identified by the present comparison; the statement concerns precisely the positive-order structure responsible for the derivative balance.

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### 1.2.3 Local well-posedness

At the flat state, the first line of Eq. (1.19) reduces to the linear symbol in Eq. (3.3). Fix  $\sigma > 0$ ,  $g > 0$ , and  $\rho_\star > 0$ . For this choice of physical parameters, define the renormalized densities

$$\rho_\alpha^- := \rho_\star \left(1 + \frac{\alpha}{2}\right), \quad \rho_\alpha^+ := \rho_\star \left(1 - \frac{\alpha}{2}\right), \quad 0 < \alpha \leq 2.$$

With  $\rho^\pm = \rho_\alpha^\pm$ , we isolate the parameter region in which the two dispersive branches are separated and purely imaginary.

**Hypothesis 1** (Strict Atwood–Weber stability).

$$\sigma > 0, \quad g\alpha > 0, \quad W_{\alpha, \bar{\Omega}} := \frac{\kappa_{\text{We}}^{\max}}{2\kappa_{\text{cap}}} = \frac{4 - \alpha^2}{16} |\bar{\Omega}|^2 \sqrt{\frac{\langle\langle \rho^\pm \rangle\rangle}{\sigma g \alpha}} < 1. \quad (1.26)$$

- $\sigma$  is the surface tension coefficient. In particular, capillarity is present at every nonzero frequency.
- $g\alpha$  is the flat Rayleigh–Taylor restoring acceleration. In particular, the heavier fluid lies below the lighter one when  $g\alpha > 0$ .
- $\kappa_{\text{cap}}$  is the positive capillary wavenumber defined by

$$\kappa_{\text{cap}} := \sqrt{\frac{(\rho^- - \rho^+)g}{\sigma}}.$$

In particular,  $\kappa_{\text{cap}} > 0$  under  $g\alpha > 0$ .

- $\kappa_{\text{We}}^{\max}$  is the maximal effective Weber wavenumber generated by the constant background shear and defined by

$$\kappa_{\text{We}}^{\max} := \frac{\rho^- \rho^+}{\sigma(\rho^- + \rho^+)} |\bar{\Omega}|^2.$$

In particular,  $\kappa_{\text{We}}^{\max} = 0$  if  $\bar{\Omega} = 0$  or if one of the two densities vanishes.

With these notations, the stability criterion Eq. (1.26) is equivalent to

$$(\kappa_{\text{We}}^{\max})^2 < 4\kappa_{\text{cap}}^2.$$

By Proposition 3.4, this implies that the two spectral branches of the flat linearized CDE are purely imaginary and simple at every nonzero frequency.

For the Cauchy statement, we introduce the phase space

$$\mathcal{X}^s := H^{s+\frac{1}{4}}(\mathbb{R}^2; \mathbb{R}) \times H^{s-\frac{1}{4}}(\mathbb{R}^2; \mathbb{R}). \quad (1.27)$$

**Definition 1.10** (Stable parameter set). Let  $\sigma > 0$ ,  $\mathfrak{g} > 0$ , and  $\rho_\star > 0$  be fixed. We define the *stable parameter set* by

$$\mathcal{S}_{\text{st}} := \{(\mathfrak{a}, \bar{\Omega}) \in (0, 2) \times \mathbb{R}^2 \mid \mathfrak{g}\mathfrak{a} > 0, \quad W_{\mathfrak{a}, \bar{\Omega}} < 1\} \cup (\{2\} \times \mathbb{R}^2).$$

Thus  $\mathcal{S}_{\text{st}}$  consists of the two-phase parameters satisfying the strict Atwood–Weber stability condition, together with the one-phase endpoint  $\mathfrak{a} = 2$ .

**Remark 1.11** (Linear stability and strict hyperbolicity). The terminology in Definition 1.10 is justified by the linearized analysis. Indeed, by Proposition 3.4, one has

$$(\mathfrak{a}, \bar{\Omega}) \in \mathcal{S}_{\text{st}} \iff \mathfrak{d}(\mathfrak{a}, \bar{\Omega}|\xi) < 0 \quad \text{for every } \xi \in \mathbb{R}^2 \setminus \{0\},$$

where  $\mathfrak{d}(\mathfrak{a}, \bar{\Omega}|\xi)$  is defined in Eq. (3.4). Equivalently, the two eigenvalues  $\lambda_\pm(\mathfrak{a}, \bar{\Omega}|\xi)$  of the flat linearized CDE, computed in Section 3, are purely imaginary and distinct at every nonzero frequency, or, equivalently,  $-iL_0(\xi)$  has two distinct real eigenvalues. Hence  $\mathcal{S}_{\text{st}}$  is precisely the parameter region of strict full-frequency linear hyperbolicity.  $\blacklozenge$

The uniform gap provided by Hypothesis 1 allows the positive-order block in Eq. (1.19) to be symmetrized and diagonalized; this reduction is carried out in Section 6. The resulting energy estimate yields the following local well-posedness theorem for the canonical CDE.

**Theorem 1.12** (Uniform local well-posedness in the Hamiltonian scale). *Let  $\sigma > 0$ ,  $\mathfrak{g} > 0$ , and  $\rho_\star > 0$ . There exists  $s_0 > 0$  such that, for every  $s \geq s_0$  and every compact set  $K \Subset \mathcal{S}_{\text{st}}$ , there exist  $\varepsilon_0, c_{s,K} > 0$ , depending only on  $s$  and  $K$ , such that the following holds.*

*For every  $(\mathfrak{a}, \bar{\Omega}) \in K$  and every real-valued datum  $Y_0 := (\eta_0, \tilde{\psi}_0) \in \mathcal{X}^s$  satisfying*

$$\|Y_0\|_{\mathcal{X}^s} \leq \varepsilon < \varepsilon_0,$$

*the canonical CDE Eq. (1.13), with densities  $\rho^\pm = \rho_\mathfrak{a}^\pm$ , admits a unique real-valued classical solution*

$$Y := (\eta, \tilde{\psi}) \in \mathcal{C}([-T_\varepsilon, T_\varepsilon]; \mathcal{X}^s) \cap \mathcal{C}^1([-T_\varepsilon, T_\varepsilon]; \mathcal{X}^{s-\frac{3}{2}}), \quad Y(0) = Y_0,$$

*with*

$$T_\varepsilon \geq c_{s,K} \varepsilon^{-1}.$$

**Remark 1.13.** The strict Atwood–Weber threshold  $W_{\mathfrak{a}, \bar{\Omega}} < 1$  is the CDE spectral form of the Kelvin–Rayleigh–Taylor stability balance isolated by Lannes [80, eq. (1.1)]. Let us spell out the comparison. In our sign convention  $\llbracket f^\pm \rrbracket = f^- - f^+$ , Lannes’ nonlinear criterion reads

$$\llbracket -\partial_z p^\pm|_{z=\zeta} \rrbracket > \frac{1}{4\sigma} \frac{(\rho^- \rho^+)^2}{(\rho^- + \rho^+)^2} c(\zeta) \left\| \llbracket V^\pm|_{z=\zeta} \rrbracket \right\|_{L^\infty}^4, \quad (1.28)$$

where  $c(\zeta)$  is the geometric prefactor entering the two-fluid symmetrizer. The left-hand side is the Rayleigh–Taylor restoring force, while the right-hand side measures the destabilizing Kelvin–Helmholtz shear, regularized by capillarity.

In the flat infinite-depth configuration one has  $c(0) = 1$ , and the two quantities entering Eq. (1.28) become exactly the constant coefficients appearing in the linearized CDE. More precisely,

$$\llbracket -\partial_z p^\pm|_{z=0} \rrbracket = (\rho^- - \rho^+) \mathfrak{g}, \quad \left\| \llbracket V^\pm|_{z=0} \rrbracket \right\|_{L^\infty} = |\bar{\Omega}|,$$

up to the fixed  $\mathbf{J}$ -rotation built into the vorticity variable  $\Omega = \nabla^\perp \llbracket \psi^\pm \rrbracket$ . Therefore Eq. (1.28) reduces to

$$(\rho^- - \rho^+) \mathfrak{g} > \frac{1}{4\sigma} \frac{(\rho^- \rho^+)^2}{(\rho^- + \rho^+)^2} |\bar{\Omega}|^4. \quad (1.29)$$

On the other hand, since

$$\frac{4 - \mathfrak{a}^2}{16} = \frac{\rho^- \rho^+}{(\rho^- + \rho^+)^2}, \quad \mathfrak{a} = \frac{2(\rho^- - \rho^+)}{\rho^- + \rho^+},$$

the inequality  $W_{\mathfrak{a}, \bar{\Omega}} < 1$  is equivalent to Eq. (1.29).

Thus the two criteria express the same principal stability mechanism: Rayleigh–Taylor restoring force and capillarity must dominate the Kelvin–Helmholtz shear. In [80] this balance is formulated as a nonlinear stability condition in physical variables, involving the actual interfacial velocity jump  $\llbracket V^\pm|_{z=\zeta} \rrbracket$ . In the present theorem the same balance is obtained from the eigenvalue calculation of the flat CDE around a state carrying the prescribed affine background  $\bar{\Omega}$ . The point is therefore not that the condition is a special case of Lannes’ theorem, nor that Lannes’ criterion excludes nonzero shear; rather, the comparison shows that our Atwood–Weber threshold is the constant-background spectral realization, in CDE variables, of Lannes’ Kelvin–Rayleigh–Taylor criterion. This is precisely the stability gap used below to construct the complex symmetrizer and diagonalizer.  $\blacklozenge$

### 1.3 Ideas of the proof

The proof follows the three layers used to present the main results. The first and the third are tied to the vortex-sheet system, whereas the middle one is an operator-level statement which is formulated and proved independently of it. We first reduce the Euler equations to an autonomous contour-dynamics system in canonical variables. We then isolate the graph integral operators generated by this reduction and paralin-earize them through a single abstract mechanism. Only after this singular-integral analysis has been completed do we return to the fluid system, assemble the real paradifferential CDE, and perform the diagonal reduction compatible with the non-homogeneous energy at frequency zero. Schematically,

$$\begin{aligned} (\mathbf{u}^\pm, p^\pm, \gamma) &\longmapsto (\eta, \tilde{\psi}), \\ (h, k), 0 < n \leq d &\longmapsto t^{(n)}(\eta)[\omega] \longmapsto \text{structure-preserving paradifferential reduction}, \\ (\eta, \tilde{\psi}) &\longmapsto \text{paradifferential CDE} \longmapsto Y_{\text{AI}} \longmapsto U \longmapsto W. \end{aligned}$$

The second line is the model-independent layer of the argument. It extracts the local homogeneous structure responsible for the symbols from the particular fluid model, and is established on  $\mathbb{R}^d$  before the vortex-sheet kernels are inserted. The critical endpoint is the case used in the vortex-sheet application; the sub-critical result in the appendix records the robustness of the same construction beyond the principal-value regime.

**From the Euler system to an autonomous CDE.** The starting point is the concentration of the vorticity on the interface. Since the velocity fields are irrotational away from the sheet, the Biot–Savart law expresses their traces through the three-dimensional Birkhoff–Rott integral. Once the interface is written as a graph, the vorticity has the form  $\omega = \left[ \frac{\Omega}{\nabla \eta \cdot \Omega} \right]$ , and the average trace of the velocity is described entirely by the two singular operators  $\mathcal{B}_0(\eta)$  and  $q(\eta)$  of Eqs. (1.15a) and (1.15b). In particular, the kinematic condition becomes

$$\eta_t = g^\star(\eta)[\Omega].$$

At this stage the system is already supported on the interface, but it is not yet closed in the canonical variables, since  $\Omega$  is still an independent unknown.

The density-weighted trace potential  $\psi$  in Eq. (1.11) is precisely the variable that allows us to eliminate  $\Omega$  as an independent unknown. Indeed, writing  $\varphi := \llbracket \psi^\pm \rrbracket$ , the Plemelj jump relations give

$$\psi = (\text{Id} + \mathfrak{a}\tau_0(\eta))\varphi, \quad \Omega = \nabla^\perp \varphi.$$

This is a second-kind boundary integral equation closely related to the one used in [49] to recover the vortex-sheet amplitude in the two-dimensional Muskat problem. There are, however, substantial differences between the two settings. The analysis of [49] concerns a one-dimensional interface in a two-dimensional fluid domain and exploits tools specific to that geometry to obtain a non-perturbative inversion result. Here the interface is two-dimensional and the corresponding operator acts on functions on  $\mathbb{R}^2$ . We work perturbatively around the flat interface: the Sobolev invertibility of  $\text{Id} + \alpha\tau_0(\eta)$  follows from the smallness of  $\eta$ , while the precise structure of its inverse required in the subsequent analysis is recovered by a multidimensional paradifferential parametrix construction, see Lemma 5.23 and Proposition 5.24. Thus the mechanism in [49] provides a useful structural analogy, but does not directly apply to the present three-dimensional problem. We therefore set

$$\Omega = S(\alpha|\eta)\psi, \quad S(\alpha|\eta)\psi := \nabla^\perp (\text{Id} + \alpha\tau_0(\eta))^{-1}\psi, \quad (1.30)$$

and substitute Eq. (1.30) into the kinematic and Bernoulli equations. This gives a closed system for  $(\eta, \psi)$ . Equivalently, the generalized Dirichlet–Neumann operator appearing in the first equation of Eq. (1.13a) is not introduced as an abstract elliptic boundary operator, but is factorized as

$$\mathcal{G}(\alpha|\eta) = \mathcal{G}^\star(\eta) \circ S(\alpha|\eta). \quad (1.31)$$

Thus both pieces of  $\mathcal{G}$  remain explicit in terms of the Birkhoff–Rott kernels. This is important later: rather than invoking the symbolic theory of the Dirichlet–Neumann operator, we recover its paradifferential structure directly from the underlying singular integrals. The factorization in Eq. (1.31) is also the point at which the fluid problem meets the abstract theory: its nonlocal building blocks are critical graph-kernel operators associated with the admissible homogeneous pairs of Definition 5.4.

The constant background requires one further step. Its potential  $\tilde{\psi} = x^\perp \cdot \tilde{\Omega}$  is affine and does not belong to the Sobolev phase space. We therefore split  $\psi = \tilde{\psi} + \tilde{\psi}$ , absorb the background contribution into

$$\tilde{\Omega}[\tilde{\Omega}](\eta) := S(\alpha|\eta)\tilde{\psi},$$

and subtract the constant Bernoulli gauge  $c_{\tilde{\Omega}}$ . The flat state is then represented by  $(\eta, \tilde{\psi}) = (0, 0)$ , while the full background vorticity remains present through  $\tilde{\Omega}[\tilde{\Omega}](\eta)$ . This gives exactly the autonomous system Eq. (1.13): all bulk quantities and the independent vorticity have disappeared, and every term in the right-hand side is a function of  $(\eta, \tilde{\psi})$  and of the fixed parameters.

Let us also stress how this reduction differs from the contour formulation used by Fefferman, Ionescu, and Lie in [63]. There the interface is a curve in a two-dimensional fluid domain, no preferred parametrization of the curve is fixed, and the boundary vorticity is retained in the singular-integral representation in order to study the geometry of a possible self-intersection. The one-dimensional character of the interface also entails structural simplifications which are no longer available in the present setting: there is a single tangential direction, whereas for a surface in three-dimensional space the tangential geometry is genuinely two-dimensional and the associated singular and paradifferential operators carry a nontrivial angular dependence in frequency space. Here we work in the graph gauge and eliminate the vorticity in favour of the canonical trace potential. The distinction is therefore not that one formulation is a contour equation and the other is not. It lies both in the geometry of the interface and in the choice of variables and purpose of the reduction: our goal is an autonomous CDE on a fixed Sobolev phase space, suitable for a direct quasilinear analysis.

The important output of this first reduction is not only the closed system itself. Its nonlocal geometry has been concentrated in a finite family of graph operators whose dependence on the interface occurs through normalized finite differences. This is the model-independent structure isolated in the next step.

**A structure-preserving parilinearization theorem for graph kernels.** This is the main methodological contribution of the paper. The singular operators produced by the contour reduction are not parilinearized one at a time. Instead, their local geometry is encoded by an admissible homogeneous pair  $(h, k)$  of order  $-n$  and vanishing order  $\ell$ , with  $0 < n \leq d$ , in the sense of Definition 5.4. Here  $h$  is the angular profile of the homogeneous kernel, whereas  $k$  describes its nonlinear dependence on the graph. At the level relevant for

the symbolic analysis, every operator has the localized homogeneous core

$$\begin{aligned} t_{\text{loc}}^{(n)}(\eta)[\omega](x) &:= \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} k(\Delta_z \eta(x)) \omega(x-z) dz, \\ \Delta_z \eta(x) &:= \frac{\eta(x) - \eta(x-z)}{|z|}, \quad \hat{z} := \frac{z}{|z|}. \end{aligned}$$

The space localization is obtained through the cut-off function  $\chi$  as in Eq. (4.7). When  $\beta = d - n = 0$ , namely  $n = d$ , the complete flat kernel is interpreted as a homogeneous principal-value distribution. The opposite parities imposed on  $h$  and  $k$  make the full graph kernel odd under the exchange of the two spatial variables and yield the exact skew-adjointness of Lemma 5.6. When  $\beta > 0$ , the homogeneous core is locally integrable and no parity condition is required. The global version treated in Appendix B is obtained by completing this local homogeneous kernel with a sufficiently regular far-field tail. The nonlinear tail contributes an order-zero paraproduct and smoothing remainders, without changing the local principal symbols. In either regime the perturbative assumption is imposed only on the graph, so that the difference quotients remain inside the analytic range of  $k$ ; no smallness is imposed on  $\omega$ .

This formulation isolates and extends the mechanisms introduced in [24, 85]. Its point is not merely to shorten the proofs of the two operators appearing below. It identifies, in arbitrary dimension, which features of the parilinearization come from homogeneity, which come from the vanishing of  $k - k(0)$ , and which are specific to the critical endpoint. The singular dependence on  $z$  and the nonlinear dependence on  $\Delta_z \eta$  occur at the same scale, and this is precisely why a standard parilinearization performed after the integration in  $z$  does not reveal the correct symbolic structure.

The proof is organized around four operations which are reusable beyond the present fluid problem.

1. *Isolation of the homogeneous core.* Set  $\tilde{k} := k - k(0)$ . The first step is to separate the contribution of the kernel near  $z = 0$  from its far-field tail. This distinction separates the singular principal symbols from the order-zero tail contribution and its smoothing remainders.

For the critical operator, the decomposition is explicit:

$$\begin{aligned} t^{\text{flat}}[\omega](x) &:= k(0) \text{ p.v. } \int_{\mathbb{R}^d} \frac{h(\hat{z})}{|z|^d} \omega(x-z) dz, \\ t^{\text{loc}}(\eta)[\omega](x) &:= \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \tilde{k}(\Delta_z \eta(x)) \omega(x-z) dz, \\ t^{\text{tail}}(\eta)[\omega](x) &:= \int_{\mathbb{R}^d} (1 - \chi(z)) \frac{h(\hat{z})}{|z|^d} \tilde{k}(\Delta_z \eta(x)) \omega(x-z) dz. \end{aligned}$$

Thus

$$t^{(d)}(\eta)[\omega] = t^{\text{flat}}[\omega] + t^{\text{loc}}(\eta)[\omega] + t^{\text{tail}}(\eta)[\omega].$$

The flat term retains the complete homogeneous principal-value multiplier. The local term contains the singularity at  $z = 0$  and produces the principal and subprincipal symbols. It is this term that is analyzed by the Bony decomposition and the local Taylor expansion below.

By contrast, the last integral only sees the far field. On  $|z| \geq 1$ ,  $|\Delta_z \eta(x)| \lesssim |z|^{-1} \|\eta\|_{L^\infty}$ , so the vanishing order  $\ell$  of  $\tilde{k}$  supplies the decay  $|z|^{-\ell}$ . By Eq. (5.32) and Proposition 4.37, for every prescribed  $\rho \geq 0$ ,

$$\begin{aligned} t^{\text{tail}}(\eta)[\omega] &= \text{Op}^{\text{BW}} \left( f_{\geq \ell}^{\text{tail}}[\omega](\eta) \right) \eta \\ &\quad + r_{[\rho, \geq \ell]}^{\text{tail}}(\eta) \omega + r_{[\rho, \geq \ell]}^{\text{tail}}[\omega](\eta) \eta. \end{aligned}$$

The function is given in Eq. (5.30). It enters the generic order-zero coefficient of  $\eta$ .

The subcritical case is organized in the same way: the homogeneous near-field component produces the finite-order symbols, while the non-homogeneous far-field contribution is treated by the corresponding tail estimates in Theorem B.1.

2. *Bony decomposition at fixed convolution scale.* For  $z \neq 0$ , set  $\omega_z(x) := \omega(x-z)$ . The localized nonlinear product is decomposed before integration in  $z$ :

$$\tilde{k}(\Delta_z \eta) \omega_z = \text{Op}^{\text{BW}}(\tilde{k}(\Delta_z \eta)) \omega_z + \text{Op}^{\text{BW}}(\omega_z) [\tilde{k}(\Delta_z \eta)] + R^{\text{BW}}[\tilde{k}(\Delta_z \eta), \omega_z].$$

The three terms are respectively the quasilinear branch, in which the high frequency remains on  $\omega$ ; the transported branch, in which the high frequency is transferred to  $\eta$ ; and the localized Bony remainder. The latter becomes arbitrarily smoothing after integration in  $z$ , by Lemma 5.10 in the critical case and Lemma B.3 in the subcritical case. Performing this decomposition before the  $z$ -integration is essential: the powers of  $|z|$  which measure the Taylor order of the graph are exactly those which determine the order of the resulting Fourier symbols.

3. *Extraction of the symbols from the local graph geometry.* On the support of  $\chi$ , Taylor's formula gives

$$\Delta_z \eta(x) = \nabla \eta(x) \cdot \hat{z} - \frac{|z|}{2} \hat{z}^\top \mathbf{H} \eta(x) \hat{z} + e_{[2]}(\eta; x, z),$$

where  $e_{[2]}$  has local order two by Eq. (5.38). Therefore

$$k(\Delta_z \eta) = k(\nabla \eta \cdot \hat{z}) - \frac{|z|}{2} k'(\nabla \eta \cdot \hat{z}) (\hat{z}^\top \mathbf{H} \eta \hat{z}) + k_{[2]}(\eta; x, z). \quad (1.32)$$

The basic order conversion is

$$\text{local kernel order } n + j \longrightarrow \text{symbol order } -\beta - j.$$

The first term in Eq. (1.32) gives the order- $(-\beta)$  quasilinear symbol, while the transported branch gives an order-one symbol linear in  $\omega$ . The remainder  $k_{[2]}$  begins two local orders later and is therefore treated by Proposition 4.35 at order  $-\beta - 2$ , modulo an arbitrarily smoothing operator.

After freezing  $p = \nabla \eta(x)$ , the leading local kernels depend on  $z$  only through the homogeneous radial factor and the angular variable  $\vartheta = \hat{z}$ . Analyticity of  $k$  allows us to separate this angular dependence from the  $x$ -dependent coefficient:

$$k(p \cdot \vartheta) = \sum_{\alpha \in \mathbb{N}^d} \frac{k^{(|\alpha|)}(0)}{\alpha!} p^\alpha \vartheta^\alpha.$$

Thus each term of the series is the product of a coefficient  $p^\alpha$  and a convolution operator with a fixed homogeneous angular amplitude  $h(\vartheta) \vartheta^\alpha$ . For instance, the leading quasilinear branch is reduced term by term to operators of the form

$$p^\alpha \text{ p.v. } \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z}) \hat{z}^\alpha}{|z|^d} \omega(x - z) dz.$$

The parity assumptions in Definition 5.4 ensure that every nonzero Taylor coefficient produces precisely the odd or even angular amplitude required by the homogeneous multiplier formulas of Proposition 4.38. The transported branch is reduced in the same way, after the symmetrization by  $\delta_z^{(2)} \eta$ , to compensated homogeneous kernels of degree  $-d - 1$ .

At this point the cutoff no longer carries symbolic information. Indeed, the localized kernel and its complete homogeneous model have the same singularity at  $z = 0$ , and Lemmas 4.41 and 4.44 shows that their high-frequency Fourier multipliers differ only by a rapidly decreasing symbol:

$$t_{h,\alpha}^{\text{loc}} f = \mathfrak{m}_{h,\alpha}(D) f + \mathfrak{s}_{h,\alpha}(D) f, \quad \mathfrak{s}_{h,\alpha} \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M},$$

where

$$t_{h,\alpha}^{\text{loc}}[\omega](x) := \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z}) \hat{z}^\alpha}{|z|^d} \omega(x - z) dz, \quad \mathfrak{m}_{h,\alpha}(D)[\omega](x) := \text{p.v.} \int_{\mathbb{R}^d} \frac{h(\hat{z}) \hat{z}^\alpha}{|z|^d} \omega(x - z) dz.$$

Consequently, all finite-order symbolic information can be computed from the complete homogeneous kernels. The degree- $(-d)$  odd kernels give purely imaginary multipliers of order zero, whereas the compensated even kernels arising in the transported branch give real multipliers of order one.

Applying Proposition 4.38 term by term and then summing back the analytic expansions reconstructs  $k$  and  $k'$ , and yields

$$\begin{aligned} a_0^{h,k;\omega}(p, \xi) &:= -\frac{\pi}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\vartheta \cdot \xi) h(\vartheta) k(p \cdot \vartheta) d\sigma(\vartheta), \\ a_1^{h,k;\eta}(p, \xi) &:= \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\vartheta \cdot \xi| h(\vartheta) k'(p \cdot \vartheta) d\sigma(\vartheta), \\ a_1^{h,k;\eta}(p, \xi) &= -\partial_p a_0^{h,k;\omega}(p, \xi) [\xi]. \end{aligned}$$

The last identity is Eq. (5.23). It shows that the two principal columns are not unrelated outputs of the calculation: they are different manifestations of the same graph kernel. In the subcritical regime, the corresponding principal symbols are the localized Fourier integrals in Eqs. (B.5) and (B.6). Their orders follow from the same local homogeneity, without an endpoint parity argument.

The angular variable survives the Fourier transform. The resulting symbols have the form

$$a(\nabla\eta(x), \hat{\xi}) |\xi|^\nu.$$

In the present application,  $d = 2$  and  $\hat{z} \in \mathbb{S}^1$ , so the graph geometry couples to a continuum of frequency directions. This is a genuinely surface-level feature: for a one-dimensional contour,  $\mathbb{S}^0 = \{-1, 1\}$  and the corresponding angular structure collapses to two directions and simplify remarkably the analysis.

4. *Preservation of the critical adjoint structure.* The first lower-order contribution in the quasilinear branch comes from two different sources. The principal symbol produces a first Bony–Weyl correction, while the next term in the Taylor expansion of  $\Delta_z \eta$  produces a geometric correction of the same order. The two contributions are equal and opposite, as shown in Eqs. (5.50) and (5.56). Hence no independent symbol of order  $-1$  survives in the critical case, and the first generic lower-order term in the  $\omega$ -column has order  $-2$ .

There is a simple geometric way to understand this cancellation. Weyl quantization is centered at the midpoint of the interaction. In the corresponding coordinates the graph difference takes the form

$$\frac{\eta(X + z/2) - \eta(X - z/2)}{|z|} = \nabla\eta(X) \cdot \hat{z} + O(|z|^2).$$

There is therefore no correction of order  $|z|$ . When the calculation is first expanded around one endpoint, such a term appears separately as the first geometric Taylor correction and the first Weyl correction; their cancellation simply restores this centered structure.

In the critical regime, the parity assumptions have an additional role: they make the principal-value operator exactly skew-adjoint. Accordingly, the principal Bony–Weyl operator is  $\operatorname{Op}^{\text{BW}}\left(\mathbf{i}a_0^{h,k;\omega}\right)$ , with  $a_0^{h,k;\omega}$  real and odd in  $\xi$ . Thus the cancellation above is also the symbolic manifestation of the adjoint structure of the original singular integral. The same centered-difference mechanism explains the corresponding gap between orders  $-\beta$  and  $-\beta-2$  in the subcritical case; there, however, no parity assumption is needed because the kernel is already locally integrable.

The fluid system enters only after this operator-level construction has been completed. By Lemma 5.13, the two scalar components of  $\mathcal{B}_0(\eta)$  and the operator  $q(\eta)$  are critical instances with  $d = 2$ ,  $n = 2$ , and therefore  $\beta = 0$ . The two components of  $\mathcal{B}_0$  have vanishing order  $\ell = 2$ , whereas  $q$  has vanishing order  $\ell = 1$ . The former correspond to an odd angular profile and an even geometric nonlinearity; the latter to an even angular profile and an odd geometric nonlinearity. Their parilinearizations Propositions 5.14 and 5.15 are therefore applications of the same critical theorem. The model-specific part is reduced to verifying admissibility, evaluating the angular integrals, and assembling the composite operators of the CDE.

We then parilinearize  $S(\mathbf{a}|\eta)$ , the velocity traces, the generalized Dirichlet–Neumann operators, and the Bernoulli terms, and combine them in Theorem 1.8. As a consistency check, the graph-kernel route recovers the geometric principal symbol

$$2\lambda_1(\eta; x, \xi) = \sqrt{\left(1 + |\nabla\eta(x)|^2\right) |\xi|^2 - (\nabla\eta(x) \cdot \xi)^2} = |\xi| \sqrt{1 + |\nabla^\perp\eta(x) \cdot \hat{\xi}|^2}.$$

Up to the normalization of the two-fluid potential, this is exactly the principal symbol of the Dirichlet-Neumann operator. The abstract approach recovers this familiar geometry without treating the Dirichlet-Neumann calculus as the point of departure, while keeping explicit control over the lower-order vorticity and background contributions. The vortex-sheet system is therefore the main application of a graph-kernel parilinearization method formulated before, and independently of, the model.

**From the paradifferential system to the Cauchy theory.** From this point onward no singular integral is treated directly. We start from the real system in Theorem 1.8. Its positive-order part contains the Alinhac block  $\mathbf{B}$ , which would lead to a loss of derivatives if it were treated perturbatively. We first remove it by the real triangular transformation

$$Y_{\text{Al}} := \mathbf{G}Y, \quad \mathbf{G} := \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}} \left( \begin{bmatrix} 0 & 0 \\ b_{\text{can}} & 0 \end{bmatrix} \right). \quad (1.33)$$

The matrix symbol in Eq. (1.33) is nilpotent, so the inverse is obtained exactly by changing its sign. The transformation leaves  $\eta$  unchanged and replaces  $\tilde{\psi}$  by the usual good unknown. It is important that this step is performed in real variables: it removes the geometric derivative loss before the two dispersive branches are mixed by the complexification; see Proposition 6.2.

Under Hypothesis 1, the two branches of the flat linearized system are simple and purely imaginary. Recall  $\mathfrak{d}(\mathfrak{a}, \bar{\Omega}|\xi)$  defined in Eq. (3.4), we denote the positive dispersive frequency by

$$\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) := \sqrt{-\mathfrak{d}_{\mathfrak{a}, \bar{\Omega}}(\xi)}.$$

For every  $\xi \neq 0$ , the corresponding flat eigenvectors suggest the complex coordinate

$$u_{\text{hom}} := \frac{1}{\sqrt{2}} \left( \mathfrak{m}_{-\frac{1}{4}}^{-1}(D) \eta_{\text{Al}} + i \mathfrak{m}_{-\frac{1}{4}}(D) \tilde{\psi}_{\text{Al}} \right), \quad \mathfrak{m}_{-\frac{1}{4}}(\mathfrak{a}, \bar{\Omega}|\xi) := \sqrt{\frac{|\xi|}{2\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi)}}.$$

This choice diagonalizes the flat system exactly away from  $\xi = 0$  and has the correct high-frequency orders. The obstruction is confined to low frequencies. Indeed, by Eqs. (3.2) and (3.4),

$$\mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) \xrightarrow{|\xi| \rightarrow 0} -\mathfrak{g}\mathfrak{a}, \quad \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \sim \sqrt{\frac{\mathfrak{g}\mathfrak{a}}{2}} |\xi|^{\frac{1}{2}}, \quad \mathfrak{m}_{-\frac{1}{4}}(\mathfrak{a}, \bar{\Omega}|\xi) \sim \frac{|\xi|^{\frac{1}{4}}}{(2\mathfrak{g}\mathfrak{a})^{\frac{1}{4}}}.$$

Hence  $\mathfrak{m}_{-\frac{1}{4}}^{-1}(\mathfrak{a}, \bar{\Omega}|\xi)$  grows like  $|\xi|^{-\frac{1}{4}}$  as  $\xi \rightarrow 0$ . The exact eigenvector normalization therefore makes the map  $\eta_{\text{Al}} \mapsto \mathfrak{m}_{-\frac{1}{4}}^{-1}(D) \eta_{\text{Al}}$  singular at the origin; in particular, it does not define a bounded Fourier multiplier from  $H^{s+\frac{1}{4}}$  to  $H^s$ . Equivalently, the squared Fourier weight carried by the  $\eta_{\text{Al}}$ -component behaves like  $|\xi|^{-\frac{1}{2}}$ .

This singularity is not present in the physical quadratic energy. At the flat state the latter contains

$$\int_{\mathbb{R}^2} \left( -\mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) |\hat{\eta}(\xi)|^2 + \frac{|\xi|}{2} |\hat{\tilde{\psi}}(\xi)|^2 \right) d\xi.$$

Since  $-\mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) \rightarrow \mathfrak{g}\mathfrak{a} > 0$ , the gravitational term controls the low-frequency part of  $\eta$  with an ordinary  $L^2$  weight. Thus the factor  $|\xi|^{-\frac{1}{4}}$  is an artifact of the exact homogeneous eigenvector normalization, rather than a degeneracy of the flat Hamiltonian. The homogeneous coordinate is therefore well adapted to the high-frequency diagonalization, but not to the natural inhomogeneous Sobolev phase space.

For this reason we switch the homogeneous diagonalizer off near the origin. With the cut-off  $\chi$  of Eq. (4.7), we set

$$\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} := \chi + (1 - \chi) \mathfrak{m}_{-\frac{1}{4}}, \quad U := \begin{bmatrix} u \\ \tilde{u} \end{bmatrix} = \mathbf{M}(\mathfrak{a}, \bar{\Omega}|D) Y_{\text{Al}}, \quad (1.34)$$

where  $\mathbf{M}$  is defined in Eq. (6.47). This is a bounded and boundedly invertible map between the real inhomogeneous phase space and the complex Sobolev scale, and it preserves the conjugation condition  $U = \begin{bmatrix} u \\ \tilde{u} \end{bmatrix}$ .

The price is that the flat operator is no longer exactly diagonal in the low-frequency region. The discrepancy is the explicit constant-coefficient multiplier  $\mathbf{A}_{\text{lo}}^{\text{C,lin}}$  of Eq. (6.51).

At high frequency the remaining diagonalization has a classical algebraic core. An order-zero matrix first diagonalizes the quasilinear block of order 3/2. The transport symbol has order one but is scalar in the two complex components, and hence it is already diagonal. After this first conjugation, the only off-diagonal positive-order term has order 1/2. It is removed by an order-minus-one generator  $\dot{\mathbf{S}}_{-1}$ , chosen so that, at the homogeneous symbolic level,

$$\left[ i\dot{\omega}_{\frac{3}{2}} \dot{\theta}_0 \mathbf{J}_d, \dot{\mathbf{S}}_{-1} \right] + \left( \mathbf{L}_{\frac{1}{2}} \right)^{\text{off}} = 0.$$

This is the standard subprincipal mechanism: commutation with the separated principal branches divides by their spectral gap and cancels the off-diagonal half-order term. In the notation of the proof, the commutator generates exactly the opposite of the term containing  $\dot{\beta}_0^{\text{C,pr}} |\xi|^{1/2} \mathbf{J}$ ; the cancellation is recorded in Eq. (6.151).

What is less standard is making this formal cancellation into an admissible change of coordinates for the problem at hand. Two kinds of difficulty arise. The first concerns the generator  $\dot{\mathbf{S}}_{-1}$  itself. Being homogeneous, it has to be cut off near  $\xi = 0$ . It is also built from the leading capillary part of the frequency, which differs from the exact frequency by symbols that are lower order but not negligible. The cut-off then creates further terms, localized on a fixed transition annulus.

The second concerns the resulting conjugation. Each conjugating operator must have a tame inverse on the non-homogeneous Sobolev scale, and its time derivative must remain of an acceptable order. The full composition, moreover, must preserve the reality condition  $U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$  of Eq. (1.34). Finally, all these bounds must be uniform as  $(\mathfrak{a}, \bar{\Omega})$  ranges over compact subsets of the stable set.

Section 6.3 carries out this program in detail and produces an equivalent unknown  $W$  for which all positive-order symbols are real and diagonal, as stated in Proposition 6.7.

The final step is the energy estimate. At high frequency, if  $d_j$  is one of the diagonal entries of the positive-order symbol, we choose the Sobolev weight to be a scalar function  $\lambda_s(d_j)$ . The potentially order-losing Poisson bracket then vanishes identically:

$$\{\lambda_s(d_j), d_j\} = \lambda'_s(d_j) \{d_j, d_j\} = 0. \quad (1.35)$$

This is the reason for diagonalizing the complete positive-order part rather than using a fixed power of  $\langle D \rangle$  in the energy.

At low frequency we return to the real Hamiltonian structure. Let

$$\mathbf{S}_{\text{lo}}^{\mathbb{R}} := \begin{bmatrix} -\mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) & 0 \\ 0 & \frac{|\xi|}{2} \end{bmatrix}, \quad \mathbf{S}_{\text{lo}}^{\mathbb{C}} := (\mathbf{M}^{-1})^* \mathbf{S}_{\text{lo}}^{\mathbb{R}} \mathbf{M}^{-1}.$$

The low-frequency quadratic form in the complex variable is therefore exactly the pullback of the real quadratic Hamiltonian, not merely an equivalent norm. The isolated defect  $\mathbf{A}_{\text{lo}}^{\text{C,lin}}$  is not skew-adjoint on the transition annulus, but it is not supposed to be considered in isolation: its symmetrizer defect is cancelled by the transition part of the flat diagonal dispersion. The complete flat linear symbol contributes exactly zero to the low-frequency energy balance, as shown in Eq. (6.181).

Combining the high-frequency cancellation Eq. (1.35) with the exact low-frequency Hamiltonian identity gives the modified energy estimate

$$\left| \frac{d}{dt} E_s(t) \right| \lesssim_s E_{s_0}^{\frac{1}{2}}(t) E_s(t). \quad (1.36)$$

The energy  $E_s$  is equivalent to the Hamiltonian low-high norm of Eq. (6.184). It controls the full inhomogeneous elevation and the differentiated low-frequency part of  $\tilde{\psi}$ . The only missing component is the undifferentiated low-frequency potential, which is propagated directly from its equation and grows at most like  $\varepsilon|t|$ . The Riccati estimate Eq. (1.36) therefore closes on an interval of size  $c\varepsilon^{-1}$ . Existence follows by Friedrichs regularization and compactness; the same reductions applied to the equation for the difference, at one lower Sobolev index, give uniqueness, and a Bona–Smith approximation completes the proof of continuous dependence.

## 1.4 Previous results

The literature on water waves, vortex sheets, and moving interfaces is vast; see, among others, the monograph of Lannes and the survey of Wu [81, 108]. We recall here the developments that are most directly connected with the Cauchy problem and with the reduction of a free-boundary evolution to a quasilinear equation on the interface. We begin with the contour-dynamics approach to incompressible flows, which provides the closest historical precedent for the point of view adopted in this paper: the bulk variables are eliminated and the dynamics is encoded directly by nonlinear singular integral operators acting on the moving interface. We then discuss the one-phase water-wave problem, where the modern paradifferential strategy was developed most systematically, and finally turn to two-fluid interfaces and to the Kelvin–Helmholtz mechanism, which is the setting of the present work.

### 1.4.1 Contour dynamics in incompressible flows

The modern contour-dynamics approach to incompressible flows emerged from the study of two-dimensional Euler vortex patches. If the vorticity is piecewise constant, the Euler transport equation preserves this structure and the Biot–Savart law allows the motion to be described entirely through the evolution of the boundaries separating regions of constant vorticity. Thus a bulk transport equation is replaced by a nonlocal evolution equation on a curve. Deem and Zabusky discovered families of steadily translating and rotating finite-area vortices, the so-called  $V$ -states [60], and Zabusky, Hughes, and Roberts subsequently formulated contour dynamics as a systematic method for the two-dimensional Euler equations [114]. This point of view became a powerful numerical tool for resolving increasingly fine filamentation of vorticity interfaces; an important development was Dritschel’s contour-dynamics and contour-surgery framework [62]. On the analytical side, Burbea placed the construction of rotating vortex patches in a bifurcation framework [35]. The global regularity theory of vortex patches was later established by Chemin and, by a different argument, by Bertozzi and Constantin [31, 44]: an initially smooth patch boundary remains smooth for all times. These results provided one of the first settings in which a genuinely nonlinear singular integral evolution of an interface could be controlled globally through geometric cancellations.

The study of coherent vortex structures generated a second major branch of the contour-dynamics theory. Building on Burbea’s formulation, Hmidi, Mateu, and Verdera proved smoothness of rotating Euler patches close to the bifurcation circle [72]. Castro, Córdoba, and Gómez-Serrano extended this program to the generalized surface quasi-geostrophic family, constructing and proving regularity of rotating patch solutions [41], and established analyticity for families of uniformly rotating patches in active scalar equations [42]. They later constructed compactly supported smooth vorticities rotating uniformly under the two-dimensional Euler flow [43]. These works illustrate a recurring feature of contour dynamics: although the geometric unknown is lower dimensional, the resulting equation contains singular nonlocal operators whose precise cancellation properties determine both the regularity of the evolution and the structure of special solutions.

A particularly influential development of this perspective, due in large part to Córdoba, Gancedo, and collaborators, concerns sharp fronts for the surface quasi-geostrophic equation and its generalized variants. Rodrigo derived a rigorous evolution equation for an SQG sharp front and obtained local existence and uniqueness for smooth data by a Nash–Moser argument [93]. Córdoba, Fontelos, Mancho, and Rodrigo subsequently gave numerical and self-similar evidence for finite-time singularity formation in a family of contour equations interpolating between Euler patches and the SQG sharp front [52]. Gancedo established local-in-time existence in Sobolev spaces for the  $\alpha$ -patch family, including the SQG endpoint [64]; Córdoba, Córdoba, and Gancedo later completed the Sobolev theory at the SQG endpoint by proving uniqueness [51]. Further developments include the local theory and blow-up criteria for generalized SQG patches of Gancedo and Patel [66], and the low-regularity theory of Gancedo, Nguyen, and Patel, which allows SQG fronts in  $H^{2+s}$ , for arbitrarily small  $s > 0$ , and hence permits unbounded curvature [65]. In the more singular generalized SQG regime, Córdoba, Gómez-Serrano, and Ionescu proved global stability of the half-plane patch under small localized perturbations [54]. Complementing these existence theories, Gancedo and Strain ruled out splash singularities for SQG sharp fronts as long as the relevant regularity remains controlled [67]. More recently, the direct parilinearization of the singular contour operator has become a tool for long-time analysis: Berti, Cuccagna, Gancedo, and Scrobogna derived a paradifferential reduction of the

$\alpha$ -SQG sharp-front equation and used it to obtain an extended lifespan for small solutions [24].

The same contour-dynamics philosophy has played a central role in the analysis of porous-medium interfaces. Córdoba and Gancedo derived a contour formulation for the three-dimensional Muskat problem and proved local well-posedness in the stable regime, together with ill-posedness in the unstable one [53]. Córdoba, Córdoba, and Gancedo developed the corresponding two-dimensional Sobolev theory under the Rayleigh–Taylor sign condition [49] and subsequently established a three-dimensional local theory [50]. Constantin, Córdoba, Gancedo, and Strain obtained global solutions under quantitative smallness and slope assumptions [47]. The contour formulation also made it possible to study the failure of the stability condition dynamically: Castro, Córdoba, Fefferman, Gancedo, and López-Fernández showed that the Rayleigh–Taylor condition may break down in finite time and, by the same mechanism, constructed turning water waves [39]; Castro, Córdoba, Fefferman, and Gancedo later constructed solutions starting from analytic data in the stable regime which enter the unstable regime and subsequently lose smoothness [38]. On the other hand, splash formation is excluded in the regular Muskat evolution by the geometric mechanism identified in [67]. In this family of problems the sign of the principal singular operator therefore distinguishes a dissipative stable evolution from an unstable one, making the extraction of its principal symbolic structure particularly transparent and important.

Contour dynamics was also used directly for inviscid free-boundary Euler flows. Córdoba, Córdoba, and Gancedo studied the interface between two irrotational incompressible fluids in two dimensions and proved local Sobolev well-posedness under the Rayleigh–Taylor condition through a contour formulation [48]. For genuine vortex sheets, Castro, Córdoba, and Gancedo analyzed the Birkhoff–Rott evolution in a parametrization adapted to finite-energy sheets, obtaining a Cauchy theory for analytic data and exhibiting the corresponding obstruction for nonanalytic amplitudes [40]. We return below to the Kelvin–Helmholtz instability and to the regularizing effect of surface tension in the discussion of the two-phase water-wave problem.

Viewed together, the Euler patch, SQG, Muskat, water-wave, and vortex-sheet theories show that contour dynamics is not merely a convenient reformulation of particular fluid models. It provides a common analytic architecture in which the evolution is driven by nonlinear singular integral operators whose kernels depend on finite differences of the unknown geometry. The degree of the singularity changes from model to model, as do the parity and adjoint properties responsible for the available cancellations, but the central problem is repeatedly the same: to extract from the nonlinear integral operator the quasilinear part which governs the highest-order dynamics while controlling the remaining terms.

This is precisely the aspect of contour dynamics that is relevant to the present paper. Our purpose is not to develop another model-specific expansion of a Birkhoff–Rott operator, but to isolate a general structure-preserving parilinearization mechanism for this class of geometric singular integrals. In this sense, the results of Theorems 5.7 and B.1 may be viewed as an abstract counterpart of the model-dependent singular integral analyses appearing throughout the contour-dynamics literature: the homogeneous singular part determines the paradifferential principal and subprincipal operators, the parity of the kernel retains the relevant adjoint structure, and sufficiently regular nonhomogeneous tails yield order-zero paraproducts modulo smoothing remainders.

#### 1.4.2 The one-phase problem

At the endpoint  $\rho^+ = 0$ , equivalently  $\alpha = 2$ , the upper phase is replaced by vacuum and Eq. (1.9) reduces to the classical water-wave problem. In the irrotational graph setting, let  $\phi$  be the velocity potential in

$$\Omega(t) := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid y < \eta(t, x)\}, \quad d \in \{1, 2\},$$

and let

$$\psi(t, x) := \phi(t, x, \eta(t, x))$$

be its trace on the free surface. In this historical discussion only, we use the conventional notation  $G(\eta)$  for the Dirichlet–Neumann operator,

$$G(\eta)\psi := (\partial_y \phi - \nabla \eta \cdot \nabla \phi)|_{y=\eta}.$$

After normalizing the fluid density, the Euler equations are equivalent to the Zakharov–Craig–Sulem system

$$\begin{cases} \partial_t \eta = G(\eta) \psi, \\ \partial_t \psi + \mathfrak{g} \eta - \sigma \operatorname{div} \left( \frac{\nabla \eta}{\sqrt{1 + |\nabla \eta|^2}} \right) + \frac{1}{2} |\nabla \psi|^2 - \frac{1}{2} \frac{(G(\eta) \psi + \nabla \eta \cdot \nabla \psi)^2}{1 + |\nabla \eta|^2} = 0. \end{cases} \quad (1.37)$$

This is the canonical Hamiltonian system

$$\partial_t \eta = \frac{\delta \mathcal{H}}{\delta \psi}, \quad \partial_t \psi = -\frac{\delta \mathcal{H}}{\delta \eta}, \quad \mathcal{H}(\eta, \psi) := \frac{1}{2} \int_{\mathbb{R}^d} \left[ \psi G(\eta) \psi + \mathfrak{g} \eta^2 + 2\sigma \left( \sqrt{1 + |\nabla \eta|^2} - 1 \right) \right] dx,$$

identified by Zakharov and subsequently developed in the Dirichlet–Neumann formulation by Craig and Sulem [57, 115]. The reduction of the bulk Euler equations to Eq. (1.37) isolates the main analytic difficulty in the nonlinear first-order elliptic operator  $G(\eta)$ .

The first rigorous Cauchy theories for the exact equations concerned analytic data, small data, or special geometries. Foundational contributions were made by Nalimov and Shinbrot [87, 99], and by Reeder and Shinbrot [91, 92]. Further perturbative theories were developed by Kano–Nishida, Yosihara, and Craig [56, 78, 112, 113]. These works established that the exact free-boundary system admits a local dynamics, but they did not yet provide a large-data Sobolev theory at the natural quasilinear level.

The decisive breakthrough was achieved by Wu. For the infinite-depth pure-gravity problem, she proved local well-posedness for Sobolev data, without a smallness assumption, first for a one-dimensional interface and then for a two-dimensional interface [106, 107]. The argument combines a boundary-integral description of the harmonic velocity field with nonlinear changes of variables adapted to the geometry of the interface. In the one-phase irrotational setting, the Taylor sign condition is generated by the equations themselves under the standard geometric assumptions, and this positivity supplies the coercive mechanism for the energy estimates. Wu later proved almost-global existence for small two-dimensional water waves and global existence for small three-dimensional water waves [109, 110], revealing additional nonlinear cancellations that are invisible in a direct Eulerian energy estimate. We also refer to the work of Beyer and Gunther [32].

A complementary geometric and Lagrangian theory was developed for the free-boundary Euler equations, including flows with vorticity. Christodoulou and Lindblad obtained intrinsic a priori estimates under the Taylor sign condition, Lindblad established the corresponding local existence theory, and Coutand and Shkoller treated the free-surface problem with or without surface tension in Lagrangian coordinates [46, 55, 83]. Shatah and Zeng subsequently formulated geometric energies for both free surfaces and fluid interfaces and established local well-posedness in a framework that does not require irrotationality or simple connectivity [96–98]. These approaches remain fundamental, especially when the bulk vorticity is part of the dynamics.

The surface formulation Eq. (1.37) led to a different route, based on Bony’s paradifferential calculus [34]. Lannes derived tame estimates and shape-derivative formulas for the Dirichlet–Neumann operator and used an Alinhac good unknown in a Nash–Moser construction of solutions under the Taylor sign condition [79]. Alazard and Métivier then obtained a precise parilinearization of  $G(\eta)$  and exposed the quasilinear structure of three-dimensional water waves [14]. Denoting Bony’s paraproduct by  $T$ , the notation customary in this literature is

$$\begin{aligned} B &:= \frac{G(\eta) \psi + \nabla \eta \cdot \nabla \psi}{1 + |\nabla \eta|^2}, & V &:= \nabla \psi - B \nabla \eta, \\ U &:= \psi - T_B \eta, & \lambda(\eta; x, \xi) &:= \sqrt{(1 + |\nabla \eta|^2) |\xi|^2 - (\nabla \eta \cdot \xi)^2}. \end{aligned} \quad (1.38)$$

The principal parilinearization takes the form

$$G(\eta) \psi = T_\lambda U - T_V \cdot \nabla \eta + R(\eta, \psi),$$

where the remainder is smoother in the tame sense relevant to the energy method. If  $a$  denotes the Taylor coefficient and  $\ell$  the order-two principal symbol of the linearized curvature operator, the evolution is

thereby reduced to

$$\begin{cases} (\partial_t + T_V \cdot \nabla) \eta - T_\lambda U = f_1, \\ (\partial_t + T_V \cdot \nabla) U + T_a \eta + \sigma T_\ell \eta = f_2. \end{cases} \quad (1.39)$$

The dispersive symbol of the symmetrized system is, at principal level,

$$\gamma \sim \sqrt{\lambda(a + \sigma \ell)};$$

it has order 1/2 in the pure-gravity regime and order 3/2 in the capillary regime. Alazard, Burq, and Zuily used this structure to establish the gravity-capillary and pure-gravity Cauchy theories by a direct paradifferential symmetrization [6, 8]. Métivier's general theory of paradifferential nonlinear systems supplies the corresponding abstract Cauchy framework [84]. In this approach, the good unknown removes the apparent loss in the shape derivative of the Dirichlet–Neumann operator, while the symbolic diagonalization turns the free-boundary Euler equations into a transport-dispersive system amenable to energy and microlocal estimates.

The reduction Eq. (1.39) became a common starting point for several further developments. It is important to distinguish the paradifferential reduction itself from the additional mechanisms used after it: depending on the problem, the latter may be semiclassical Strichartz estimates, Fourier resonances, normal forms, holomorphic coordinates, or KAM reducibility.

At the local level, Alazard, Burq, and Zuily developed semiclassical Strichartz estimates, a Cauchy theory for nonlocalized data, and lower-regularity gravity theories [7, 9, 10]. De Poyferré and Nguyen carried the gravity-capillary reduction down to the quasilinear Sobolev threshold and combined it with Strichartz estimates to allow non-Lipschitz initial velocity traces [58, 59]. Nguyen proved Hadamard well-posedness for the gravity flow constructed by the paradifferential method [88]. Using quantitative paracomposition, he also established a sharp two-dimensional gravity-capillary Cauchy theory [89]. Ai further improved the Sobolev thresholds for gravity waves through refined parametrices and lossless Strichartz estimates [1, 2]. More recently, Ai, Ifrim, and Tataru developed balanced cubic energy estimates and used them in a low-regularity local and global theory for two-dimensional gravity waves [3, 4]. These works progressively move the Cauchy theory below the threshold accessible by a purely Lipschitz-coefficient energy estimate.

For small and localized data on Euclidean domains, the quasilinear analysis—written either in paradifferential form or in equivalent boundary and holomorphic coordinates—has been combined with dispersive decay and nonlinear normal forms. Large-time control in long-wave regimes was established by Alvarez-Samaniego and Lannes [18]. In the pure-gravity problem, Germain, Masmoudi, and Shatah proved global existence in three space dimensions, while Ionescu and Pusateri, Alazard and Delort, Wang obtained global solutions and modified scattering in two space dimensions [12, 13, 68, 76, 103]. We refer to [11, 102, 104] when the system is above a flat bottom and [71] when the system is irrotational with finite bottom. Hunter, Ifrim, and Tataru gave a holomorphic-coordinate proof of local and almost-global existence, which was followed by the global theory of Ifrim and Tataru [73, 74]. For long-term regularity results in three space dimensions, we refer to [70, 116]. Moreover, Wu proved the case that the interface is not in  $C^1$  [111]. For capillary waves, Germain, Masmoudi, and Shatah proved global existence in three space dimensions, Ifrim and Tataru obtained a cubic lifespan in two space dimensions, and Ionescu and Pusateri proved global regularity and modified scattering for the full two-dimensional system [69, 75, 77], and Wang proved global regularity, scattering, and the non-existence of small traveling waves for the three-dimensional finite depth system in [105]. Deng, Ionescu, Pausader, and Pusateri established the corresponding global theory for the three-dimensional gravity-capillary system [61]. In [117], Zhu proved the propagation result for gravity-capillary system of general dimension. The proofs use different nonlinear structures, but all require an accurate identification of the quasilinear principal and subprincipal dynamics before the dispersive analysis can begin.

On compact spatial domains, where linear dispersion does not yield time decay, paradifferential conjugations instead enter Nash–Moser, KAM, and Birkhoff normal-form schemes. Alazard and Baldi constructed gravity-capillary standing waves, Baldi, Berti, Haus, and Montalto constructed time-quasiperiodic pure-gravity waves in finite depth, and Berti and Montalto obtained quasiperiodic gravity-capillary standing waves [5, 22, 30]. Berti and Delort proved almost-global existence for periodic gravity-capillary waves, while Berti, Feola, and Pusateri derived the quartic Birkhoff normal form and the associated long-time stability for periodic pure-gravity waves [25, 26]. For gravity-capillary waves with constant vorticity, Berti, Maspero, and

Murgante developed a Hamiltonian paradifferential Birkhoff normal form and obtained almost-global existence for generic surface tension [29]. Paradifferential reducibility has also been combined with constant-vorticity formulations in the construction of traveling quasiperiodic waves [27]; for a local Hamiltonian formulation with general vorticity and its shallow-water stability, see also [21]. Thus, in the one-phase theory, paradifferential calculus has become the standard device that separates the universal quasilinear geometry from the problem-specific dispersive or Hamiltonian argument. For recent results about paradifferential reducibility and KAM theory, We refer to [16, 17].

### 1.4.3 The two-phase problem

When both densities are positive, the interface carries two harmonic velocity potentials, or equivalently two Dirichlet–Neumann operators, and the jump of tangential velocity becomes an independent dynamical quantity. This changes the principal symbol in an essential way: in addition to the Rayleigh–Taylor restoring mechanism, it contains the destabilizing Kelvin–Helmholtz contribution generated by the velocity jump. The Hamiltonian structure of the two-fluid problem was clarified by Benjamin and Bridges [23], but the analytic behavior differs sharply from the one-phase endpoint.

In the genuine vortex-sheet regime, a nonzero tangential velocity jump and the absence of surface tension produce a high-frequency Kelvin–Helmholtz instability [86]. Local solutions can be constructed in analytic classes, as in the work of Sulem, Sulem, Bardos, and Frisch [100, 101]; Caflisch and Orellana subsequently obtained a long-time analytic theory for small perturbations of a flat sheet [36]. The complementary construction of singular solutions by Caflisch and Orellana proves ill-posedness in Sobolev spaces [37]. Lebeau later showed that, under a priori Hölder regularity assumptions, any local two-dimensional Kelvin–Helmholtz solution must in fact have an analytic interface and analytic sheet strength [82]; see also Wu’s survey [108]. This rigidity explains why one cannot expect a generic Sobolev Cauchy theory for unregularized vortex sheets.

Surface tension suppresses the short-wave instability and restores a quasilinear energy theory. Ambrose established local well-posedness for two-dimensional periodic vortex sheets with surface tension, and Ambrose and Masmoudi treated three-dimensional vortex sheets [19, 20]. Cheng, Coutand, and Shkoller proved local well-posedness for three-dimensional vortex sheets with surface tension in the presence of bulk vorticity [45]. The geometric framework of Shatah and Zeng gives a further local theory for fluid interfaces with surface tension without assuming irrotationality or simple connectivity [97, 98]. Pusateri obtained estimates uniform in the surface-tension coefficient and in the density of the lighter phase, and used them to justify the simultaneous one-fluid and zero-surface-tension limit [90].

A central paradifferential contribution in the two-phase setting is Lannes’ nonlinear stability criterion [80]. Working with the two Dirichlet–Neumann operators, Lannes derived a symbolic condition that combines the Rayleigh–Taylor coefficient, the tangential velocity jump, the density contrast, and surface tension. The criterion shows that gravity can control the shear at low and intermediate frequencies, whereas capillarity controls the high-frequency Kelvin–Helmholtz component; it also yields estimates adapted to singular asymptotic regimes and requires a symbolic decomposition of the bottom contribution into a principal part and an infinitely smoothing tail. This work provides the closest Dirichlet–Neumann analogue of the stability mechanism used below. Two-fluid geometry also imposes restrictions absent from the one-phase problem: for instance, locally smooth two-dimensional interfaces with two positive densities cannot develop a splash singularity [63].

The energy theories of Ambrose and of Ambrose–Masmoudi are already direct interface arguments. What remained much less developed than in the Dirichlet–Neumann setting was a systematic symbolic procedure that converts the nonlinear Birkhoff–Rott operators themselves into paradifferential operators while retaining their adjoint and parity structure. A recent step in this direction is the work of Murgante, Roulley, and Scrobogna on vortex sheets close to a circle, where a direct parilinearization of nonlinear singular integral operators is combined with Hamiltonian Birkhoff normal forms to obtain almost-global stability below a Weber threshold [85].

The present paper addresses the corresponding methodological problem for three-dimensional graph interfaces. Rather than first replacing the contour dynamics by a Dirichlet–Neumann system, we parilinearize the singular integral operators themselves. The abstract results of Theorems 5.7 and B.1 isolate the homogeneous singular structure, preserve the adjoint and parity information needed by the energy method,

and reduce nonhomogeneous tails to order-zero paraproducts modulo smoothing remainders. Applied to the two-phase contour-dynamics equation, this yields the complete quasilinear symbolic system and the Sobolev local well-posedness result proved in the sequel.

## 1.5 Structure of the paper

In Section 2, we recast Eq. (1.9) as an autonomous Contour Dynamics Equation (CDE) in canonical surface variables. Section 3 is devoted to the linearization of the canonical CDE at the zero equilibrium and to the analysis of the associated dispersion and stability conditions. In Section 4, we introduce the functional setting and the paradifferential tools used throughout the paper.

The main methodological part of the analysis is contained in Section 5. We first develop an abstract structure-preserving parilinearization theorem for critical homogeneous graph integral operators on  $\mathbb{R}^d$ . We then apply this result to the singular integral operators entering the contour dynamics and derive the complete paradifferential reduction of the canonical CDE. In Section 6, we introduce the Alinhac good unknown, reduce the resulting system to a diagonal complex form, establish the modified energy estimates, and conclude the proof of Theorem 1.12. Finally, Appendix A is devoted to prove some technical lemmas and Appendix B gives the subcritical locally integrable counterpart of the abstract parilinearization theorem.

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## 2 Recasting Eq. (1.9) as a Contour Dynamic Equation (CDE)

It is well-known that the irrotationality assumption in Eq. (1.9) allows us to recast Eq. (1.9) as a system of contour dynamics equations of the interface  $\Gamma(t)$ . Our purpose here is to describe this formulation. We consider

$$\mathbf{v}^\pm(t, x) := \mathbf{u}^\pm(t, \boldsymbol{\gamma}(t, x)), \quad (2.1a)$$

$$q^\pm(t, x) := p^\pm(t, \boldsymbol{\gamma}(t, x)),$$

$$\psi^\pm(t, x) := \phi^\pm(t, \boldsymbol{\gamma}(t, x)), \quad (2.1b)$$

respectively, the trace of the velocity field, the pressure and the potential onto the interface  $\Gamma(t)$ .

Next, following [80], we introduce the quantity

$$\psi := \frac{\llbracket \rho^\pm \psi^\pm \rrbracket}{\langle\langle \rho^\pm \rangle\rangle}. \quad (2.2)$$

**Notation 2.1.** Throughout this section, we shall make appeal to the following notations.

**i** Let us define

$$\delta_z f(x) := f(x) - f(x - z), \quad \Delta_z f(x) := \frac{f(x) - f(x - z)}{|z|}, \quad x, z \in \mathbb{R}^2.$$

ii Let  $\eta \in B_{\dot{W}^{1,\infty}(\mathbb{R}^2;\mathbb{R})}(0,1)$  and a scalar function  $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ , let us define the following operators of nonlinear-convolution type

$$\mathcal{B}_0(\eta)[\omega](x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{z}{\left(|z|^2 + (\delta_z \eta(x))^2\right)^{\frac{3}{2}}} \omega(x-z) \, dz, \quad (2.3a)$$

$$q(\eta)[\omega](x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\delta_z \eta(x)}{\left(|z|^2 + (\delta_z \eta(x))^2\right)^{\frac{3}{2}}} \omega(x-z) \, dz, \quad (2.3b)$$

where  $\delta_z \eta$  is defined in Item i. We also denote by  $\mathfrak{R}$  the vector-valued Riesz transform,

$$\mathfrak{R}\omega := \begin{bmatrix} \mathfrak{R}_1 \omega \\ \mathfrak{R}_2 \omega \end{bmatrix}, \quad \mathfrak{R}_j \omega(x) := \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{z_j}{|z|^3} \omega(x-z) \, dz, \quad (2.4)$$

that is a Fourier multiplier given by

$$\widehat{\mathfrak{R}_j \omega}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{\omega}(\xi).$$

In particular, we have

$$\mathcal{B}_0(0) = \frac{\mathfrak{R}}{2}, \quad q(0) = 0. \quad (2.5)$$

As a consequence,

$$\mathcal{B}_0(0)[1] \equiv 0. \quad (2.6)$$

The main goal of this section is to prove the following theorem, which is about reformulating the system Eq. (1.9) into the CDEs for  $\eta$  and  $\psi$ .

**Theorem 2.2** (The Vortex Sheet equations as a canonical CDE with background vorticity). *Let  $\sigma, \rho^\pm \geq 0$ ,  $\mathbf{g} \in \mathbb{R}$ ,  $\bar{\Omega} \in \mathbb{R}^2$ , and let  $(\mathbf{u}^\pm, p^\pm, \boldsymbol{\gamma})$  be a classical solution of Eq. (1.9). Let  $\eta$  and  $\psi$  be defined as in Eq. (1.1) and Eq. (2.2). Define the affine background potential*

$$\bar{\psi}(x) := x^\perp \cdot \bar{\Omega}, \quad \nabla^\perp \bar{\psi} = \bar{\Omega} \quad (2.7)$$

and the canonical perturbation

$$\tilde{\psi} := \psi - \bar{\psi}. \quad (2.8)$$

There exist  $s_0 > 0$  and  $r_0 > 0$  such that, if

$$\|\eta\|_{s_0, I} \leq r_0,$$

then the set of equations Eq. (1.9) is equivalent to the canonical contour-dynamics system

$$\begin{cases} \eta_t = \mathcal{G}^*(\eta)[\bar{\Omega}[\bar{\Omega}](\eta)] + \mathcal{G}(\mathfrak{a}|\eta)[\tilde{\psi}], \\ \tilde{\psi}_t = \tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}). \end{cases} \quad (2.9a)$$

$$(2.9b)$$

The objects appearing in Eq. (2.9) are defined as follows.

- The vorticity variable carried by the canonical slice  $\{\tilde{\psi} = 0\}$  is

$$\bar{\Omega}[\bar{\Omega}](\eta) := S(\mathfrak{a}|\eta)\bar{\psi}, \quad (2.10)$$

where

- the potential-to-vorticity operator  $S(\mathfrak{a}|\eta)$  is defined by

$$S(\mathfrak{a}|\eta)\psi := \nabla^\perp (1 + \mathfrak{a}\tau_0(\eta))^{-1} \psi. \quad (2.11)$$

- the operator  $\tau_0(\eta)$  is defined as

$$\tau_0(\eta)\psi := \Delta^{-1} \nabla^\perp \cdot (\mathcal{B}_0(\eta)[\nabla \eta \cdot \nabla^\perp \psi] - q(\eta) \nabla^\perp \psi + (\mathcal{B}_0(\eta) \cdot \nabla \psi) \nabla^\perp \eta), \quad (2.12)$$

where  $\mathcal{B}_0(\eta)$  and  $q(\eta)$  are defined in Eq. (2.3a) and Eq. (2.3b), respectively.

- The operator  $g^\star(\eta)$  is the normal velocity operator acting on vorticity variables and is defined by

$$g^\star(\eta) \Omega := \mathbf{v}^\star[\Omega](\eta) \cdot \begin{bmatrix} \nabla \eta \\ -1 \end{bmatrix}, \quad (2.13)$$

where the auxiliary velocity trace  $\mathbf{v}^\star[\Omega](\eta)$  is

$$\mathbf{v}^\star[\Omega](\eta) := \begin{bmatrix} V^\star[\Omega](\eta) \\ \nu_3^\star[\Omega](\eta) \end{bmatrix} = \begin{bmatrix} (\mathcal{B}_0(\eta) [\nabla \eta \cdot \Omega] - q(\eta) \Omega)^\perp \\ -\mathcal{B}_0(\eta)^\perp \cdot \Omega \end{bmatrix}, \quad (2.14)$$

where  $\mathcal{B}_0(\eta)$  and  $q(\eta)$  are defined in Eq. (2.3a) and Eq. (2.3b), respectively.

- The operator  $g(\mathfrak{a}|\eta)$  is the generalized Dirichlet–Neumann operator in canonical potential variables and is defined by

$$g(\mathfrak{a}|\eta) \psi := g^\star(\eta) [S(\mathfrak{a}|\eta) \psi]. \quad (2.15)$$

Equivalently,

$$\mathbf{v}[\psi](\eta) := \begin{bmatrix} V[\psi](\mathfrak{a}|\eta) \\ \nu_3[\psi](\mathfrak{a}|\eta) \end{bmatrix} := \mathbf{v}^\star[S(\mathfrak{a}|\eta) \psi](\eta),$$

and

$$g(\mathfrak{a}|\eta) \psi = \mathbf{v}[\psi](\eta) \cdot \begin{bmatrix} \nabla \eta \\ -1 \end{bmatrix}.$$

- The shifted Bernoulli function in canonical variables is

$$\tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}) := \pi^\star(\rho^\pm|\eta, \bar{\Omega}[\bar{\Omega}](\eta) + S(\mathfrak{a}|\eta) \tilde{\psi}) - c_{\bar{\Omega}}, \quad (2.16)$$

where  $\pi^\star$  is given by

$$\pi^\star(\rho^\pm|\eta, \Omega) := \pi_0(\rho^\pm|\eta) + \pi_2^\star(\mathfrak{a}|\eta) [\Omega, \Omega],$$

with

- the gravity-curvature term  $\pi_0$  is defined as

$$\pi_0(\rho^\pm|\eta) := \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} \nabla \cdot \mathcal{K}(\nabla \eta) - \mathfrak{g} \mathfrak{a} \eta, \quad (2.17)$$

where  $\mathcal{K}(\nabla \eta)$  is defined in Eq. (1.7).

- the symmetric bilinear form  $\pi_2^\star$  is

$$\pi_2^\star(\mathfrak{a}|\eta) [\Omega_1, \Omega_2] := -\pi_{2;\text{I}}^\star(\eta) [\Omega_1, \Omega_2] - \pi_{2;\text{II}}^\star(\mathfrak{a}|\eta) [\Omega_1, \Omega_2], \quad (2.18)$$

with

$$\pi_{2;\text{I}}^\star(\eta) [\Omega_1, \Omega_2] := \frac{\Omega_1^\perp \cdot V^\star[\Omega_2](\eta) + \Omega_2^\perp \cdot V^\star[\Omega_1](\eta)}{2}, \quad (2.19)$$

$$\begin{aligned} \pi_{2;\text{II}}^\star(\mathfrak{a}|\eta) [\Omega_1, \Omega_2] := & \frac{\mathfrak{a}}{4} \left[ 2 V^\star[\Omega_1](\eta) \cdot V^\star[\Omega_2](\eta) \right. \\ & - \nu_3^\star[\Omega_1](\eta) (\nu_3^\star[\Omega_2](\eta) - 2 V^\star[\Omega_2](\eta) \cdot \nabla \eta) \\ & - \nu_3^\star[\Omega_2](\eta) (\nu_3^\star[\Omega_1](\eta) - 2 V^\star[\Omega_1](\eta) \cdot \nabla \eta) \\ & \left. + \frac{1}{2} \frac{\Omega_1^\top (\text{Id}_{\mathbb{R}^2} + \nabla \eta \otimes \nabla \eta) \Omega_2}{1 + |\nabla \eta|^2} \right]. \end{aligned} \quad (2.20)$$

- the constant gauge is

$$c_{\bar{\Omega}} := \pi^\star(\rho^\pm|0, \bar{\Omega}) = -\frac{\mathfrak{a}}{8} |\bar{\Omega}|^2. \quad (2.21)$$

In particular, subtracting  $c_{\bar{\Omega}}$  in Eq. (2.16) makes  $(\eta, \tilde{\psi}) = (0, 0)$  a stationary trivial solution.

Note that  $\tilde{\pi}$  decomposes according to homogeneity in  $\tilde{\psi}$  as

$$\tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}) = \tilde{\pi}_0(\rho^\pm, \bar{\Omega}|\eta) + \tilde{\pi}_1(\alpha, \bar{\Omega}|\eta)[\tilde{\psi}] + \tilde{\pi}_2(\alpha|\eta)[\tilde{\psi}, \tilde{\psi}], \quad (2.22)$$

where

$$\tilde{\pi}_0(\rho^\pm, \bar{\Omega}|\eta) := \pi_0(\rho^\pm|\eta) + \pi_2^*(\alpha|\eta)[\tilde{\Omega}[\bar{\Omega}](\eta), \tilde{\Omega}[\bar{\Omega}](\eta)] - c_{\bar{\Omega}}, \quad (2.23a)$$

$$\tilde{\pi}_1(\alpha, \bar{\Omega}|\eta)[\tilde{\psi}] := 2\pi_2^*(\alpha|\eta)[\tilde{\Omega}[\bar{\Omega}](\eta), S(\alpha|\eta)\tilde{\psi}], \quad (2.23b)$$

$$\tilde{\pi}_2(\alpha|\eta)[\tilde{\psi}_1, \tilde{\psi}_2] := \pi_2^*(\alpha|\eta)[S(\alpha|\eta)\tilde{\psi}_1, S(\alpha|\eta)\tilde{\psi}_2]. \quad (2.23c)$$

In particular, for fixed  $\eta$  and  $\bar{\Omega}$ , the term  $\tilde{\pi}_j$  is  $j$ -homogeneous in  $\tilde{\psi}$  for  $j = 0, 1, 2$ .

**Remark 2.3.** The formulation Eq. (2.9) constitutes a primary novelty of this work. Departing from verifying abstract properties of the Dirichlet-to-Neumann operator, we provide in Eq. (2.15) an explicit representation of  $\mathcal{J}(\alpha|\eta)$  in terms of the singular integral operators  $\mathcal{B}_0$  and  $q$ . While this approach introduces significant algebraic complexity, it is essential for capturing the precise parilinear structure of the system in the presence of vorticity.  $\blacklozenge$

Let us outline the structure of the rest of this section.

- In Section 2.1, we describe the trace of the velocity field on the interface.
- In Section 2.2, we derive the CDE for the elevation function  $\eta$ .
- In Section 2.3, we derive the CDE for the vorticity variable  $\Omega$ .
- In Section 2.4, we derive the CDE for the weighted trace of the potential  $\psi$ .
- In Section 2.5, we recast the potential in term of vorticity.
- In Section 2.6, we prove the main result, Theorem 2.2.
- In Section 2.7, we record the canonical background formulation summarized in Proposition 2.12.

**Remark 2.4.** The operators  $\mathcal{B}_0(\eta)$  and  $q(\eta)$  defined in Eq. (2.3a) and Eq. (2.3b) are crucial since they are the only elements of Eqs. (2.9a) and (2.9b) that resist standard parilinearization techniques (Bony paraproduct decomposition, Bony parilinearization formula, etc.). Hence a major part of the present manuscript consists on the analysis of these two operators. For this reason in the present remark we lay out some informal computations that are useful in the forthcoming analysis. Notice that using the simplification  $\delta_z \eta \approx \nabla \eta \cdot z$  we obtain from Eqs. (2.9a) and (2.9b) that

$$q(\eta)\omega \approx \nabla \eta \cdot \mathcal{B}_0(\eta)\omega, \quad (2.24)$$

modulo lower order terms. The approximation in Eq. (2.24) is useful in the sense that it relates the operators in Eqs. (2.3a) and (2.3b) via a concrete relation, hence suggesting that the parilinearization of  $\mathcal{B}_0$  entails the parilinearization for  $q$ , thus cutting in half the computations required for the manuscript. The relation Eq. (2.24) shall be made formal, and transparent, at the paradifferential level in Lemma 5.17 below.  $\blacklozenge$

## 2.1 Trace of the velocity field

To prove Theorem 2.2, we need to start by understanding the velocity field on the interface. This is a classical fact that for vortex-sheets (see for instance [33, 94, 95, 108] for the bidimensional case) this latter can be expressed in terms of the Birkhoff-Rott operator. From Eq. (1.4), we have

$$[\![\mathbf{v}^\pm]\!] \cdot \mathbf{v} = 0.$$

Consider the vortex sheet strength  $\omega$  defined on  $\Gamma(t)$  as

$$\omega := -[\![\mathbf{v}^\pm]\!] \times \mathbf{v} := \begin{bmatrix} \Omega \\ \omega_3 \end{bmatrix}. \quad (2.25)$$

Together with the irrotationality conditions in Eq. (1.9), the global vorticity field  $\vec{\omega}(t, \mathbf{x})$ , defined in the sense of distributions as the curl of the global velocity field  $\mathbf{u}^\pm$ , is supported entirely on the interface  $\Gamma(t)$  and is described by the following Dirac distribution:

$$\vec{\omega}(t, \mathbf{x}) = \omega(t, x) \delta_0(\mathbf{x} - \boldsymbol{\gamma}(t, x)).$$

The definition Eq. (2.25) combined with the orthogonality properties of the cross product imply that

$$\boldsymbol{\omega} \cdot \mathbf{v} = 0. \quad (2.26)$$

Therefore, putting together Eqs. (1.4), (2.25) and (2.26) give

$$\omega_3 = \nabla \eta \cdot \Omega \quad \text{and} \quad \frac{\mathbf{v} \times \boldsymbol{\omega}}{|\mathbf{v}|^2} \times \mathbf{v} = \boldsymbol{\omega}. \quad (2.27)$$

Next, for  $\mathbf{x} \in \Omega^\pm(t)$ , we define

$$\mathbf{u}^\pm(t, \mathbf{x}) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\mathbf{x} - \boldsymbol{\gamma}(t, y)}{|\mathbf{x} - \boldsymbol{\gamma}(t, y)|^3} \times \boldsymbol{\omega}(t, y) \, dy. \quad (2.28)$$

Let us define the *Birkhoff-Rott integral operator*

$$\text{BR}(\boldsymbol{\gamma})[\omega](t, x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\boldsymbol{\gamma}(t, x) - \boldsymbol{\gamma}(t, y)}{|\boldsymbol{\gamma}(t, x) - \boldsymbol{\gamma}(t, y)|^3} \times \boldsymbol{\omega}(t, y) \, dy. \quad (2.29)$$

Our aim is to prove the following lemma about the velocity field on the interface.

**Lemma 2.5** (Evaluation of the trace of the velocity field). *Let  $(\mathbf{u}^\pm, p^\pm, \boldsymbol{\gamma})$  be a classical solution of Eq. (1.9). Let  $\mathbf{v}^\pm, \boldsymbol{\omega}, \mathbf{u}^\pm, \text{BR}(\cdot)$  be defined as in Eqs. (2.1a), (2.25), (2.28) and (2.29). Then we have, for any  $(t, \mathbf{x}) \in I \times \Omega(t)^\pm$ , that*

$$\mathbf{u}^\pm(t, \mathbf{x}) = \mathbf{u}^\pm(t, \mathbf{x}).$$

Moreover, we have

$$\mathbf{v}^\pm = \text{BR}(\boldsymbol{\gamma})\boldsymbol{\omega} \pm \frac{1}{2} \frac{\mathbf{v} \times \boldsymbol{\omega}}{|\mathbf{v}|^2}. \quad (2.30)$$

*Proof.* For simplicity of notation, we assume that  $t \in I$  is fixed in the rest of the argument, and we drop the  $t$  dependence of the functions. In the rest of the proof, the convergence is always point-wise for all  $x \in \mathbb{R}^2$ . From direct calculations, we have

$$\text{div}_x \mathbf{u}^\pm = 0, \quad \nabla_x \times \mathbf{u}^\pm = 0 \quad \text{on} \quad \Omega^\pm.$$

Since we assume  $\boldsymbol{\gamma}$  to be at least  $\mathcal{C}^1$ , we have that there exists a  $\varepsilon_0 > 0$ , such that for any  $\varepsilon \in (0, \varepsilon_0)$  and  $x \in \mathbb{R}^2$ ,

$$\boldsymbol{\gamma}(x) \pm \varepsilon \mathbf{v}(x) \in \Omega^\pm.$$

Next, we study the limit  $\varepsilon \downarrow 0$ . For any  $x \in \mathbb{R}^2$ , we calculate the limit  $\lim_{\varepsilon \downarrow 0} \mathbf{u}^\pm(\boldsymbol{\gamma}(x) \pm \varepsilon \mathbf{v}(x))$ . We can split

$$\mathbf{u}^-(\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x)) = I_1^\varepsilon(x) + I_2^\varepsilon(x),$$

where

$$I_1^\varepsilon(x) = \frac{1}{4\pi} \text{p.v.} \int_{|y-x| > \varepsilon^\delta |\mathbf{v}(x)|} \frac{\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)}{|\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)|^3} \times \boldsymbol{\omega}(y) \, dy,$$

$$I_2^\varepsilon(x) = \frac{1}{4\pi} \text{p.v.} \int_{|y-x| \leq \varepsilon^\delta |\mathbf{v}(x)|} \frac{\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)}{|\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)|^3} \times \boldsymbol{\omega}(y) \, dy.$$

Here, we have introduced a parameter  $\delta \in (\frac{2}{3}, 1)$ .

**Part 1** (Estimate of  $I_1^\varepsilon$ ). We first show that  $I_1^\varepsilon \rightarrow \text{BR}(\boldsymbol{\gamma})\omega$  point-wise as  $\varepsilon \downarrow 0$ . Observe from Eq. (1.1), that

$$|\boldsymbol{\gamma}(y) - \boldsymbol{\gamma}(x)| \geq |y - x|. \quad (2.31)$$

Hence, for  $|y - x| > \varepsilon^\delta |\mathbf{v}(x)|$ , we have

$$\frac{\varepsilon |\mathbf{v}(x)|}{|\boldsymbol{\gamma}(y) - \boldsymbol{\gamma}(x)|} \leq \varepsilon^{1-\delta} \rightarrow 0 \quad \text{as} \quad \varepsilon \downarrow 0. \quad (2.32)$$

Define the function  $f_i(u) := \frac{u_i}{|u|^3}$ . Then, by Taylor integral formula, we have

$$f_i(\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)) - f_i(\boldsymbol{\gamma}(x) - \boldsymbol{\gamma}(y)) = -\varepsilon \int_0^1 \mathbf{v}(x) \cdot \nabla f_i(\boldsymbol{\gamma}(x) - t\varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)) dt.$$

Thus, denoting  $f := (f_1, f_2)$ , we obtain that

$$I_1^\varepsilon(x) := I_{1,1}^\varepsilon(x) + I_{1,2}^\varepsilon(x),$$

where

$$I_{1,1}^\varepsilon(x) := \frac{1}{4\pi} \text{p.v.} \int_{|y-x| > \varepsilon^\delta |\mathbf{v}(x)|} \frac{\boldsymbol{\gamma}(x) - \boldsymbol{\gamma}(y)}{|\boldsymbol{\gamma}(x) - \boldsymbol{\gamma}(y)|^3} \times \omega(y) dy,$$

and

$$I_{1,2}^\varepsilon(x) := -\frac{\varepsilon}{4\pi} \text{p.v.} \int_{|y-x| > \varepsilon^\delta |\mathbf{v}(x)|} \int_0^1 df(\boldsymbol{\gamma}(x) - t\varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)) [\mathbf{v}(x)] dt \times \omega(y) dy.$$

By definition of the principle value, we have  $I_{1,1}^\varepsilon(x) \rightarrow \text{BR}(\boldsymbol{\gamma})[\omega](x)$  as  $\varepsilon \downarrow 0$ . For the term  $I_{1,2}^\varepsilon$ , first we have for  $i, j = 1, 2, 3$ ,  $\partial_j f_i(u) = O(|u|^{-3})$ . So together with Eq. (2.31), Eq. (2.32) and  $t \in [0, 1]$ , we obtain that

$$\|df(\boldsymbol{\gamma}(x) - t\varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y))\|_{\mathcal{L}(\mathbb{R}^3)} \lesssim |\boldsymbol{\gamma}(x) - t\varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)|^{-3} \lesssim |\boldsymbol{\gamma}(x) - \boldsymbol{\gamma}(y)|^{-3} \lesssim |y - x|^{-3}.$$

Notice that

$$\int_{\varepsilon^\delta}^\infty \frac{\varepsilon}{r^2} dr = \varepsilon^{1-\delta} \rightarrow 0, \quad \text{as} \quad \varepsilon \downarrow 0.$$

Therefore, passing to the polar coordinates  $y - x = r \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ , we obtain that  $I_{1,2}^\varepsilon(x) \rightarrow 0$  as  $\varepsilon \downarrow 0$ , which means that  $I_1^\varepsilon(x) \rightarrow \text{BR}(\boldsymbol{\gamma})[\omega](x)$  as  $\varepsilon \downarrow 0$ .

**Part 2** (Estimate of  $I_2^\varepsilon$ ). We write

$$I_2^\varepsilon(x) := I_{2,1}^\varepsilon(x) + I_{2,2}^\varepsilon(x),$$

where

$$I_{2,1}^\varepsilon(x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \mathbf{1}_{|y-x| \leq \varepsilon^\delta |\mathbf{v}(x)|} \frac{\boldsymbol{\gamma}(x) - \boldsymbol{\gamma}(y)}{|\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)|^3} \times \omega(y) dy,$$

and

$$I_{2,2}^\varepsilon(x) := -\frac{\varepsilon}{4\pi} \mathbf{v}(x) \times \text{p.v.} \int_{\mathbb{R}^2} \omega(y) \mathbf{1}_{|y-x| \leq \varepsilon^\delta |\mathbf{v}(x)|} \frac{1}{|\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)|^3} dy.$$

We first estimate  $I_{2,1}^\varepsilon(x)$ . From Eq. (1.1) and Eq. (1.4), we can write

$$\begin{aligned} |\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)|^2 &= |y - x - \varepsilon \nabla \eta(x)|^2 + (\eta(y) - \eta(x) + \varepsilon)^2 \\ &= |y - x|^2 + \varepsilon^2 |\mathbf{v}(x)|^2 + |\eta(y) - \eta(x)|^2 + 2\varepsilon (\eta(y) - \eta(x) - \nabla \eta(x) \cdot (y - x)). \end{aligned}$$

By Taylor expansion, we have

$$\eta(y) = \eta(x) + \nabla \eta(x) \cdot (y - x) + O(|y - x|^2).$$

Moreover,

$$|y - x|^2 + (\nabla \eta(x) \cdot (y - x))^2 = \sum_{i=1}^2 \left(1 + |\partial_i \eta(x)|^2\right) (y_i - x_i)^2 + 2\partial_1 \eta(x) \partial_2 \eta(x) (y_1 - x_1) (y_2 - x_2).$$

For the symmetric matrix

$$\begin{pmatrix} 1 + |\partial_1 \eta(x)|^2 & \partial_1 \eta(x) \partial_2 \eta(x) \\ \partial_1 \eta(x) \partial_2 \eta(x) & 1 + |\partial_2 \eta(x)|^2 \end{pmatrix},$$

its two eigenvalues are  $|\mathbf{v}(x)|^2$  and 1. Under the orthogonal transformation, we only need to consider  $|\mathbf{v}(x)|^2 (y_1 - x_1)^2 + (y_2 - x_2)^2$ . Let  $\tilde{y} := (\tilde{y}_1, \tilde{y}_2)$ , with

$$\tilde{y}_1 := |\mathbf{v}(x)|^{\frac{1}{2}} (y_1 - x_1), \quad \tilde{y}_2 := |\mathbf{v}(x)|^{-\frac{1}{2}} (y_2 - x_2).$$

Then, we have

$$|\mathbf{v}(x)|^2 (y_1 - x_1)^2 + (y_2 - x_2)^2 = |\mathbf{v}(x)| |\tilde{y}|^2.$$

Remind that  $\eta \in H^{s_0}$ , we have  $\|\nabla \eta\|_{L^\infty} < \infty$ . Denote  $c := \left(1 + \|\nabla \eta\|_{L^\infty}^2\right)^{\frac{1}{4}}$ , then  $|\mathbf{v}(x)|^{\frac{1}{2}} \leq c$  for any  $x \in \mathbb{R}^2$ . Thus

$$|y - x|^2 (\nabla \eta(x) \cdot (y - x)) = O(|y - x|^3).$$

Recalling that  $|y - x| < \varepsilon^\delta |\mathbf{v}(x)|$  and  $\delta > \frac{2}{3}$ , we obtain that

$$|\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x) - \boldsymbol{\gamma}(y)|^2 \approx |\mathbf{v}(x)| |\tilde{y}|^2 + |\mathbf{v}(x)|^2 \varepsilon^2.$$

The integration region of  $\tilde{y}$  is an ellipse, located between two circles with radii  $\frac{1}{c} \varepsilon^\delta |\mathbf{v}(x)|$  and  $c \varepsilon^\delta |\mathbf{v}(x)|$ .

Next, we denote the polar coordinates  $\tilde{y} = r \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$ . For any constant  $C > 0$  that does not depend on  $\varepsilon$ ,

From symmetry, we have

$$\int_0^{C\varepsilon^\delta} \int_0^{2\pi} \frac{r^2 (\cos \theta, \sin \theta)}{(|\mathbf{v}(x)| r^2 + |\mathbf{v}(x)|^2 \varepsilon^2)^{\frac{3}{2}}} dr d\theta = 0.$$

Moreover, as  $\varepsilon \downarrow 0$ , we have

$$\int_0^{C\varepsilon^\delta} \frac{r^3}{(|\mathbf{v}(x)| r^2 + |\mathbf{v}(x)|^2 \varepsilon^2)^{\frac{3}{2}}} dr < \frac{1}{|\mathbf{v}(x)|^{3/2}} \int_0^{C\varepsilon^\delta} 1 dr \leq C\varepsilon^\delta \rightarrow 0.$$

As a consequence,  $I_{2,1}^\varepsilon(x) \rightarrow 0$  as  $\varepsilon \downarrow 0$ . To compute the limit of  $I_{2,2}^\varepsilon(x)$ , we only need to study the limit of

$$-\frac{\mathbf{v}(x)}{4\pi} \times \text{p.v.} \int_0^{2\pi} \int_0^{C\varepsilon^\delta} \frac{\varepsilon r \omega(x + |\mathbf{v}(x)| r (\cos \theta, \sin \theta))}{(|\mathbf{v}(x)| r^2 + |\mathbf{v}(x)|^2 \varepsilon^2)^{\frac{3}{2}}} dr d\theta$$

as  $\varepsilon \downarrow 0$ . Notice that

$$\begin{aligned} \int_0^{C\varepsilon^\delta} \frac{\varepsilon r}{(|\mathbf{v}(x)| r^2 + |\mathbf{v}(x)|^2 \varepsilon^2)^{\frac{3}{2}}} dr &\stackrel{r=\varepsilon |\mathbf{v}(x)|^{\frac{1}{2}} r'}{\longrightarrow} \frac{1}{|\mathbf{v}(x)|^2} \int_0^{\frac{C\varepsilon^{\delta-1}}{|\mathbf{v}(x)|^{1/2}}} \frac{r'}{(r'^2 + 1)^{\frac{3}{2}}} dr' \\ &\stackrel{\varepsilon \downarrow 0}{\longrightarrow} \frac{1}{|\mathbf{v}(x)|^2} \int_0^\infty \frac{r'}{(r'^2 + 1)^{\frac{3}{2}}} dr' = \frac{1}{|\mathbf{v}(x)|^2}. \end{aligned}$$

So we obtain that

$$\lim_{\varepsilon \rightarrow 0^+} I_2^\varepsilon(x) = \lim_{\varepsilon \rightarrow 0^+} I_{2,2}^\varepsilon(x) = -\frac{1}{2} \frac{\mathbf{v}(x) \times \omega(x)}{|\mathbf{v}(x)|^2}.$$

**Part 3 (Synthesis).** Combining the foregoing calculations, we conclude that

$$\lim_{\varepsilon \downarrow 0} \mathbf{u}^- (\boldsymbol{\gamma}(x) - \varepsilon \mathbf{v}(x)) = \text{BR}(\boldsymbol{\gamma})[\omega](x) - \frac{1}{2} \frac{\mathbf{v}(x) \times \omega(x)}{|\mathbf{v}(x)|^2}. \quad (2.33)$$

Similarly, we obtain

$$\lim_{\varepsilon \downarrow 0} \mathbf{u}^+ (\boldsymbol{\gamma}(x) + \varepsilon \mathbf{v}(x)) = \text{BR}(\boldsymbol{\gamma})[\omega](x) + \frac{1}{2} \frac{\mathbf{v}(x) \times \omega(x)}{|\mathbf{v}(x)|^2}. \quad (2.34)$$

Let  $\mathbf{w}^\pm := \mathbf{u}^\pm - \mathbf{u}^\pm$  in  $\Omega^\pm$ . By the foregoing trace formulas Eqs. (2.33) and (2.34),

$$\llbracket \mathbf{u}^\pm \rrbracket_\Gamma = -\frac{\mathbf{v} \times \boldsymbol{\omega}}{|\mathbf{v}|^2}.$$

On the other hand, recalling that  $\llbracket \mathbf{v}^\pm \rrbracket \cdot \mathbf{v} = 0$ , from Eq. (2.25) we obtain

$$\llbracket \mathbf{v}^\pm \rrbracket = -\frac{\mathbf{v} \times \boldsymbol{\omega}}{|\mathbf{v}|^2}.$$

Therefore  $\llbracket \mathbf{w}^\pm \rrbracket_\Gamma = 0$ , and we may define the continuous vector field

$$\mathbf{w} := \mathbf{w}^- \mathbf{1}_{\Omega^-} + \mathbf{w}^+ \mathbf{1}_{\Omega^+} \quad \text{on } \mathbb{R}^3.$$

Since both  $\mathbf{u}^\pm$  and  $\mathbf{u}^\pm$  are divergence-free and irrotational in  $\Omega^\pm$ , and since  $\mathbf{w}$  has no jump across  $\Gamma$ , we have  $\operatorname{div}_x \mathbf{w} = 0$  and  $\nabla_x \times \mathbf{w} = 0$  in the sense of distributions on  $\mathbb{R}^3$ . Consequently,

$$\Delta_x \mathbf{w} = \nabla_x (\operatorname{div}_x \mathbf{w}) - \nabla_x \times (\nabla_x \times \mathbf{w}) = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3).$$

Hence  $\mathbf{w}$  is harmonic on  $\mathbb{R}^3$ . By the standing far-field normalization,  $\mathbf{w}$  is bounded. Liouville's theorem therefore implies that  $\mathbf{w}$  is constant. The same normalization fixes the common constant velocity to be zero, and hence  $\mathbf{w} \equiv 0$ . Thus  $\mathbf{u}^\pm = \mathbf{u}^\pm$  in  $\Omega^\pm$ , and the trace formulas computed above yield Eq. (2.30).  $\square$

## 2.2 The CDE for the elevation $\eta$

Our next goal is to find the CDE for the elevation function  $\eta$ . It is obtained by particularizing the transport equation Eq. (1.4) in our context via the explicit computation of the Birkhoff-Rott operator Eq. (2.29). More precisely, the result reads as follows.

**Lemma 2.6** (Derivation of the CDE for the elevation function). *Let the interface  $\gamma$  be a graph onto  $\mathbb{R}^2$  as in Eq. (1.1), there exist  $s_0 > 0$  and  $r_0 > 0$  such that if  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  then we have that*

$$\eta_t = g^\star(\eta) \Omega, \tag{2.35}$$

where

- $\Omega$  is defined in Eq. (2.25);
- the operator  $g^\star(\eta)$  is defined as

$$g^\star(\eta) \Omega := \mathbf{v}^\star[\Omega](\eta) \cdot \begin{bmatrix} \nabla \eta \\ -1 \end{bmatrix},$$

and  $\mathbf{v}^\star[\Omega](\eta)$  is defined in Eq. (2.14).

*Proof.* First observe that, by virtue of Eq. (1.1) and Eq. (1.4), one has

$$\boldsymbol{\gamma}_t \cdot \mathbf{v} = \eta_t. \tag{2.36}$$

Besides, the relation Eq. (2.30) gives

$$\mathbf{v}^\pm \cdot \mathbf{v} = \operatorname{BR}(\boldsymbol{\gamma}) \boldsymbol{\omega} \cdot \mathbf{v}. \tag{2.37}$$

But, the integral operator Eq. (2.29) can be simplified using Eqs. (1.1), (2.27), (2.3a) and (2.3b) as

$$\begin{aligned} \operatorname{BR}(\boldsymbol{\gamma}) \boldsymbol{\omega}(x) \Big|_{\boldsymbol{\gamma}(x)=(x, \eta(x))} &= \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \left[ \frac{\delta_z \eta(x) \Omega^\perp(x-z) - z^\perp \omega_3(x-z)}{z^\perp \cdot \Omega(x-z)} \right] \frac{dz}{\left(|z|^2 + (\delta_z \eta(x))^2\right)^{\frac{3}{2}}} \\ &= \left[ \frac{-\mathcal{B}_0(\eta)^\perp [\nabla \eta \cdot \Omega] + q(\eta) \Omega^\perp}{\mathcal{B}_0(\eta)^\perp \cdot \Omega} \right](x). \end{aligned}$$

Inserting Eq. (2.36), Eq. (2.37) and Eq. (2.1a) into Eq. (1.4) gives the desired result.  $\square$

### 2.3 The CDE for the vorticity $\omega$

It is clear from Lemma 2.5, Eq. (2.28) and Eq. (1.1) that we can describe completely the velocity flow in terms of  $\eta$  and  $\omega$ . Moreover, thanks to Eq. (2.27), we obtain that the evolution of the system Eq. (1.9) is completely determined by the evolution of  $\eta$  and  $\Omega$ . We now provide the evolution equation of the quantity  $\Omega := (\omega_1, \omega_2)$  in terms of boundary contributions only.

**Notation 2.7.** Let us now define the following auxiliary variables in order to simplify the computations in the present section:

$$\mathbf{v}_N := \text{BR}(\boldsymbol{\gamma}) \boldsymbol{\omega}, \quad \mathbf{v}_T := \frac{\mathbf{v} \times \boldsymbol{\omega}}{|\mathbf{v}|^2}, \quad (2.38)$$

so that, by virtue of Eq. (2.30),

$$\mathbf{v}^\pm = \mathbf{v}_N \pm \frac{\mathbf{v}_T}{2}.$$

**Notation 2.8.** Throughout the manuscript, we denote the Jacobian matrix and the Hessian matrix of a vector field  $V : \mathbb{R}^2 \rightarrow \mathbb{R}^p$ , with  $p \in \mathbb{N}^*$ , as

$$JV := (\nabla V^\top)^\top \quad \text{and} \quad J^\perp V := (\nabla^\perp V^\top)^\top.$$

Notice that accordingly to this notation, given a scalar function  $v$  then  $\nabla v$  is a column vector, while  $Jv = (\nabla v)^\top$  is a row vector. We also denote

$$\mathbf{H}V := \nabla \nabla^\top V.$$

An analogous definition holds for the operators  $J_x$  and  $\mathbf{H}_x$ .

With these notations in hand, we can state our result.

**Lemma 2.9** (Derivation of the CDE for the vorticity). *Let  $\sigma \geq 0$ ,  $\mathfrak{g} \in \mathbb{R}$ . Let  $x \mapsto \boldsymbol{\gamma}(x)$  be a  $\mathcal{C}^2$  surface and let  $\omega$  be defined as in Eq. (2.25).*

1. **General Surface Case.** *In the case in which  $\boldsymbol{\gamma}$  is a generic surface, the vorticity  $\omega$  evolves accordingly to the evolutionary equation*

$$\partial_t ((1 + \alpha \tilde{\mathbf{m}}(\boldsymbol{\gamma})) [\omega]) = -\boldsymbol{\pi}^\dagger(\rho^\pm | \boldsymbol{\gamma}, \omega) + \tilde{\mathbf{n}}(\boldsymbol{\gamma}) [\omega], \quad (2.39)$$

where

- $\alpha$  is the Atwood number defined in Eq. (1.5);
- the nonlinear terms in Eq. (2.39) are explicitly defined as

$$\begin{aligned} \tilde{\mathbf{m}}(\boldsymbol{\gamma}) [\omega] &:= -J^\perp \boldsymbol{\gamma} (J\boldsymbol{\gamma})^\top \mathbf{v}_N, \\ \tilde{\mathbf{n}}(\boldsymbol{\gamma}) [\omega] &:= J^\perp \boldsymbol{\gamma}_t (J\boldsymbol{\gamma})^\top (\mathbf{v}_T - \alpha \mathbf{v}_N), \\ \boldsymbol{\pi}^\dagger(\rho^\pm | \boldsymbol{\gamma}, \omega) &:= \boldsymbol{\pi}_0^\dagger(\rho^\pm | \boldsymbol{\gamma}) + \boldsymbol{\pi}_2^\dagger(\alpha | \boldsymbol{\gamma}) [\omega, \omega], \end{aligned} \quad (2.40a)$$

with

$$\begin{aligned} \boldsymbol{\pi}_0^\dagger(\rho^\pm | \boldsymbol{\gamma}) &:= J\boldsymbol{\gamma} \nabla^\perp \left( \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} \kappa(\boldsymbol{\gamma}) + \mathfrak{g} \alpha \boldsymbol{\gamma}_3 \right), \\ \boldsymbol{\pi}_2^\dagger(\alpha | \boldsymbol{\gamma}) [\omega, \omega] &:= - \underbrace{(J^\perp \boldsymbol{\gamma} \nabla (\mathbf{v}_T \cdot \boldsymbol{\gamma}_t) + J\boldsymbol{\gamma} \nabla^\perp (\mathbf{v}_N \cdot \mathbf{v}_T))}_{=:\boldsymbol{\pi}_{2,I}^\dagger(\boldsymbol{\gamma}) [\omega, \omega]} \\ &\quad + \underbrace{\alpha \left( \frac{1}{2} J\boldsymbol{\gamma} \nabla^\perp \left( |\mathbf{v}_N|^2 + \frac{|\mathbf{v}_T|^2}{4} \right) + J^\perp \boldsymbol{\gamma} \nabla (\mathbf{v}_N \cdot \boldsymbol{\gamma}_t) \right)}_{=:\boldsymbol{\pi}_{2,II}^\dagger(\alpha | \boldsymbol{\gamma}) [\omega, \omega]}, \end{aligned}$$

where  $\mathbf{v}_N$  and  $\mathbf{v}_T$  are defined in Eq. (2.38) and  $\kappa(\boldsymbol{\gamma})$  is the mean curvature of the surface  $x \mapsto \boldsymbol{\gamma}(x)$ .

2. **Graph Case.** In the case in which  $\gamma$  is a graph onto  $\mathbb{R}^2$  as per Eq. (1.1), there exist  $s_0 > 0$  and  $r_0 > 0$  such that if  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  the equation Eq. (2.39) becomes

$$\partial_t \left( (1 + \alpha T(\eta)) \Omega \right) = \nabla^\perp \pi^\star (\rho^\pm | \eta, \Omega), \quad (2.41)$$

where  $\Omega$  corresponds to the first two components of  $\omega$  (cf. Eq. (2.25)) and:

- the operator  $T(\eta)$  is defined by

$$T(\eta) \Omega := \mathcal{B}_0(\eta) [\nabla \eta \cdot \Omega] - q(\eta) \Omega + \left( \mathcal{B}_0(\eta)^\perp \cdot \Omega \right) \nabla^\perp \eta,$$

with  $\mathcal{B}_0$  and  $q$  as in Eq. (2.3a) and Eq. (2.3b);

- the nonlinear term  $\pi^\star (\rho^\pm | \eta, \Omega)$  is defined as

$$\pi^\star (\rho^\pm | \eta, \Omega) := \pi_0 (\rho^\pm | \eta) + \pi_2^\star (\alpha | \eta) [\Omega, \Omega], \quad (2.42a)$$

where  $\pi_0$  is defined in Eq. (2.17), and the bilinear form  $\pi_2^\star (\alpha | \eta) [\Omega_1, \Omega_2]$  is defined as

$$\pi_2^\star (\alpha | \eta) [\Omega_1, \Omega_2] := -\pi_{2;I}^\star (\eta) [\Omega_1, \Omega_2] - \pi_{2;II}^\star (\alpha | \eta) [\Omega_1, \Omega_2],$$

where

$$\begin{aligned} \pi_{2;I}^\star (\eta) [\Omega_1, \Omega_2] &:= \frac{\Omega_1^\perp \cdot V^\star [\Omega_2] (\eta) + \Omega_2^\perp \cdot V^\star [\Omega_1] (\eta)}{2}, \\ \pi_{2;II}^\star (\alpha | \eta) [\Omega_1, \Omega_2] &:= \frac{\alpha}{4} \left[ 2V^\star [\Omega_1] (\eta) \cdot V^\star [\Omega_2] (\eta) \right. \\ &\quad - v_3^\star [\Omega_1] (\eta) (v_3^\star [\Omega_2] (\eta) - 2V^\star [\Omega_2] (\eta) \cdot \nabla \eta) \\ &\quad - v_3^\star [\Omega_2] (\eta) (v_3^\star [\Omega_1] (\eta) - 2V^\star [\Omega_1] (\eta) \cdot \nabla \eta) \\ &\quad \left. + \frac{1}{2} \frac{\Omega_1^\top (\text{Id}_{\mathbb{R}^2} + \nabla \eta \otimes \nabla \eta) \Omega_2}{(1 + |\nabla \eta|^2)} \right]. \end{aligned}$$

with  $\tilde{\mathbf{v}}[\Omega] (\eta) = (V^\star, v_3^\star)$  as in Eq. (2.14).

*Proof.* We divide the proof into two steps.

**Step 1** (The evolution equation for a generic parametrization  $\gamma$  of the interface). First observe that for any vector field  $\mathbf{f} := [f_1 \ f_2 \ f_3]^\top \in \mathbb{R}^3$ , from Eq. (1.4), we have

$$\mathbf{f} \times \mathbf{v} = (\mathbf{f} \cdot \partial_2 \gamma) \partial_1 \gamma - (\mathbf{f} \cdot \partial_1 \gamma) \partial_2 \gamma = -J\gamma \left( (J^\perp \gamma)^\top \mathbf{f} \right). \quad (2.43)$$

Therefore, in view of Eq. (2.25) and Eq. (2.43), we have

$$\omega = -(\llbracket \mathbf{v}^\pm \rrbracket \cdot \partial_2 \gamma) \partial_1 \gamma + (\llbracket \mathbf{v}^\pm \rrbracket \cdot \partial_1 \gamma) \partial_2 \gamma.$$

Consequently, taking the derivative in time, we obtain

$$\begin{aligned} \partial_t \omega &= -\partial_t (\llbracket \mathbf{v}^\pm \rrbracket \cdot \partial_2 \gamma) \partial_1 \gamma - (\llbracket \mathbf{v}^\pm \rrbracket \cdot \partial_2 \gamma) \partial_1 \partial_t \gamma \\ &\quad + \partial_t (\llbracket \mathbf{v}^\pm \rrbracket \cdot \partial_1 \gamma) \partial_2 \gamma + (\llbracket \mathbf{v}^\pm \rrbracket \cdot \partial_1 \gamma) \partial_2 \partial_t \gamma. \end{aligned}$$

Using Eq. (2.43), we have

$$\partial_t (\mathbf{v}^\pm \times \mathbf{v}) = \partial_t (\mathbf{v}^\pm \cdot \partial_2 \gamma) \partial_1 \gamma - \partial_t (\mathbf{v}^\pm \cdot \partial_1 \gamma) \partial_2 \gamma + (\mathbf{v}^\pm \cdot \partial_2 \gamma) \partial_t \partial_1 \gamma - (\mathbf{v}^\pm \cdot \partial_1 \gamma) \partial_t \partial_2 \gamma. \quad (2.44)$$

From the definitions of  $\mathbf{v}^\pm$  in Eq. (2.1a), we have for any  $j = 1, 2$ ,

$$\partial_t (\mathbf{v}^\pm \cdot \partial_j \gamma) = \mathbf{v}^\pm \cdot \partial_t \partial_j \gamma + (\partial_t \mathbf{u}^\pm) (\gamma) \cdot \partial_j \gamma + (J_x \mathbf{u}^\pm (\gamma) \partial_t \gamma) \cdot \partial_j \gamma, \quad (2.45)$$

and

$$\partial_j (\mathbf{v}^\pm \cdot \partial_t \gamma) = \mathbf{v}^\pm \cdot \partial_t \partial_j \gamma + (J_x \mathbf{u}^\pm (\gamma) \partial_j \gamma) \cdot \partial_t \gamma, \quad (2.46)$$

and

$$\partial_j \left( \frac{|\mathbf{v}^\pm|^2}{2} \right) = (\mathbf{v}^\pm)^\top J_x \mathbf{u}^\pm(\boldsymbol{\gamma}) \partial_j \boldsymbol{\gamma}. \quad (2.47)$$

Observe that from Eq. (2.45) and Eq. (2.46) we obtain

$$\partial_t (\mathbf{v}^\pm \cdot \partial_j \boldsymbol{\gamma}) - \partial_j (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) = (\partial_t \mathbf{u}^\pm)(\boldsymbol{\gamma}) \cdot \partial_j \boldsymbol{\gamma} + (\partial_j \boldsymbol{\gamma})^\top (J_x \mathbf{u}^\pm(\boldsymbol{\gamma}) \partial_t \boldsymbol{\gamma}) - (\partial_t \boldsymbol{\gamma})^\top (J_x \mathbf{u}^\pm(\boldsymbol{\gamma}) \partial_j \boldsymbol{\gamma}). \quad (2.48)$$

Notice that thanks to Eq. (1.10) we have that  $J_x \mathbf{u}^\pm = \mathbf{H}_x \phi^\pm$ . Hence by Schwarz Theorem  $J_x \mathbf{u}^\pm$  is a symmetric matrix so that

$$(\partial_j \boldsymbol{\gamma})^\top (J_x \mathbf{u}^\pm(\boldsymbol{\gamma}) \partial_t \boldsymbol{\gamma}) - (\partial_t \boldsymbol{\gamma})^\top (J_x \mathbf{u}^\pm(\boldsymbol{\gamma}) \partial_j \boldsymbol{\gamma}) = 0. \quad (2.49)$$

Plugging Eq. (2.49) into Eq. (2.48), we infer

$$\partial_t (\mathbf{v}^\pm \cdot \partial_j \boldsymbol{\gamma}) - \partial_j (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) = (\partial_t \mathbf{u}^\pm)(\boldsymbol{\gamma}) \cdot \partial_j \boldsymbol{\gamma}. \quad (2.50)$$

The trace of the first equation of Eq. (1.9) onto  $\Gamma(t)$  is

$$(\partial_t \mathbf{u}^\pm)(\boldsymbol{\gamma}) = - \left( J_x \mathbf{u}^\pm(\boldsymbol{\gamma}) \mathbf{v}^\pm + \frac{1}{\rho^\pm} \nabla_x p^\pm(\boldsymbol{\gamma}) + \mathbf{g} \mathbf{e}_3 \right).$$

Inserting this into Eq. (2.50) then combining with Eq. (2.47) and the fact that the matrix  $J_x \mathbf{u}^\pm(\boldsymbol{\gamma})$  is symmetric, we get

$$\partial_t (\mathbf{v}^\pm \cdot \partial_j \boldsymbol{\gamma}) - \partial_j (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) = - \partial_j \left( \frac{|\mathbf{v}^\pm|^2}{2} \right) - \left( \frac{1}{\rho^\pm} \nabla_x p^\pm(\boldsymbol{\gamma}) + \mathbf{g} \mathbf{e}_3 \right) \cdot \partial_j \boldsymbol{\gamma}.$$

Remark that

$$\left( \frac{1}{\rho^\pm} \nabla_x p^\pm(\boldsymbol{\gamma}) + \mathbf{g} \mathbf{e}_3 \right) \cdot \partial_j \boldsymbol{\gamma} = \nabla_{x,z} \left( \frac{p^\pm(\boldsymbol{\gamma})}{\rho^\pm} + \mathbf{g} z \right) \cdot \partial_j \boldsymbol{\gamma} = \partial_j \left( \frac{q^\pm}{\rho^\pm} + \mathbf{g} \boldsymbol{\gamma}_3 \right).$$

Thus,

$$\partial_t (\mathbf{v}^\pm \cdot \partial_j \boldsymbol{\gamma}) - \partial_j (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) = - \partial_j \left( \frac{|\mathbf{v}^\pm|^2}{2} + \frac{q^\pm}{\rho^\pm} + \mathbf{g} \boldsymbol{\gamma}_3 \right), \quad j = 1, 2. \quad (2.51)$$

Hence, Eq. (2.51) and Eq. (2.43) imply that

$$(\partial_t (\mathbf{v}^\pm \cdot \partial_2 \boldsymbol{\gamma}) \partial_1 \boldsymbol{\gamma} - \partial_t (\mathbf{v}^\pm \cdot \partial_1 \boldsymbol{\gamma}) \partial_2 \boldsymbol{\gamma}) - (\partial_2 (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) \partial_1 \boldsymbol{\gamma} - \partial_1 (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) \partial_2 \boldsymbol{\gamma}) = J \boldsymbol{\gamma} \nabla^\perp \left( \frac{|\mathbf{v}^\pm|^2}{2} + \frac{q^\pm}{\rho^\pm} + \mathbf{g} \boldsymbol{\gamma}_3 \right).$$

Combined with Eq. (2.44) and obtain that

$$\partial_t (\mathbf{v}^\pm \times \mathbf{v}) + \mathcal{I}^\pm = J \boldsymbol{\gamma} \nabla^\perp \left( \frac{|\mathbf{v}^\pm|^2}{2} + \frac{q^\pm}{\rho^\pm} + \mathbf{g} \boldsymbol{\gamma}_3 \right), \quad (2.52)$$

where

$$\begin{aligned} \mathcal{I}^\pm &:= \mathcal{I}_1^\pm + \mathcal{I}_2^\pm, \\ \mathcal{I}_1^\pm &:= (\mathbf{v}^\pm \cdot \partial_1 \boldsymbol{\gamma}) \partial_t \partial_2 \boldsymbol{\gamma} - (\mathbf{v}^\pm \cdot \partial_2 \boldsymbol{\gamma}) \partial_t \partial_1 \boldsymbol{\gamma} &= -J^\perp \boldsymbol{\gamma}_t (J \boldsymbol{\gamma})^\top \mathbf{v}^\pm, \\ \mathcal{I}_2^\pm &:= -\partial_2 (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) \partial_1 \boldsymbol{\gamma} + \partial_1 (\mathbf{v}^\pm \cdot \partial_t \boldsymbol{\gamma}) \partial_2 \boldsymbol{\gamma} &= -J^\perp \boldsymbol{\gamma} \nabla (\mathbf{v}^\pm \cdot \boldsymbol{\gamma}_t). \end{aligned}$$

Thus, we have

$$\mathcal{I}^\pm = - (J^\perp \boldsymbol{\gamma} \nabla (\mathbf{v}^\pm \cdot \boldsymbol{\gamma}_t) + J^\perp \boldsymbol{\gamma}_t (J \boldsymbol{\gamma})^\top \mathbf{v}^\pm). \quad (2.53)$$

We insert Eq. (2.53) into Eq. (2.52) and obtain that

$$M_1 [\boldsymbol{\gamma}, \boldsymbol{\gamma}] [\mathbf{v}^\pm] := \partial_t (\mathbf{v}^\pm \times \mathbf{v}) - (J^\perp \boldsymbol{\gamma} \nabla (\mathbf{v}^\pm \cdot \boldsymbol{\gamma}_t) + J^\perp \boldsymbol{\gamma}_t (J \boldsymbol{\gamma})^\top \mathbf{v}^\pm) = J \boldsymbol{\gamma} \nabla^\perp \left( \frac{|\mathbf{v}^\pm|^2}{2} + \frac{q^\pm}{\rho^\pm} + \mathbf{g} \boldsymbol{\gamma}_3 \right). \quad (2.54)$$

From Eq. (2.54) and Eq. (1.6) it is immediate that

$$M_1[\boldsymbol{\gamma}, \boldsymbol{\gamma}] [\llbracket \rho^\pm \mathbf{v}^\pm \rrbracket] = J\boldsymbol{\gamma} \nabla^\perp \left( \frac{\llbracket \rho^\pm |\mathbf{v}^\pm|^2 \rrbracket}{2} + \sigma \kappa(\boldsymbol{\gamma}) + \mathfrak{g} \llbracket \rho^\pm \rrbracket \boldsymbol{\gamma}_3 \right). \quad (2.55)$$

From Eq. (2.38), Eq. (2.30) and Eq. (1.5), we have that

$$\llbracket \rho^\pm \mathbf{v}^\pm \rrbracket = \llbracket \rho^\pm \rrbracket \mathbf{v}_N - \langle \rho^\pm \rangle \mathbf{v}_T.$$

Therefore, using Eq. (2.38) and Eq. (2.27), we infer

$$\partial_t (\llbracket \rho^\pm \mathbf{v}^\pm \rrbracket \times \mathbf{v}) = - \langle \rho^\pm \rangle \partial_t \boldsymbol{\omega} + \llbracket \rho^\pm \rrbracket \partial_t (\mathbf{v}_N \times \mathbf{v}). \quad (2.56)$$

We use now Eq. (2.38) and Eq. (2.30) in order to argue that

$$\llbracket \rho^\pm |\mathbf{v}^\pm|^2 \rrbracket = \llbracket \rho^\pm \rrbracket \left( \mathbf{v}_N^2 + \frac{\mathbf{v}_T^2}{4} \right) - 2 \langle \rho^\pm \rangle \mathbf{v}_N \cdot \mathbf{v}_T. \quad (2.57)$$

Inserting Eq. (2.56), Eq. (2.57) and Eq. (1.5) into Eq. (2.55), we get

$$\begin{aligned} -\boldsymbol{\omega}_t + \mathfrak{a} \partial_t (\mathbf{v}_N \times \mathbf{v}) &= J^\perp \boldsymbol{\gamma} \nabla \left( (-\mathbf{v}_T + \mathfrak{a} \mathbf{v}_N) \cdot \boldsymbol{\gamma}_t \right) + J^\perp \boldsymbol{\gamma}_t (J\boldsymbol{\gamma})^\top (-\mathbf{v}_T + \mathfrak{a} \mathbf{v}_N) \\ &\quad + J\boldsymbol{\gamma} \nabla^\perp \left( \frac{\mathfrak{a}}{2} \left( \mathbf{v}_N^2 + \frac{\mathbf{v}_T^2}{4} \right) - \mathbf{v}_N \cdot \mathbf{v}_T + \frac{\sigma}{\langle \rho^\pm \rangle} \kappa(\boldsymbol{\gamma}) + \mathfrak{g} \mathfrak{a} \boldsymbol{\gamma}_3 \right). \end{aligned}$$

With standard algebraic manipulations we obtain Eq. (2.39).

**Step 2** (The evolution under the graph assumption Eq. (1.1)). Under the graph assumption Eq. (1.1), we have that

$$J\boldsymbol{\gamma} = \begin{bmatrix} \text{Id}_{\mathbb{R}^2} \\ (\nabla \eta)^\top \end{bmatrix}, \quad J^\perp \boldsymbol{\gamma} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ (\nabla^\perp \eta)^\top \end{bmatrix}, \quad (2.58)$$

so that

$$J\boldsymbol{\gamma} \nabla^\perp = \begin{bmatrix} \nabla^\perp \\ \nabla \eta \cdot \nabla^\perp \end{bmatrix}, \quad J^\perp \boldsymbol{\gamma} \nabla = \begin{bmatrix} -\nabla^\perp \\ -\nabla \eta \cdot \nabla^\perp \end{bmatrix}.$$

Combined with the definition of  $\boldsymbol{\pi}^\dagger(\rho^\pm) := \begin{bmatrix} \Pi^\dagger(\rho^\pm) \\ \pi_3^\dagger(\rho^\pm) \end{bmatrix}$  given in Eq. (2.40a) implies that

$$\begin{aligned} \Pi^\dagger(\rho^\pm | (\bullet, \eta), \boldsymbol{\omega}) &= \nabla^\perp \phi_\Pi(\rho^\pm | \eta, \boldsymbol{\omega}), \\ \phi_\Pi(\rho^\pm | \eta, \boldsymbol{\omega}) &:= \phi_{\Pi;0}(\rho^\pm | \eta) + \phi_{\Pi;2}(\mathfrak{a} | \eta) [\boldsymbol{\omega}, \boldsymbol{\omega}], \\ \phi_{\Pi;0}(\rho^\pm | \eta) &:= -\frac{\sigma}{\langle \rho^\pm \rangle} \nabla \cdot \mathcal{K}(\nabla \eta) + \mathfrak{g} \mathfrak{a} \eta, \\ \phi_{\Pi;2}(\mathfrak{a} | \eta) [\boldsymbol{\omega}, \boldsymbol{\omega}] &:= \underbrace{\mathbf{v}_T \cdot \left( \begin{bmatrix} 0_{\mathbb{R}^2} \\ \eta_t \end{bmatrix} - \mathbf{v}_N \right)}_{=: \phi_{\Pi;1;2}(\eta) [\boldsymbol{\omega}, \boldsymbol{\omega}]} + \underbrace{\frac{\mathfrak{a}}{2} \left( \mathbf{v}_N^2 + \frac{\mathbf{v}_T^2}{4} - 2 \mathbf{v}_{N;3} \eta_t \right)}_{=: \phi_{\Pi;1;2}(\mathfrak{a} | \eta) [\boldsymbol{\omega}, \boldsymbol{\omega}]}. \end{aligned} \quad (2.59)$$

Next, we have that

$$J^\perp \boldsymbol{\gamma}_t = \begin{bmatrix} 0_{\mathbb{R}^{2 \times 2}} \\ (\nabla^\perp \eta_t)^\top \end{bmatrix} \text{ and } \tilde{\mathbf{n}} = \begin{bmatrix} \tilde{N} \\ \tilde{n}_3 \end{bmatrix} \Rightarrow \tilde{N}((\bullet, \eta), \boldsymbol{\omega}) \equiv 0. \quad (2.60)$$

Moreover letting  $\mathbf{v}_N := \begin{bmatrix} V_N \\ v_{N;3} \end{bmatrix}$  we have that

$$(J\boldsymbol{\gamma})^\top \mathbf{v}_N = V_N + v_{N;3} \nabla \eta,$$

which we combine with Eq. (2.58) and  $\tilde{\mathbf{m}} = \begin{bmatrix} \tilde{M} \\ m_3 \end{bmatrix}$  in order to obtain that

$$\tilde{M}(\boldsymbol{\gamma}) \boldsymbol{\omega} \big|_{\boldsymbol{\gamma}(x)=(x,\eta(x))} = (V_N + \nu_{N;3} \nabla \eta)^\perp. \quad (2.61)$$

Now using Eq. (2.38), Eq. (2.14) and Eq. (2.27), we get

$$\begin{aligned} T(\eta) \Omega &:= \tilde{M}(\boldsymbol{\gamma}) \boldsymbol{\omega} \big|_{\substack{\boldsymbol{\gamma}(x)=(x,\eta(x)) \\ \omega_3=\nabla \eta \cdot \Omega}} \\ &= \mathcal{B}_0(\eta) [\nabla \eta \cdot \Omega] - q(\eta) \Omega + \left( \mathcal{B}_0(\eta)^\perp \cdot \Omega \right) \nabla^\perp \eta \\ &= - \left( V^\star [\Omega] (\eta) + \nu_3^\star [\Omega] (\eta) \nabla \eta \right)^\perp. \end{aligned} \quad (2.62)$$

Plugging Eq. (2.61), Eq. (2.62), Eq. (2.60) and Eq. (2.59) into Eq. (2.39), we infer

$$\partial_t \left( (1 + \alpha T(\eta)) \Omega \right) = - \nabla^\perp \phi_\Pi (\rho^\pm | \eta, \boldsymbol{\omega}) \big|_{\omega_3=\nabla \eta \cdot \Omega}. \quad (2.63)$$

We use now the special structure of the interface  $\boldsymbol{\gamma}$  and  $\boldsymbol{\omega}$ , given in Eq. (1.1) and Eq. (2.27), in order to simplify the term  $\phi_\Pi (\rho^\pm | \eta, \boldsymbol{\omega}) \big|_{\omega_3=\nabla \eta \cdot \Omega}$  in Eq. (2.63).

1. **Analysis of  $\phi_{\Pi;0}(\rho^\pm | \eta)$ :** We define

$$\pi_0(\rho^\pm | \eta) := -\phi_{\Pi;0}(\rho^\pm | \eta).$$

2. **Analysis of  $\phi_{\Pi;I;2}(\eta) [\boldsymbol{\omega}, \boldsymbol{\omega}]$ :** From Eq. (2.35) and Eq. (2.13), we have  $\eta_t + \nu_3^\star = \nabla \eta \cdot V^\star$ . Observe that for any  $A, B, C \in \mathbb{R}^2$ , we have

$$(A \cdot B) C + (C^\perp \cdot B) A^\perp = (A \cdot C) B. \quad (2.64)$$

We combine Eqs. (2.14), (2.35), (2.64) and (2.9a) and obtain that

$$\pi_{2;I}^\star(\eta) [\Omega, \Omega] := \phi_{\Pi;I;2}(\eta) [\boldsymbol{\omega}, \boldsymbol{\omega}] \big|_{\omega_3=\nabla \eta \cdot \Omega} = V_T \cdot V^\star + (\nu \times \boldsymbol{\omega})_3 (\nabla \eta \cdot V^\star).$$

Making appeal to Eq. (2.38), Eq. (1.4) and Eq. (2.27), we find

$$\begin{aligned} \pi_{2;I}^\star(\eta) [\Omega, \Omega] &= \frac{1}{1 + |\nabla \eta|^2} \left[ (-\omega_2 - \partial_2 \eta (\nabla \eta \cdot \Omega)) V_1^\star + (\omega_1 + \partial_1 \eta (\nabla \eta \cdot \Omega)) V_2^\star + (-\partial_1 \eta \omega_2 + \partial_2 \eta \omega_1) (\nabla \eta \cdot V^\star) \right] \\ &= \frac{1}{1 + |\nabla \eta|^2} \left[ - \left( 1 + |\nabla \eta|^2 \right) \omega_2 V_1^\star + \left( 1 + |\nabla \eta|^2 \right) \omega_1 V_2^\star \right] \\ &= \Omega^\perp \cdot V^\star [\Omega] (\eta). \end{aligned}$$

Therefore,

$$\pi_{2;I}^\star(\eta) [\Omega, \Omega] = \Omega^\perp \cdot V^\star [\Omega] (\eta). \quad (2.65)$$

By polarization, we obtain Eq. (2.19). Since Eq. (2.63) gives

$$\partial_t \left( (1 + \alpha T(\eta)) \Omega \right) = - \nabla^\perp \phi_\Pi,$$

the quantity in Eq. (2.65) enters the Hamiltonian-coordinate right-hand side through  $\pi^\star = -\phi_\Pi$ . Consequently, the contribution of  $\pi_{2;I}^\star$  to  $\pi_2^\star$  carries the minus sign displayed in Eq. (2.18). After the substitution  $\Omega = S(\alpha | \eta) \psi$ , this is the origin of the minus sign in the definition of  $\pi_{2;I}(\alpha | \eta)$ .

3. **Analysis of  $\phi_{\Pi;II;2}(\alpha | \eta) [\boldsymbol{\omega}, \boldsymbol{\omega}]$ :** Let us recall from Eq. (2.59) that

$$\phi_{\Pi;II;2}(\alpha | \eta) [\boldsymbol{\omega}, \boldsymbol{\omega}] = \frac{\alpha}{2} \left( \boldsymbol{\nu}_N^2 + \frac{\boldsymbol{\nu}_T^2}{4} - 2 \boldsymbol{\nu}_N \cdot \boldsymbol{\gamma}_t \right) \bigg|_{\substack{\boldsymbol{\gamma}(x)=(x,\eta(x)) \\ \omega_3=\nabla \eta \cdot \Omega}}.$$

From  $\boldsymbol{\omega} \cdot \boldsymbol{\nu} = 0$ , we have

$$|\boldsymbol{\nu} \times \boldsymbol{\omega}|^2 = |\boldsymbol{\nu}|^2 |\boldsymbol{\omega}|^2 - (\boldsymbol{\nu} \cdot \boldsymbol{\omega})^2 \iff |\boldsymbol{\nu} \times \boldsymbol{\omega}| = |\boldsymbol{\nu}| |\boldsymbol{\omega}|.$$

So

$$|\mathbf{v}_T|^2 \Big|_{\substack{\boldsymbol{\gamma}(x)=(x,\eta(x)) \\ \omega_3=\nabla\eta\cdot\Omega}} = \frac{|\boldsymbol{\omega}|^2}{|\mathbf{v}|^2} \Big|_{\substack{\boldsymbol{\gamma}(x)=(x,\eta(x)) \\ \omega_3=\nabla\eta\cdot\Omega}} = \frac{|\Omega|^2 + |\nabla\eta\cdot\Omega|^2}{1 + |\nabla\eta|^2}. \quad (2.66)$$

For the term  $|\mathbf{v}_N|^2$  using Eq. (2.14) we have that

$$|\mathbf{v}_N|^2 \Big|_{\substack{\boldsymbol{\gamma}(x)=(x,\eta(x)) \\ \omega_3=\nabla\eta\cdot\Omega}} = |\mathbf{v}^\star[\Omega](\eta)|^2 = |V^\star[\Omega](\eta)|^2 + |\nu_3^\star[\Omega](\eta)|^2. \quad (2.67)$$

For the term  $\mathbf{v}_N \cdot \boldsymbol{\gamma}_t$  we have by virtue of Eq. (2.13)

$$\begin{aligned} \mathbf{v}_N \cdot \boldsymbol{\gamma}_t \Big|_{\substack{\boldsymbol{\gamma}(x)=(x,\eta(x)) \\ \omega_3=\nabla\eta\cdot\Omega}} &= -\nu_3^\star[\Omega](\eta) \eta_t = -\nu_3^\star[\Omega](\eta) g^\star(\eta) \Omega \\ &= -\nu_3^\star[\Omega](\eta) (V^\star[\Omega](\eta) \cdot \nabla\eta - \nu_3^\star[\Omega](\eta)). \end{aligned} \quad (2.68)$$

We combine Eqs. (2.66) to (2.68) in the definition of  $\tilde{\pi}_2(\mathfrak{a}|\boldsymbol{\gamma}, \omega)$  (cf. Eq. (2.59)) and obtain that

$$\begin{aligned} \pi_{2;\text{II}}^\star(\mathfrak{a}|\eta)[\Omega, \Omega] &:= \phi_{\text{II},\text{II};2}(\mathfrak{a}|\eta)[\boldsymbol{\omega}, \boldsymbol{\omega}] \Big|_{\omega_3=\nabla\eta\cdot\Omega} \\ &= \frac{\mathfrak{a}}{2} \left( |V^\star[\Omega](\eta)|^2 - \nu_3^\star[\Omega](\eta) (\nu_3^\star[\Omega](\eta) - 2V^\star[\Omega](\eta) \cdot \nabla\eta) + \frac{1}{4} \frac{|\Omega|^2 + |\nabla\eta\cdot\Omega|^2}{(1 + |\nabla\eta|^2)} \right). \end{aligned}$$

Once again, the associated bilinear form Eq. (2.20) is obtained via polarization.

Eq. (2.18) follows by summation, and we conclude. □

## 2.4 The CDE for the variable $\psi$

In this subsection, we prove the following lemma dealing with the evolution of the weighted trace of the potential  $\psi$ .

**Lemma 2.10** (Evolution of the weighted trace of the velocity potential). *The function  $\psi$  defined via Eqs. (2.1b) and (2.2), satisfies*

$$\Omega = S(\mathfrak{a}|\eta)\psi, \quad (2.69)$$

where  $S(\mathfrak{a}|\eta)$  is the operator defined in Eq. (2.11). Moreover, there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  the function  $\psi$  solves the evolutionary equation Eq. (2.9b), i.e.

$$\psi_t = \pi(\rho^\pm|\eta, \psi), \quad (2.70)$$

where the right hand-side is defined by

$$\pi(\rho^\pm|\eta, \psi) := \pi^\star(\rho^\pm|\eta, S(\mathfrak{a}|\eta)\psi), \quad (2.71)$$

with  $\pi^\star(\rho^\pm|\eta, \Omega)$  as in Eq. (2.42a).

*Proof.* Let us recall that  $\psi$  is defined in Eq. (2.2). Let us define the *modified vorticity*

$$\boldsymbol{\zeta} := \begin{bmatrix} Z \\ \zeta_3 \end{bmatrix} := -\frac{\llbracket \rho^\pm \mathbf{v}^\pm \rrbracket \times \mathbf{v}}{\langle\langle \rho^\pm \rangle\rangle}.$$

Standard computations combined with Eqs. (2.25) and (2.30) and the first identity of Eq. (2.40) show that

$$\boldsymbol{\zeta} = \boldsymbol{\omega} - \mathfrak{a}\text{BR}(\boldsymbol{\gamma}) \boldsymbol{\omega} \times \mathbf{v} = (1 + \mathfrak{a}\tilde{\mathbf{m}}(\boldsymbol{\gamma}))[\boldsymbol{\omega}]. \quad (2.72)$$

We plug Eq. (2.72) into Eq. (2.39) and obtain that

$$\boldsymbol{\zeta}_t = -\boldsymbol{\pi}^\dagger(\rho^\pm|\boldsymbol{\gamma}, \boldsymbol{\omega}) + \tilde{\mathbf{n}}(\boldsymbol{\gamma})[\boldsymbol{\omega}]. \quad (2.73)$$

Let us now reduce the first two components Eq. (2.73) to the case in which  $\gamma$  satisfies the graph assumption Eq. (1.1). The transformed r.h.s. of Eq. (2.73) is given by the r.h.s. of Eq. (2.41), so that

$$\partial_t Z = \nabla^\perp \pi^\star (\rho^\pm | \eta, \Omega). \quad (2.74)$$

Notice now that, defining  $\nu^\pm := \begin{bmatrix} V^\pm \\ v_3^\pm \end{bmatrix}$ , we have

$$\nabla \psi^\pm(x) = \nabla \phi^\pm(x, \eta(x)) + \partial_{x_3} \phi^\pm(x, \eta(x)) \nabla \eta(x) = V^\pm(x) + v_3^\pm(x) \nabla \eta(x), \quad (2.75)$$

so that

$$\nabla \llbracket \psi^\pm \rrbracket = \llbracket V^\pm \rrbracket + \llbracket v_3^\pm \rrbracket \nabla \eta. \quad (2.76)$$

Now from Eq. (2.25) we have that

$$\omega = \begin{bmatrix} (\llbracket V^\pm \rrbracket + \llbracket v_3^\pm \rrbracket \nabla \eta)^\perp \\ -\nabla^\perp \eta \cdot \llbracket V^\pm \rrbracket \end{bmatrix}, \quad (2.77)$$

so that, from Eqs. (2.76) and (2.77), we obtain that

$$\Omega = \nabla^\perp \llbracket \psi^\pm \rrbracket. \quad (2.78)$$

Similar computations moreover give that

$$Z = \nabla^\perp \psi. \quad (2.79)$$

We plug Eqs. (2.78) and (2.79) into Eq. (2.74) in order to express the evolution of  $\psi$  in terms of  $\eta$  and  $\llbracket \psi^\pm \rrbracket$ , this gives

$$\psi_t = \pi^\star (\rho^\pm | \eta, \nabla^\perp \llbracket \psi^\pm \rrbracket). \quad (2.80)$$

Let us now relate  $\psi$  and  $\llbracket \psi^\pm \rrbracket$  via an invertible, bounded operator. We combine Eqs. (2.62) and (2.72) and obtain that

$$Z = (1 + \alpha T(\eta)) [\Omega]. \quad (2.81)$$

Putting together Eqs. (2.78), (2.79) and (2.81), we obtain that

$$\nabla^\perp \psi = (1 + \alpha T(\eta)) [\nabla^\perp \llbracket \psi^\pm \rrbracket]. \quad (2.82)$$

Eq. (2.82) has no curl-free component, hence we can apply the operator  $\Delta^{-1} \nabla^\perp \cdot$  to it and obtain that

$$\psi = (1 + \alpha \tau_0(\eta)) [\llbracket \psi^\pm \rrbracket],$$

where (cf. Eq. (2.62))

$$\tau_0(\eta) := \Delta^{-1} \nabla^\perp \cdot T(\eta) \nabla^\perp.$$

Notice that there exist  $s_0 > 0$  and  $r_0 > 0$  such that for all  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  the operator  $(1 + \alpha \tau_0(\eta)) \in \mathcal{L}(H^s)$  is invertible for all  $s \geq s_0$  (the proof is postponed to Lemma 5.23), so we have that

$$\llbracket \psi^\pm \rrbracket = (1 + \alpha \tau_0(\eta))^{-1} \psi. \quad (2.83)$$

It will be convenient to distinguish the unweighted trace jump from the canonical weighted trace. We therefore set

$$\varphi := \llbracket \psi^\pm \rrbracket = (1 + \alpha \tau_0(\eta))^{-1} \psi.$$

Thus

$$\psi = (1 + \alpha \tau_0(\eta)) \varphi, \quad \Omega = \nabla^\perp \varphi.$$

All these identities are understood in homogeneous spaces, hence modulo additive constants. Plugging Eq. (2.83) into Eq. (2.80) and defining  $\pi(\rho^\pm | \eta, \psi)$  as in the statement we obtain Eq. (2.9b).  $\square$

## 2.5 Recasting the potential in term of vorticity

Recall  $\psi^\pm(t, x)$  defined in Eq. (2.1b). Let us now express  $\psi^\pm, \psi$  in terms of  $\Omega$ .

**Lemma 2.11.** *Let  $\psi^\pm$  and  $\psi$  be as in Eqs. (2.1b) and (2.2) respectively, there exists a  $s_0 > 0$  and a  $r_0 > 0$  such that if  $\eta \in B_{s_0, \mathbb{R}}(I; r_0), \Omega \in (B_{s_0, \mathbb{R}}(I; r_0))^2$  then*

$$\psi^\pm(x) = \Delta^{-1} \nabla^\perp \cdot \left( \mp \frac{1}{2} + T(\eta) \right) [\Omega], \quad (2.84a)$$

$$\psi(x) = \Delta^{-1} \nabla^\perp \cdot (1 + \alpha T(\eta)) [\Omega], \quad (2.84b)$$

where  $T(\eta)$  is defined in Eq. (2.62). Moreover, we have

$$T(\eta) \Omega = \nabla^\perp \left( \frac{\psi^+ + \psi^-}{2} \right) (x). \quad (2.85)$$

*Proof.* By Eq. (2.75) and Eq. (2.30), we get

$$V^\pm(x) + v_3^\pm(x) \nabla \eta(x) = \left( \text{BR}(\gamma) \Omega \pm \frac{1}{2} \frac{v \times \Omega}{|v|^2} \right)_h + \left( \text{BR}(\gamma) \Omega \pm \frac{1}{2} \frac{v \times \Omega}{|v|^2} \right)_3 \nabla \eta. \quad (2.86)$$

Remind Eq. (2.14) and Eq. (2.62), we have

$$\begin{aligned} (\text{BR}(\gamma) \Omega)_h + (\text{BR}(\gamma) \Omega)_3 \nabla \eta &= -\mathcal{B}_0(\eta)^\perp [\nabla \eta \cdot \Omega] + q(\eta) \Omega^\perp + (\mathcal{B}_0(\eta)^\perp \cdot \Omega) \nabla \eta \\ &= -(T(\eta) \Omega)^\perp, \end{aligned} \quad (2.87)$$

and

$$\left( \frac{v \times \Omega}{|v|^2} \right)_h + \left( \frac{v \times \Omega}{|v|^2} \right)_3 \nabla \eta = \begin{bmatrix} -\Omega_2 \\ \Omega_1 \end{bmatrix} = \Omega^\perp.$$

So

$$\nabla \cdot (\text{BR}(\gamma) \Omega)_h + (\text{BR}(\gamma) \Omega)_3 \nabla \eta = T(\eta) \Omega, \quad \nabla \cdot \left( \left( \frac{v \times \Omega}{|v|^2} \right)_h + \left( \frac{v \times \Omega}{|v|^2} \right)_3 \nabla \eta \right) = \nabla \cdot \Omega^\perp = -\nabla^\perp \cdot \Omega, \quad (2.88)$$

and we directly take divergence of  $\nabla \psi^\pm(x)$  and use Eq. (2.88) to obtain Eq. (2.84a). Eq. (2.84b) is a consequence of Eqs. (2.2) and (2.84a).

Finally, we obtain from Eq. (2.86) and Eq. (2.87) that

$$\begin{aligned} T(\eta) \Omega &= ((\text{BR}(\gamma) \Omega)_h + (\text{BR}(\gamma) \Omega)_3 \nabla \eta)^\perp \\ &= \left( \frac{V^+(x) + V^-(x)}{2} + \frac{v_3^+(x) + v_3^-(x)}{2} \nabla \eta(x) \right)^\perp. \end{aligned} \quad (2.89)$$

Thus we obtain Eq. (2.85) from Eqs. (2.75) and (2.89).  $\square$

**The normal velocity in Hamiltonian coordinates.** We can now rewrite Eq. (2.14) in the Hamiltonian variable  $\psi$ , indeed we have that

$$\Omega = S(\alpha|\eta) \psi, \quad \omega_3 = \nabla \eta \cdot \Omega = \nabla \eta \cdot S(\alpha|\eta) \psi$$

so that

$$\begin{aligned} v[\psi](\alpha|\eta) &:= \begin{bmatrix} V[\psi](\alpha|\eta) \\ v_3[\psi](\alpha|\eta) \end{bmatrix} := v^* [S(\alpha|\eta) \psi](\eta) \\ &= -\text{BR}(\gamma) \left[ \frac{S(\alpha|\eta) \psi}{\nabla \eta \cdot S(\alpha|\eta) \psi} \right] \Big|_{\gamma(x)=(x, \eta(x))} \\ &= \begin{bmatrix} \mathcal{B}_0(\eta)^\perp [\nabla \eta \cdot S(\alpha|\eta) \psi] - q(\eta) (S(\alpha|\eta) \psi)^\perp \\ -\mathcal{B}_0(\eta)^\perp \cdot S(\alpha|\eta) \psi \end{bmatrix}. \end{aligned} \quad (2.90)$$

The Hamiltonian structure of the canonical CDE is proved in Theorem 2.2.

## 2.6 Proof of Theorem 2.2

The proof combines the CDE in vorticity variables with the canonical background shift.

*Proof.* By Lemma 2.6, the elevation equation is

$$\eta_t = g^\star(\eta) \Omega.$$

By Eq. (2.70), the potential equation is

$$\psi_t = \pi(\rho^\pm | \eta, \psi).$$

We fix once and for all the Bernoulli gauge. Recall that the equation for the weighted trace is determined only up to an additive function of time, namely

$$\psi_t = \pi(\rho^\pm | \eta, \psi) + c(t).$$

In the background formulation we choose this gauge by imposing  $c(t) \equiv -c_{\tilde{\Omega}}$ . Equivalently, one may replace the weighted trace by  $\psi - c_{\tilde{\Omega}} t$ . With this convention, setting  $\psi = \tilde{\psi} + \tilde{\psi}$ , and using that  $\tilde{\psi}$  is time-independent, we obtain

$$\tilde{\psi}_t = \pi(\rho^\pm | \eta, \tilde{\psi} + \tilde{\psi}) - c_{\tilde{\Omega}}.$$

By Eq. (2.91), Eq. (2.71) and the definition of  $\tilde{\pi}$ , this gives

$$\tilde{\psi}_t = \tilde{\pi}(\rho^\pm, \tilde{\Omega} | \eta, \tilde{\psi}).$$

Now set  $\psi = \tilde{\psi} + \tilde{\psi}$ . Since  $\tilde{\psi}$  is independent of time,  $\psi_t = \tilde{\psi}_t$ . Moreover, by Eq. (2.10),

$$\Omega = \tilde{\Omega}[\tilde{\Omega}](\eta) + S(a|\eta) \tilde{\psi}. \quad (2.91)$$

Therefore

$$\eta_t = g^\star(\eta) [\tilde{\Omega}[\tilde{\Omega}](\eta)] + g^\star(\eta) [S(a|\eta) \tilde{\psi}] = g^\star(\eta) [\tilde{\Omega}[\tilde{\Omega}](\eta)] + g(a|\eta) [\tilde{\psi}],$$

which is Eq. (2.9a). On the other hand,

$$\tilde{\psi}_t = \pi(\rho^\pm | \eta, \tilde{\psi} + \tilde{\psi}) = \tilde{\pi}(\rho^\pm, \tilde{\Omega} | \eta, \tilde{\psi}), \quad (2.92)$$

after subtracting the constant gauge  $c_{\tilde{\Omega}}$  as in Eq. (2.16). This proves Eq. (2.9b).

It remains only to justify the homogeneous decomposition of  $\tilde{\pi}$ . Using Eq. (2.71), Eq. (2.91), and the bilinearity of  $\pi_2^\star$ , we get

$$\begin{aligned} \pi^\star(\rho^\pm | \eta, \tilde{\Omega}[\tilde{\Omega}](\eta) + S(a|\eta) \tilde{\psi}) - c_{\tilde{\Omega}} &= \pi_0(\rho^\pm | \eta) + \pi_2^\star(a|\eta) [\tilde{\Omega}[\tilde{\Omega}](\eta), \tilde{\Omega}[\tilde{\Omega}](\eta)] - c_{\tilde{\Omega}} \\ &\quad + 2\pi_2^\star(a|\eta) [\tilde{\Omega}[\tilde{\Omega}](\eta), S(a|\eta) \tilde{\psi}] \\ &\quad + \pi_2^\star(a|\eta) [S(a|\eta) \tilde{\psi}, S(a|\eta) \tilde{\psi}]. \end{aligned}$$

The three lines are exactly Eq. (2.23a), Eq. (2.23b), and Eq. (2.23c). This concludes the proof.  $\square$

## 2.7 Canonical background variables

The canonical background variables used in Theorem 2.2 are introduced there because they are part of the final CDE formulation. In this subsection we record the elementary identities which justify the canonical rewriting.

The affine background potential  $\tilde{\psi}$  and the canonical perturbation  $\tilde{\psi}$  are defined in Eqs. (2.7) and (2.8). Since  $\tilde{\psi}$  is independent of  $\eta$ , Eq. (2.92) holds, and therefore the change of variables preserves the canonical symplectic form.

By Eq. (2.69), applied with  $\psi = \tilde{\psi}$ , the vorticity carried by the slice  $\{\tilde{\psi} = 0\}$  is

$$\tilde{\Omega}[\tilde{\Omega}](\eta) = S(a|\eta) \tilde{\psi}.$$

Consequently, inserting  $\psi = \tilde{\psi} + \tilde{\psi}$  into Eq. (2.69) gives the decomposition Eq. (2.91). Finally, the gauge  $c_{\tilde{\Omega}}$  in Eq. (2.21) removes the constant value of the Bernoulli function at the flat state.

**Proposition 2.12** (Canonical background formulation). *Assume the hypotheses of Theorem 2.2. Let  $\bar{\psi}$ ,  $\tilde{\psi}$ ,  $\bar{\Omega}[\bar{\Omega}](\eta)$ , and  $c_{\bar{\Omega}}$  be defined by Eqs. (2.7), (2.8), (2.10) and (2.21). Then the change of variables*

$$\psi = \bar{\psi} + \tilde{\psi}$$

*preserves the canonical two-form in the sense of Eq. (2.92). In these variables, the canonical CDE and the homogeneous decomposition of its shifted Bernoulli function are those in Eqs. (2.9) and (2.22). In particular, for fixed  $\eta$  and  $\bar{\Omega}$ , the term  $\tilde{\pi}_j$  is  $j$ -homogeneous in  $\tilde{\psi}$  for  $j = 0, 1, 2$ .*

*Proof.* The shift  $\psi \mapsto \tilde{\psi} = \psi - \bar{\psi}$  is a translation by the  $\eta$ -independent function  $\bar{\psi}$ , hence it preserves the canonical two-form by Eq. (2.92). By Eqs. (2.13), (2.15) and (2.91), substituting  $\psi = \bar{\psi} + \tilde{\psi}$  gives the first equation in Eq. (2.9). Since  $\tilde{\psi}$  is time-independent,  $\psi_t = \tilde{\psi}_t$ , and the second equation follows from Eq. (2.16). Finally, Eq. (2.22) gives the asserted homogeneous decomposition.  $\square$

**Remark 2.13** (The equilibrium  $(0, 0)$ ). At  $(\eta, \tilde{\psi}) = (0, 0)$  one has

$$\bar{\Omega}[\bar{\Omega}](0) = \bar{\Omega}. \quad (2.93)$$

by Eq. (2.6),  $\mathcal{J}(\alpha|0)\tilde{\psi} = 0$ , and  $\pi(\rho^\pm|0, \tilde{\psi}) - c_{\bar{\Omega}} = 0$  by Eq. (2.21). The flat configuration is therefore a stationary point of Eq. (2.9).  $\blacklozenge$

### 3 Linearization at the trivial state

The purpose of this section is to linearize the canonical CDE Eq. (2.9) at the equilibrium state  $(\eta, \tilde{\psi}) = (0, 0)$  and to analyse the resulting dispersion relation. The two main results are Proposition 3.1, which gives the explicit linearized matrix operator together with its eigenvalues, and Proposition 3.4, which characterizes the purely imaginary part of the spectrum in terms of the stability scales introduced in Definition 3.3.

**Proposition 3.1** (Linearization of the canonical CDE at the trivial state). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle\langle \rho^\pm \rangle\rangle > 0$ ,  $\mathfrak{g} \in \mathbb{R}$ , and  $\bar{\Omega} \in \mathbb{R}^2$ . The linearization of the canonical CDE Eq. (2.9) at  $(\eta, \tilde{\psi}) = (0, 0)$  in the direction  $(h, \varphi)$  is*

$$\partial_t \begin{bmatrix} h \\ \varphi \end{bmatrix} = \mathbf{L}_0(D) \begin{bmatrix} h \\ \varphi \end{bmatrix}, \quad (3.1)$$

where

$$\mathbf{L}_0(D) := \frac{\alpha}{4} \bar{\Omega}^\perp \cdot \nabla \text{Id}_{\mathbb{C}^2} + \begin{bmatrix} 0 & \frac{1}{2}|D| \\ \mathfrak{q}(\alpha, \bar{\Omega}|D) & 0 \end{bmatrix}.$$

Here

$$\mathfrak{q}(\alpha, \bar{\Omega}|\xi) := -\mathfrak{g}\alpha - \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2 + \frac{4 - \alpha^2}{8} \frac{(\bar{\Omega}^\perp \cdot \xi)^2}{|\xi|}. \quad (3.2)$$

Equivalently, in Fourier variables,

$$\partial_t \begin{bmatrix} \hat{h} \\ \hat{\varphi} \end{bmatrix} = \mathbf{L}_0(\xi) \begin{bmatrix} \hat{h} \\ \hat{\varphi} \end{bmatrix}, \quad \mathbf{L}_0(\xi) := i \frac{\alpha}{4} (\bar{\Omega}^\perp \cdot \xi) \text{Id}_{\mathbb{C}^2} + \begin{bmatrix} 0 & \frac{1}{2}|\xi| \\ \mathfrak{q}(\alpha, \bar{\Omega}|\xi) & 0 \end{bmatrix}. \quad (3.3)$$

The characteristic polynomial is

$$\det(\omega \text{Id}_{\mathbb{C}^2} - \mathbf{L}_0(\xi)) = \left( \omega - i \frac{\alpha}{4} (\bar{\Omega}^\perp \cdot \xi) \right)^2 - \mathfrak{d}(\alpha, \bar{\Omega}|\xi),$$

where

$$\mathfrak{d}(\alpha, \bar{\Omega}|\xi) := \frac{|\xi|}{2} \mathfrak{q}(\alpha, \bar{\Omega}|\xi) = \frac{4 - \alpha^2}{16} (\bar{\Omega}^\perp \cdot \xi)^2 - \frac{\mathfrak{g}\alpha}{2} |\xi| - \frac{\sigma}{2\langle\langle \rho^\pm \rangle\rangle} |\xi|^3. \quad (3.4)$$

Consequently, the two eigenvalues are

$$\lambda_\pm(\alpha, \bar{\Omega}|\xi) := i \frac{\alpha}{4} (\bar{\Omega}^\perp \cdot \xi) \pm \sqrt{\mathfrak{d}(\alpha, \bar{\Omega}|\xi)}. \quad (3.5)$$

*Proof.* We linearize the two equations in Eq. (2.9) separately. The argument is organized as follows. We first record the flat-state values of the operators  $\mathcal{B}_0$ ,  $q$ ,  $\tau_0$ ,  $S$ , and their first variations, which feed into every subsequent computation. We then linearize the canonical background vorticity  $\tilde{\Omega}[\tilde{\Omega}](\eta)$ , which couples the two equations. With this input we linearize successively the  $\eta$ -equation and the  $\tilde{\psi}$ -equation, and finally we assemble the matrix form and read off the spectrum.

The vector-valued Riesz transform  $\mathfrak{R}$  used throughout has principal symbol  $-i\xi/|\xi|$  and is defined as a singular integral operator in Eq. (2.4).

**Part 1** (Flat identities). Combining Eq. (2.5) and Eq. (2.14), we get

$$\mathbf{v}^*[\Omega](0) = \begin{bmatrix} V^*[\Omega](0) \\ \nu_3^*[\Omega](0) \end{bmatrix} = \begin{bmatrix} 0_{\mathbb{R}^2} \\ -\frac{1}{2}\mathfrak{R}^\perp \cdot \Omega \end{bmatrix}, \quad g^*(0)[\Omega] = \frac{1}{2}\mathfrak{R}^\perp \cdot \Omega. \quad (3.6)$$

Moreover, Eq. (2.5) and Eq. (2.12) gives

$$\tau_0(0) = 0, \quad S(\mathbf{a}|0) = \nabla^\perp. \quad (3.7)$$

Consequently, by virtue of Eq. (2.15), Eq. (3.6) and Eq. (3.7), we find

$$g(\mathbf{a}|0)[\varphi] = g^*(0)[\nabla^\perp \varphi] = \frac{1}{2}\mathfrak{R}^\perp \cdot \nabla^\perp \varphi = \frac{1}{2}|D|\varphi, \quad (3.8)$$

the last equality following from the symbolic identity  $\mathfrak{R}^\perp \cdot \nabla^\perp = |D|$ . We shall also need the first variations. Differentiating Eq. (2.3a) and Eq. (2.3b), we find

$$d_\eta \mathcal{B}_0(0)[h]\tilde{\Omega} = 0 \quad \text{and} \quad d_\eta q(0)[h]\tilde{\Omega} = \frac{\tilde{\Omega}}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\delta_z h}{|z|^3} dz = \frac{1}{2}|D|h\tilde{\Omega}. \quad (3.9)$$

The last identity of Eq. (3.9) is from

$$\mathcal{F}_{x \rightarrow \xi} \left( \text{p.v.} \int_{\mathbb{R}^2} \frac{h(x) - h(x-z)}{|z|^3} dz \right) (\xi) = 2\pi|\xi|\hat{h}(\xi).$$

**Part 2** (Linearization of the canonical background vorticity). Recall the definition Eq. (2.10),

$$\tilde{\Omega}[\tilde{\Omega}](\eta) = S(\mathbf{a}|\eta)\tilde{\psi}.$$

Since  $S(\mathbf{a}|\eta) = \nabla^\perp (1 + \mathbf{a}\tau_0(\eta))^{-1}$  and  $\tau_0(0) = 0$ , we obtain

$$d_\eta \tilde{\Omega}[\tilde{\Omega}](0)[h] = -\mathbf{a}\nabla^\perp (d_\eta \tau_0(0)[h]\tilde{\psi}). \quad (3.10)$$

We compute the scalar factor in Eq. (3.10). By Eq. (2.12), Eq. (2.5), and Eq. (3.9),

$$\begin{aligned} d_\eta \tau_0(0)[h]\tilde{\psi} &= \Delta^{-1} \nabla^\perp \cdot (\mathcal{B}_0(0)[\nabla h \cdot \tilde{\Omega}] - d_\eta q(0)[h]\tilde{\Omega}) \\ &= \Delta^{-1} \nabla^\perp \cdot \left( \frac{1}{2}\mathfrak{R}[\nabla h \cdot \tilde{\Omega}] - \frac{1}{2}|D|h\tilde{\Omega} \right) \\ &= -\frac{1}{2}\tilde{\Omega} \cdot \mathfrak{R}^\perp h. \end{aligned}$$

Indeed, the first term in the second line is annihilated by  $\nabla^\perp \cdot \mathfrak{R} = 0$ , while the second is rewritten using  $\Delta^{-1} \nabla^\perp \cdot (|D|h\tilde{\Omega}) = \tilde{\Omega} \cdot \mathfrak{R}^\perp h$ . Therefore

$$d_\eta \tilde{\Omega}[\tilde{\Omega}](0)[h] = \frac{\mathbf{a}}{2} \nabla^\perp (\tilde{\Omega} \cdot \mathfrak{R}^\perp h). \quad (3.11)$$

Equivalently, in Fourier variables,

$$\mathcal{F}_{x \rightarrow \xi} (d_\eta \tilde{\Omega}[\tilde{\Omega}](0)[h])(\xi) = \frac{\mathbf{a}}{2} \frac{(\tilde{\Omega} \cdot \xi^\perp)}{|\xi|} \xi^\perp \hat{h}(\xi).$$

**Part 3** (Linearization of the  $\eta$ -equation). The first equation in Eq. (2.9) reads

$$\eta_t = g^\star(\eta) [\tilde{\Omega}[\bar{\Omega}](\eta)] + g(\mathfrak{a}|\eta) [\tilde{\psi}].$$

Its linearization in the direction  $(h, \varphi)$  is

$$\begin{aligned} h_t &= d_\eta(g^\star(\eta) [\tilde{\Omega}[\bar{\Omega}](\eta)])_{\eta=0}[h] + g(\mathfrak{a}|0) [\varphi] \\ &= d_\eta g^\star(0) [h] [\bar{\Omega}] + g^\star(0) [d_\eta \tilde{\Omega}[\bar{\Omega}](0) [h]] + g(\mathfrak{a}|0) [\varphi]. \end{aligned} \quad (3.12)$$

Let us deal with the first contribution. Recall from Eq. (2.13) and Eq. (2.14) that

$$g^\star(\eta) [\Omega] = V^\star[\Omega](\eta) \cdot \nabla \eta - v_3^\star[\Omega](\eta).$$

Now, according to Eq. (3.6), Eq. (2.14) and Eq. (3.9), we get

$$V^\star[\bar{\Omega}](0) = 0 \quad \text{and} \quad d_\eta v_3^\star[\bar{\Omega}](0) [h] = 0. \quad (3.13)$$

Therefore, the first contribution vanishes, namely

$$d_\eta g^\star(0) [h] [\bar{\Omega}] = 0. \quad (3.14)$$

Using Eq. (3.6) and Eq. (3.11) and the fact that

$$\mathfrak{R}^\perp \cdot \nabla^\perp = |D|, \quad \bar{\Omega} \cdot \mathfrak{R}^\perp = -\bar{\Omega}^\perp \cdot \mathfrak{R},$$

we obtain

$$\begin{aligned} g^\star(0) [d_\eta \tilde{\Omega}[\bar{\Omega}](0) [h]] &= \frac{1}{2} \mathfrak{R}^\perp \cdot \left[ \frac{\mathfrak{a}}{2} \nabla^\perp (\bar{\Omega} \cdot \mathfrak{R}^\perp h) \right] \\ &= \frac{\mathfrak{a}}{4} |D| (\bar{\Omega} \cdot \mathfrak{R}^\perp h) \\ &= \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp \cdot \nabla h. \end{aligned} \quad (3.15)$$

Inserting Eq. (3.14), Eq. (3.15), and Eq. (3.8) into Eq. (3.12) yields

$$h_t = \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp \cdot \nabla h + \frac{1}{2} |D| \varphi, \quad (3.16)$$

which is the first component of Eq. (3.1).

**Part 4** (Linearization of the  $\tilde{\psi}$ -equation). The second equation in Eq. (2.9) reads

$$\tilde{\psi}_t = \tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}).$$

By Eq. (2.16) and Eq. (2.42a), this can be rewritten as

$$\tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}) = \pi_0(\rho^\pm|\eta) + \pi_2^\star(\mathfrak{a}|\eta) [\Omega_{\text{Tot}}, \Omega_{\text{Tot}}] - c_{\bar{\Omega}},$$

where the canonical total vorticity is

$$\Omega_{\text{Tot}} := \Omega_{\text{Tot}}(\eta, \tilde{\psi}) := \tilde{\Omega}[\bar{\Omega}](\eta) + S(\mathfrak{a}|\eta) \tilde{\psi}.$$

Recall from Eq. (2.93) that  $\Omega_{\text{Tot}}(0, 0) = \bar{\Omega}$ , we deduce that

$$d_{(\eta, \tilde{\psi})} \tilde{\pi}(\rho^\pm, \bar{\Omega}|0, 0) [h, \varphi] = d_\eta \pi_0(\rho^\pm|\eta) [h] + (d_\eta \pi_2^\star(\mathfrak{a}|0)) [h] [\bar{\Omega}, \bar{\Omega}] + 2\pi_2^\star(\mathfrak{a}|0) \left[ \bar{\Omega}, d_{(\eta, \tilde{\psi})} \Omega_{\text{Tot}}(0, 0) [h, \varphi] \right].$$

The gravity-capillary contribution is directly obtained from the definition of  $\pi_0$  in Eq. (2.17),

$$d_\eta \pi_0(\rho^\pm|0) [h] = \frac{\sigma}{\llbracket \rho^\pm \rrbracket} \Delta h - \mathfrak{g} \mathfrak{a} h. \quad (3.17)$$

The first variation of the canonical total vorticity is  $(\eta, \tilde{\psi}) = (0, 0)$  is, by virtue of Eq. (3.11),

$$\begin{aligned} d_{(\eta, \tilde{\psi})} \Omega_{\text{Tot}}(0, 0) [h, \varphi] &= d_{\eta} \tilde{\Omega} [\tilde{\Omega}] (0) [h] + \nabla^{\perp} \varphi \\ &= \frac{\mathfrak{a}}{2} \nabla^{\perp} (\tilde{\Omega} \cdot \mathfrak{R}^{\perp} h) + \nabla^{\perp} \varphi. \end{aligned} \quad (3.18)$$

Since  $\tilde{\Omega} \in \mathbb{R}^2$ , one gets from Eq. (3.6) that

$$\nu_3^* [\tilde{\Omega}] (0) = 0. \quad (3.19)$$

Therefore, coming back to the definition of  $\pi_2^*$  in Eq. (2.18)-Eq. (2.19)-Eq. (2.20) and using Eq. (3.13), we get for any  $\Omega$ ,

$$\pi_2^* (\mathfrak{a} | 0) [\tilde{\Omega}, \Omega] = -\frac{\mathfrak{a}}{8} \tilde{\Omega} \cdot \Omega. \quad (3.20)$$

We now substitute the two pieces of Eq. (3.18) into Eq. (3.20). For the contribution  $\Omega = \nabla^{\perp} \varphi$  we obtain, using  $\tilde{\Omega} \cdot \xi^{\perp} = -\tilde{\Omega}^{\perp} \cdot \xi$ ,

$$2\pi_2^* (\mathfrak{a} | 0) [\tilde{\Omega}, \nabla^{\perp} \varphi] = -\frac{\mathfrak{a}}{4} \tilde{\Omega} \cdot \nabla^{\perp} \varphi = \frac{\mathfrak{a}}{4} \tilde{\Omega}^{\perp} \cdot \nabla \varphi. \quad (3.21)$$

For the contribution  $\Omega = d_{\eta} \tilde{\Omega} [\tilde{\Omega}] (0) [h]$  we obtain, by Eq. (3.11),

$$\begin{aligned} 2\pi_2^* (\mathfrak{a} | 0) [\tilde{\Omega}, d_{\eta} \tilde{\Omega} [\tilde{\Omega}] (0) [h]] &= -\frac{\mathfrak{a}}{4} \tilde{\Omega} \cdot d_{\eta} \tilde{\Omega} [\tilde{\Omega}] (0) [h] \\ &= -\frac{\mathfrak{a}^2}{8} \tilde{\Omega} \cdot \nabla^{\perp} (\tilde{\Omega} \cdot \mathfrak{R}^{\perp} h), \end{aligned}$$

which in Fourier variables reads

$$\mathcal{F}_{x \rightarrow \xi} (2\pi_2^* (\mathfrak{a} | 0) [\tilde{\Omega}, d_{\eta} \tilde{\Omega} [\tilde{\Omega}] (0) [h]]) (\xi) = -\frac{\mathfrak{a}^2}{8} \frac{(\tilde{\Omega}^{\perp} \cdot \xi)^2}{|\xi|} \hat{h}(\xi). \quad (3.22)$$

It remains to differentiate  $\pi_2^* (\mathfrak{a} | \eta) [\tilde{\Omega}, \tilde{\Omega}]$  with respect to  $\eta$  at fixed background vorticity argument. By Eq. (2.20), Eq. (3.13) and Eq. (3.19), the term  $\pi_{2;\text{II}}^*$  has zero first variation at  $\eta = 0$ . Now recall from Eq. (2.19) that

$$\pi_{2;\text{I}}^* (\eta) [\tilde{\Omega}, \tilde{\Omega}] = \tilde{\Omega}^{\perp} \cdot V^* [\tilde{\Omega}] (\eta).$$

By virtue of Eq. (2.14) and Eq. (3.9), we find

$$d_{\eta} V^* [\tilde{\Omega}] (0) [h] = \left( \frac{1}{2} \mathfrak{R} [\nabla h \cdot \tilde{\Omega}] - \frac{1}{2} |D| h \tilde{\Omega} \right)^{\perp}.$$

Consequently,

$$\begin{aligned} d_{\eta} \pi_{2;\text{I}}^* (0) [h] [\tilde{\Omega}, \tilde{\Omega}] &= \tilde{\Omega}^{\perp} \cdot d_{\eta} V^* [\tilde{\Omega}] (0) [h] \\ &= \frac{1}{2} \tilde{\Omega} \cdot \mathfrak{R} [\nabla h \cdot \tilde{\Omega}] - \frac{1}{2} |\tilde{\Omega}|^2 |D| h. \end{aligned}$$

In Fourier variables,

$$\mathcal{F}_{x \rightarrow \xi} (\tilde{\Omega} \cdot \mathfrak{R} [\nabla h \cdot \tilde{\Omega}]) (\xi) = \frac{(\tilde{\Omega} \cdot \xi)^2}{|\xi|} \hat{h}(\xi),$$

so that, using

$$|\tilde{\Omega}|^2 |\xi| = |\xi|^{-1} \left[ (\tilde{\Omega} \cdot \xi)^2 + (\tilde{\Omega} \cdot \xi^{\perp})^2 \right],$$

we deduce

$$\mathcal{F}_{x \rightarrow \xi} (d_{\eta} \pi_2^* (\mathfrak{a} | 0) [h] [\tilde{\Omega}, \tilde{\Omega}]) (\xi) = \mathcal{F}_{x \rightarrow \xi} \left( -d_{\eta} \pi_{2;\text{I}}^* (0) [h] [\tilde{\Omega}, \tilde{\Omega}] \right) (\xi) = \frac{1}{2} \frac{(\tilde{\Omega} \cdot \xi^{\perp})^2}{|\xi|} \hat{h}(\xi). \quad (3.23)$$

Combining Eq. (3.17), Eq. (3.21), Eq. (3.22), and Eq. (3.23), we obtain

$$\varphi_t = \frac{\mathfrak{a}}{4} \tilde{\Omega}^{\perp} \cdot \nabla \varphi + \frac{\sigma}{\langle \rho^{\pm} \rangle} \Delta h - \mathfrak{g} \mathfrak{a} h + \frac{4 - \mathfrak{a}^2}{8} \mathcal{F}_{\xi \rightarrow x}^{-1} \left( \frac{(\tilde{\Omega} \cdot \xi^{\perp})^2}{|\xi|} \hat{h}(\xi) \right), \quad (3.24)$$

which is the second component of Eq. (3.1).

**Part 5** (Matrix form and spectrum). Combining Eq. (3.16) and Eq. (3.24), we obtain

$$\partial_t \begin{bmatrix} h \\ \varphi \end{bmatrix} = \begin{bmatrix} \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp \cdot \nabla \text{Id}_{\mathbb{C}^2} + \begin{bmatrix} 0 & \frac{1}{2} |D| \\ \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|D) & 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} h \\ \varphi \end{bmatrix},$$

with  $\mathfrak{q}$  defined in Eq. (3.2); taking the Fourier transform yields Eq. (3.3). Since the transport part is scalar, the characteristic polynomial is

$$\det(\omega \text{Id}_{\mathbb{C}^2} - \mathbf{L}_0(\xi)) = \left( \omega - i \frac{\mathfrak{a}}{4} (\bar{\Omega}^\perp \cdot \xi) \right)^2 - \frac{|\xi|}{2} \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi),$$

and the identity

$$\frac{|\xi|}{2} \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) = \frac{4 - \mathfrak{a}^2}{16} (\bar{\Omega}^\perp \cdot \xi)^2 - \frac{\mathfrak{g}\mathfrak{a}}{2} |\xi| - \frac{\sigma}{2 \langle\langle \rho^\pm \rangle\rangle} |\xi|^3$$

is precisely Eq. (3.4). The two roots of the characteristic polynomial are Eq. (3.5).

This concludes the proof of Proposition 3.1.  $\square$

**Remark 3.2** (Structural consistency). The transport  $\frac{\mathfrak{a}}{4} \bar{\Omega}^\perp \cdot \nabla$  appears in both rows of  $\mathbf{L}_0(D)$  for two structurally distinct reasons: in the  $\eta$ -equation it arises from the canonical background vorticity  $\tilde{\Omega}[\tilde{\Omega}](\eta) = S(\mathfrak{a}|\eta)\tilde{\psi}$ , whereas in the  $\tilde{\psi}$ -equation it arises from the bilinear term  $2\pi_2^*(\mathfrak{a}|0)[\tilde{\Omega}, \nabla^\perp \varphi]$ . That the same transport speed appears in both rows is the direct CDE-level manifestation of the canonical Hamiltonian structure.  $\blacklozenge$

**Definition 3.3** (Linear stability scales). We introduce the following stability scales attached to the linear dispersion relation Eq. (3.4).

1. The *reduced gravity*, or *Rayleigh–Taylor acceleration*, is

$$\mathfrak{g}'_\alpha := \frac{\mathfrak{g}\mathfrak{a}}{2}.$$

2. The *capillary coefficient* is

$$c_\sigma := \frac{\sigma}{2 \langle\langle \rho^\pm \rangle\rangle} = \frac{\sigma}{\rho^- + \rho^+}.$$

3. The *directional Kelvin–Helmholtz shear coefficient* is

$$\text{KH}_{\mathfrak{a}, \bar{\Omega}}(\vartheta) := \frac{4 - \mathfrak{a}^2}{16} (\bar{\Omega} \cdot \vartheta^\perp)^2, \quad \vartheta \in \mathbb{R}^2, \quad |\vartheta| = 1.$$

4. If  $\sigma > 0$ , the *directional effective Weber wave number* is

$$\kappa_{\text{We}}(\vartheta) := \frac{\text{KH}_{\mathfrak{a}, \bar{\Omega}}(\vartheta)}{c_\sigma} = \frac{\rho^- \rho^+}{\sigma(\rho^- + \rho^+)} (\bar{\Omega} \cdot \vartheta^\perp)^2.$$

For a prescribed length scale  $L > 0$ , the dimensionless quantity  $L \kappa_{\text{We}}(\vartheta)$  is the corresponding effective directional Weber number.

5. If  $\sigma > 0$ , the *signed inverse squared capillary length* is

$$\kappa_{\text{cap}}^2 := \frac{\mathfrak{g}'_\alpha}{c_\sigma} = \frac{(\rho^- - \rho^+) \mathfrak{g}}{\sigma}.$$

When  $(\rho^- - \rho^+) \mathfrak{g} > 0$ , this is positive and

$$\ell_{\text{cap}} := \frac{1}{\kappa_{\text{cap}}} = \sqrt{\frac{\sigma}{(\rho^- - \rho^+) \mathfrak{g}}}$$

is the capillary length; for a prescribed length scale  $L > 0$ , the dimensionless quantity  $L^2 \kappa_{\text{cap}}^2$  is the corresponding Bond, or Eötvös, number.

6. The maximal Kelvin–Helmholtz and Weber wavenumbers are

$$\text{KH}_{\alpha, \bar{\Omega}}^{\max} := \sup_{|\vartheta|=1} \text{KH}_{\alpha, \bar{\Omega}}(\vartheta) = \frac{4 - \alpha^2}{16} |\bar{\Omega}|^2 = \frac{\rho^- \rho^+}{(\rho^- + \rho^+)^2} |\bar{\Omega}|^2$$

and, if  $\sigma > 0$ ,

$$\kappa_{\text{We}}^{\max} := \sup_{|\vartheta|=1} \kappa_{\text{We}}(\vartheta) = \frac{\rho^- \rho^+}{\sigma(\rho^- + \rho^+)} |\bar{\Omega}|^2.$$

**Proposition 3.4** (Purely imaginary linear frequencies). *Assume the hypotheses of Proposition 3.1, and let the stability scales be those introduced in Definition 3.3. Then, for every  $\xi \in \mathbb{R}^2 \setminus \{0\}$ , with  $\hat{\xi} := \xi / |\xi|$ , one has*

$$\mathfrak{d}(\alpha, \bar{\Omega} | \xi) = |\xi| [\text{KH}_{\alpha, \bar{\Omega}}(\hat{\xi}) |\xi| - g'_\alpha - c_\sigma |\xi|^2]. \quad (3.25)$$

Consequently,

$$\lambda_\pm(\alpha, \bar{\Omega} | \xi) \in i\mathbb{R} \iff c_\sigma |\xi|^2 - \text{KH}_{\alpha, \bar{\Omega}}(\hat{\xi}) |\xi| + g'_\alpha \geq 0. \quad (3.26)$$

More precisely, the following alternatives hold.

1. If  $\sigma > 0$ , then

$$\lambda_\pm(\alpha, \bar{\Omega} | \xi) \in i\mathbb{R} \iff |\xi|^2 - \kappa_{\text{We}}(\hat{\xi}) |\xi| + \kappa_{\text{cap}}^2 \geq 0. \quad (3.27)$$

In particular, if

$$(\kappa_{\text{We}}^{\max})^2 \leq 4\kappa_{\text{cap}}^2, \quad (3.28)$$

then both dispersion branches are purely imaginary for every  $\xi \in \mathbb{R}^2 \setminus \{0\}$ .

Conversely, if there exists  $\vartheta \in \mathbb{R}^2$  with  $|\vartheta| = 1$  such that

$$\kappa_{\text{We}}(\vartheta)^2 > 4\kappa_{\text{cap}}^2,$$

then the frequencies in the direction  $\vartheta$  are not purely imaginary for

$$|\xi| \in (\max\{0, \kappa_-(\vartheta)\}, \kappa_+(\vartheta)), \quad \hat{\xi} = \vartheta, \quad (3.29)$$

where

$$\kappa_\pm(\vartheta) := \frac{\kappa_{\text{We}}(\vartheta) \pm \sqrt{\kappa_{\text{We}}(\vartheta)^2 - 4\kappa_{\text{cap}}^2}}{2}. \quad (3.30)$$

Moreover, independently of Eq. (3.28), capillarity stabilizes the high-frequency region: both branches are purely imaginary for all  $|\xi| \geq \kappa_{\text{hi}}$ , where

$$\kappa_{\text{hi}} := \begin{cases} \frac{\kappa_{\text{We}}^{\max} + \sqrt{(\kappa_{\text{We}}^{\max})^2 - 4\kappa_{\text{cap}}^2}}{2}, & \text{if } (\kappa_{\text{We}}^{\max})^2 > 4\kappa_{\text{cap}}^2, \\ 0, & \text{otherwise.} \end{cases} \quad (3.31)$$

2. If  $\sigma = 0$ , then

$$\lambda_\pm(\alpha, \bar{\Omega} | \xi) \in i\mathbb{R} \iff \text{KH}_{\alpha, \bar{\Omega}}(\hat{\xi}) |\xi| \leq g'_\alpha. \quad (3.32)$$

Hence, if  $g'_\alpha < 0$ , no nonzero frequency has purely imaginary branches. If  $g'_\alpha = 0$ , the branches are purely imaginary precisely in the directions satisfying  $\text{KH}_{\alpha, \bar{\Omega}}(\hat{\xi}) = 0$ . If  $g'_\alpha > 0$ , then in every direction satisfying  $\text{KH}_{\alpha, \bar{\Omega}}(\hat{\xi}) > 0$  the purely imaginary range is exactly

$$0 < |\xi| \leq \frac{g'_\alpha}{\text{KH}_{\alpha, \bar{\Omega}}(\hat{\xi})}. \quad (3.33)$$

In particular, if  $g'_\alpha > 0$  and  $\text{KH}_{\alpha, \bar{\Omega}}^{\max} > 0$ , then both branches are purely imaginary for every frequency satisfying

$$0 < |\xi| \leq \frac{g'_\alpha}{\text{KH}_{\alpha, \bar{\Omega}}^{\max}}. \quad (3.34)$$

Full-frequency stability in the capillarity-free case holds if and only if

$$g'_a \geq 0, \quad KH_{a,\bar{\Omega}}^{\max} = 0; \quad (3.35)$$

equivalently,

$$ga \geq 0, \quad (4 - a^2) |\bar{\Omega}|^2 = 0.$$

*Proof.* The two branches differ from the purely imaginary scalar  $i\frac{a}{4}(\bar{\Omega}^\perp \cdot \xi)$  only by  $\pm \sqrt{\mathfrak{d}_{a,\bar{\Omega}}(\xi)}$ , where  $\mathfrak{d}_{a,\bar{\Omega}}(\xi) \in \mathbb{R}$ . Hence both branches are purely imaginary if and only if  $\mathfrak{d}_{a,\bar{\Omega}}(\xi) \leq 0$ . We first rewrite the discriminant in the dispersion-physics variables of Definition 3.3, then separately analyse the cases  $\sigma > 0$  and  $\sigma = 0$ .

Let  $\xi \in \mathbb{R}^2 \setminus \{0\}$  and  $\hat{\xi} := \xi / |\xi|$ . Since  $\xi = |\xi| \hat{\xi}$ ,

$$(\bar{\Omega}^\perp \cdot \xi)^2 = |\xi|^2 (\bar{\Omega}^\perp \cdot \hat{\xi})^2.$$

Using the definitions of  $KH_{a,\bar{\Omega}}$ ,  $g'_a$ , and  $c_\sigma$  in Definition 3.3, this gives

$$\mathfrak{d}(a, \bar{\Omega} | \xi) = |\xi| [KH_{a,\bar{\Omega}}(\hat{\xi}) |\xi| - g'_a - c_\sigma |\xi|^2],$$

which is Eq. (3.25). Since  $|\xi| > 0$ , the condition  $\mathfrak{d}_{a,\bar{\Omega}}(\xi) \leq 0$  is equivalent to Eq. (3.26).

**Part 1** (The capillary case  $\sigma > 0$ ). If  $\sigma > 0$ , then  $c_\sigma > 0$ . Dividing Eq. (3.26) by  $c_\sigma$  and using

$$\kappa_{\text{We}}(\hat{\xi}) = \frac{KH_{a,\bar{\Omega}}(\hat{\xi})}{c_\sigma}, \quad \kappa_{\text{cap}}^2 = \frac{g'_a}{c_\sigma},$$

yields Eq. (3.27).

Fix a direction  $\vartheta \in \mathbb{R}^2$  with  $|\vartheta| = 1$ , and consider the quadratic polynomial

$$P_\vartheta(X) := X^2 - \kappa_{\text{We}}(\vartheta)X + \kappa_{\text{cap}}^2, \quad X \in \mathbb{R},$$

with discriminant  $\Delta_\vartheta := \kappa_{\text{We}}^2(\vartheta) - 4\kappa_{\text{cap}}^2$ . If  $(\kappa_{\text{We}}^{\max})^2 \leq 4\kappa_{\text{cap}}^2$ , then  $\Delta_\vartheta \leq 0$  for every  $\vartheta$ . Since  $P_\vartheta$  has positive leading coefficient,  $P_\vartheta(X) \geq 0$  for every  $X \in \mathbb{R}$ , hence for every  $X > 0$ . Applying this with  $X = |\xi|$  and  $\vartheta = \hat{\xi}$  proves Eq. (3.28).

Conversely, assume that for some direction  $\vartheta$ ,  $\kappa_{\text{We}}^2(\vartheta) > 4\kappa_{\text{cap}}^2$ . Then  $P_\vartheta$  has two real roots  $\kappa_\pm(\vartheta)$  given by Eq. (3.30); since the leading coefficient is positive,  $P_\vartheta$  is negative precisely between its two roots. Intersecting with the positive half-line gives

$$P_\vartheta(X) < 0 \iff X \in (\max\{0, \kappa_-(\vartheta)\}, \kappa_+(\vartheta)),$$

which in view of Eq. (3.27) is exactly the unstable band Eq. (3.29)–Eq. (3.30).

It remains to prove the high-frequency statement. Since  $\kappa_{\text{We}}(\vartheta) \leq \kappa_{\text{We}}^{\max}$  for every direction, we have, for every  $X > 0$ ,

$$P_\vartheta(X) \geq X^2 - \kappa_{\text{We}}^{\max} X + \kappa_{\text{cap}}^2.$$

If  $(\kappa_{\text{We}}^{\max})^2 \leq 4\kappa_{\text{cap}}^2$ , the right-hand side is nonnegative for every  $X$  and one may take  $\kappa_{\text{hi}} = 0$ . Otherwise, the right-hand side is nonnegative for every  $X$  larger than its largest root, namely for

$$X \geq \frac{\kappa_{\text{We}}^{\max} + \sqrt{(\kappa_{\text{We}}^{\max})^2 - 4\kappa_{\text{cap}}^2}}{2},$$

which proves Eq. (3.31).

**Part 2** (The capillarity-free case  $\sigma = 0$ ). If  $\sigma = 0$ , then  $c_\sigma = 0$ , and Eq. (3.26) reduces to

$$KH_{a,\bar{\Omega}}(\hat{\xi}) |\xi| \leq g'_a,$$

which is Eq. (3.32).

Since  $\text{KH}_{a,\bar{\Omega}}(\hat{\xi}) \geq 0$  and  $|\xi| > 0$ , the preceding inequality has no solution if  $g'_a < 0$ , proving the first assertion. If  $g'_a = 0$ , the inequality becomes  $\text{KH}_{a,\bar{\Omega}}(\hat{\xi})|\xi| \leq 0$ ; since both factors on the left-hand side are nonnegative and  $|\xi| > 0$ , this is equivalent to  $\text{KH}_{a,\bar{\Omega}}(\hat{\xi}) = 0$ , proving the second assertion.

Assume now  $g'_a > 0$ . If  $\text{KH}_{a,\bar{\Omega}}(\hat{\xi}) > 0$ , solving  $\text{KH}_{a,\bar{\Omega}}(\hat{\xi})|\xi| \leq g'_a$  for  $|\xi|$  gives exactly Eq. (3.33). If  $\text{KH}_{a,\bar{\Omega}}(\hat{\xi}) = 0$ , the inequality is satisfied for every  $|\xi| > 0$  in that direction.

Taking the maximum over directions gives the uniform low-frequency criterion: if  $\text{KH}_{a,\bar{\Omega}}^{\max} > 0$  and  $|\xi|$  satisfies  $0 < |\xi| \leq g'_a / \text{KH}_{a,\bar{\Omega}}^{\max}$ , then

$$\text{KH}_{a,\bar{\Omega}}(\hat{\xi})|\xi| \leq \text{KH}_{a,\bar{\Omega}}^{\max}|\xi| \leq g'_a,$$

proving Eq. (3.34).

Finally, full-frequency stability in the capillarity-free case means that  $\text{KH}_{a,\bar{\Omega}}(\vartheta)X \leq g'_a$  for every direction  $\vartheta$  and every  $X > 0$ , which holds if and only if  $\text{KH}_{a,\bar{\Omega}}(\vartheta) = 0$  for every  $\vartheta$  and  $g'_a \geq 0$ . By Definition 3.3, this is equivalent to  $(4 - a^2)|\bar{\Omega}|^2 = 0$  and  $ga \geq 0$ , proving Eq. (3.35).

This concludes the proof of Proposition 3.4.  $\square$

**Remark 3.5** (Dimensionless Atwood–Weber threshold). When  $ga > 0$ , one has  $\kappa_{\text{cap}}^2 > 0$ , and the full-frequency sufficient condition Eq. (3.28) can be recast, with  $W_{a,\bar{\Omega}}$  defined in Eq. (1.26), as

$$W_{a,\bar{\Omega}} \leq 1.$$

Thus the Atwood number has two distinct stabilizing effects: it generates the Rayleigh–Taylor restoring term  $\kappa_{\text{cap}}^2$  when  $ga > 0$ , and it weakens the Kelvin–Helmholtz contribution through the factor  $4 - a^2$ . In the Kelvin–Helmholtz endpoint  $a = 0$ , one has  $\kappa_{\text{cap}}^2 = 0$ , so no nonzero background  $\bar{\Omega}$  can satisfy a full-frequency stability condition on  $\mathbb{R}^2$ . In that endpoint one can only obtain a high-frequency stability statement, which by Eq. (3.31) reduces to

$$|\xi| \geq \kappa_{\text{We}}^{\max}|a=0| = \frac{\langle\langle \rho^\pm \rangle\rangle}{2\sigma} |\bar{\Omega}|^2.$$

This is the flat-interface analogue of the Weber threshold in the periodic Kelvin–Helmholtz problem, but without the discrete low-frequency gap present in the circular geometry, see [85].  $\blacklozenge$

## 4 Paradifferential toolkit

Throughout this section,  $d \geq 1$ . In accordance with Notation 1.1, all definitions and results are stated for scalar functions, operators, and symbols; the vector- and matrix-valued versions follow componentwise. Standard spatial, frequency, and integration variables retain the usual lower-case notation; in particular,

$$x, \xi, z, \zeta, v, v', \zeta \in \mathbb{R}^d.$$

All multi-indices belong to  $\mathbb{N}^d$ . The symbols  $\nabla_x$ ,  $\nabla_\xi$ ,  $\nabla_x^2$ ,  $\Delta_x$ , and  $\Delta_z$  denote the Euclidean gradients, Hessian, and Laplacians in dimension  $d$ . Moreover, we denote

$$\delta_z f(x) := f(x) - f(x - z), \quad \Delta_z f(x) := \frac{\delta_z f(x)}{|z|}, \quad \hat{z} := \frac{z}{|z|} \in \mathbb{S}^{d-1}. \quad (4.1)$$

**Notation 4.1.** We use the standard surface measure  $d\sigma$  on  $\mathbb{S}^{d-1}$ ; for  $d = 1$ , it is counting measure on  $\mathbb{S}^0 = \{-1, 1\}$ . The Fourier convention is

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx, \quad \mathcal{F}(m(D)f)(\xi) = m(\xi) \hat{f}(\xi),$$

with inverse transform

$$f(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \hat{f}(\xi) d\xi.$$

Every functional space below is displayed with its domain and codomain. In operator spaces, omission of the codomain means that domain and codomain coincide. For brevity, in norm subscripts we suppress the ambient domain and codomain. In particular,  $H^s$  there stands for  $H^s(\mathbb{R}^d; \mathbb{C})$ . Real scalar backgrounds belong to  $H^s(\mathbb{R}^d; \mathbb{R})$ , and we use the convention

$$\begin{aligned} \|u\|_s &:= \|u\|_{H^s(\mathbb{R}^d; \mathbb{C})}, & \|u\|_{s,I} &:= \sup_{t \in I} \|u(t)\|_s, \\ \|\tilde{U}_p\|_{s,I} &:= \max_{1 \leq j \leq p} \|u_j\|_{s,I}, & B_{s,\mathbb{R}}(I; r) &:= B_{C(I; H^s(\mathbb{R}^d; \mathbb{R}))}(0, r). \end{aligned}$$

This section collects the paradifferential formalism that will be used in Section 5 to parilinearize the generalized water-waves system Eq. (2.9). We first remind the general notion of smoothing operators and symbols. They depend on an external function and we keep track of the homogeneity with respect to this latter. Then, we introduce the Bony–Weyl quantization and recall its main para-composition rules. In a second part, we develop an approach to parilinearize general singular integral operators on the whole space. This uses localized and tail kernel functions to deal with integrability near the origin and at infinity, respectively.

#### 4.1 General paradifferential calculus

In this subsection, we recall the basic definitions and propositions in paradifferential calculus, such as symbols, functions, smoothing operators, and Bony–Weyl quantization.

##### *p*-linear bounded maps on Fréchet spaces, continuously differentiable and analytic maps.

**Definition 4.2** (*p*-linear bounded maps in locally convex vector spaces). Let  $\Lambda \subset \mathbb{R}$ ,  $X$  be a Banach space,  $Y$  a locally convex topological vector space with topology generated by a family of seminorms  $\{p_\lambda\}_{\lambda \in \Lambda}$ , and  $p \in \mathbb{N}^*$ . We denote by  $\mathcal{L}^p(X; Y)$  the space of continuous *p*-linear maps  $T : X^p \rightarrow Y$ , i.e. maps that are linear in each argument separately and satisfy

$$p_\lambda(T(u_1, \dots, u_p)) \leq C_\lambda \prod_{j=1}^p \|u_j\|_X, \quad \forall u_1, \dots, u_p \in X, \lambda \in \Lambda,$$

for constants  $C_\lambda > 0$  depending on  $\lambda$  but not on  $u_1, \dots, u_p$ . We write

$$\mathcal{L}(X; Y) := \mathcal{L}^1(X; Y), \quad \mathcal{L}^p(X) := \mathcal{L}^p(X; X).$$

We shall also use real-analytic functions with values in a Fréchet space; we recall the definition for the reader's convenience.

**Definition 4.3** ( $\mathcal{C}^k$  maps into locally convex spaces). Let  $X$  be a Banach space, let  $Y$  be a locally convex space whose topology is generated by a family of seminorms  $\{p_\lambda\}_{\lambda \in \Lambda}$ , and let  $U \subseteq X$  be open. A map  $f : U \rightarrow Y$  is of class  $\mathcal{C}^1$  if there exists a map  $df : U \rightarrow \mathcal{L}(X; Y)$  such that, for every  $x_0 \in U$  and every seminorm  $p_\lambda$ ,

$$p_\lambda(f(x_0 + h) - f(x_0) - df(x_0)h) = o(\|h\|_X) \quad \text{as } \|h\|_X \rightarrow 0,$$

and such that  $x \mapsto df(x)$  is continuous for the topology of uniform convergence on bounded subsets of  $X$ , seminorm by seminorm in  $Y$ . Inductively,  $f \in \mathcal{C}^k(U; Y)$  if  $f \in \mathcal{C}^1$  and  $df \in \mathcal{C}^{k-1}(U; \mathcal{L}(X; Y))$ . We set  $\mathcal{C}^\infty(U; Y) := \bigcap_{k \geq 1} \mathcal{C}^k(U; Y)$ .

**Definition 4.4** (Real-analytic maps into locally convex spaces). Let  $X$  be a Banach space,  $Y$  a locally convex space with topology generated by seminorms  $\{p_\lambda\}_{\lambda \in \Lambda}$ , and  $U \subseteq X$  open. We say that  $f : U \rightarrow Y$  is *real-analytic*, and write  $f \in \mathcal{C}^\omega(U; Y)$ , if for every  $x_0 \in U$  there exists  $r = r(x_0) > 0$  such that  $B_X(x_0, r) \subseteq U$  and

$$f(x_0 + h) = \sum_{k=0}^{+\infty} \frac{1}{k!} d^k f(x_0)[h]^k, \quad \forall h \in B_X(0, r),$$

where  $d^k f(x_0) \in \mathcal{L}^k(X; Y)$  is the *k*-th Fréchet differential of  $f$  at  $x_0$  (cf. Definition 4.3),

$$d^k f(x_0)[h]^k := d^k f(x_0)[h, \dots, h]$$

denotes its evaluation on the  $k$ -fold diagonal, and the series converges in  $Y$ , i.e. for every  $\lambda \in \Lambda$ ,

$$\lim_{N \rightarrow +\infty} p_\lambda \left( f(x_0 + h) - \sum_{k=0}^N \frac{1}{k!} \mathrm{d}^k f(x_0) [h]^k \right) = 0, \quad \forall h \in B_X(0, r). \quad (4.2)$$

We say that the series converges *uniformly* on  $B_X(0, r)$  if the limit in Eq. (4.2) is uniform in  $h \in B_X(0, r)$  for every  $\lambda$ .

### Tame smoothing operators.

**Definition 4.5** (Tame smoothing operators with vanishing order). Let  $\varrho \in \mathbb{R}$  and  $q \in \mathbb{N}$ . We say that  $r_\varrho \in \mathcal{R}_{\geq q}^\varrho$  if, for some fixed  $p \in \mathbb{N}^\star$ , there exists  $s_0 > 0$  such that, for every  $s \geq s_0$ , there exists a radius  $r_s^{\varrho} > 0$  with the following property. For every  $0 < r \leq r_s^{\varrho}$  and every

$$\vec{U}_p \in (B_{s_0}(I; r))^p,$$

we have

$$r_\varrho(\vec{U}_p) \in \mathcal{L}(H^{s-\varrho}(\mathbb{R}^d; \mathbb{C}); H^s(\mathbb{R}^d; \mathbb{C})),$$

and, for every  $h \in H^{s-\varrho}(\mathbb{R}^d; \mathbb{C})$ ,

$$\|r_\varrho(\vec{U}_p) h\|_{H^s} \lesssim_{s, \varrho} \|\vec{U}_p\|_{s_0}^q \|h\|_{H^{s-\varrho}}. \quad (4.3)$$

We call  $r_\varrho$  real-to-real if  $r_\varrho(\vec{U}_p)h$  is real-valued whenever the backgrounds and  $h$  are real-valued. We use the convention  $\mathcal{R}^\varrho := \mathcal{R}_{\geq 1}^\varrho$ .

### Fourier multipliers, symbols and functions.

**Definition 4.6** (Fourier multiplier). Let  $m \in \mathbb{R}$ , we denote with  $\tilde{\Gamma}^m$  the space of maps

$$\xi \mapsto m(\xi) \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{C}),$$

such that for any  $\beta \in \mathbb{N}^d$  and  $|\xi| \geq 1/2$  we have that

$$|\partial_\xi^\beta m(\xi)| \lesssim_\beta |\xi|^{m-|\beta|}.$$

**Definition 4.7** (Symbols with finite regularity and order of vanishing). Let  $m \in \mathbb{R}$  and let  $q \in \mathbb{N}^\star$ .

1. We denote by  $\Gamma_{\geq q}^m$  the space of maps

$$a : (B_{s_0}(I; r_0))^p \rightarrow L_{\text{loc}}^\infty(\mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}); \mathbb{C})$$

$$\vec{U}_p := (u_1, \dots, u_p)^\top \mapsto \{(x, \xi) \mapsto a(\vec{U}_p; x, \xi)\}$$

defined for some  $s_0 > d/2$ ,  $r_0 > 0$ ,  $p \in \mathbb{N}^\star$ , and satisfying the following properties.

- (a) For every  $M \in \mathbb{N}$ , there exists  $\mu_M^a \geq 0$  such that, for every  $s \geq s_0$ , there is  $0 < r_{s, M}^a \leq r_0$  such that, for every

$$\vec{U}_p \in \left( B_{s_0 + \mu_M^a, \mathbb{R}}(I; r) \cap \mathcal{C}\left(I; H^{s + \mu_M^a}(\mathbb{R}^d; \mathbb{R})\right) \right)^p, \quad 0 < r \leq r_{s, M}^a,$$

and every  $x \in \mathbb{R}^d$ , the map

$$\xi \mapsto a(\vec{U}_p; x, \xi)$$

belongs to  $C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$ .

- (b) With the losses fixed above, for every  $M \in \mathbb{N}$ , every  $s \geq s_0$ , every multi-index  $\beta \in \mathbb{N}^d$  with  $|\beta| \leq M$ , every

$$\vec{U}_p \in \left( B_{s_0 + \mu_M^a, \mathbb{R}}(I; r) \cap \mathcal{C}\left(I; H^{s + \mu_M^a}(\mathbb{R}^d; \mathbb{R})\right) \right)^p, \quad 0 < r \leq r_{s, M}^a,$$

and every  $|\xi| \geq \frac{1}{2}$ , one has

$$\left\| \partial_\xi^\beta a(\vec{U}_p; \bullet, \xi) \right\|_{s, I} \lesssim_{s, \beta, M} \|\vec{U}_p\|_{s_0 + \mu_M^a, I}^{q-1} \|\vec{U}_p\|_{s + \mu_M^a, I} |\xi|^{m-|\beta|}. \quad (4.4)$$

The losses  $\mu_M^a$  may be chosen nondecreasing in  $M$ .

2. We use the convention

$$\Gamma^m := \Gamma_{\geq 1}^m.$$

3. We define

$$\Gamma_{\geq 0}^m := \tilde{\Gamma}^m + \Gamma_{\geq 1}^m.$$

Equivalently,  $a \in \Gamma_{\geq 0}^m$  if it can be written as

$$a(\vec{U}_p; x, \xi) = a_0(\xi) + a_{\geq 1}(\vec{U}_p; x, \xi), \quad a_0 \in \tilde{\Gamma}^m, \quad a_{\geq 1} \in \Gamma_{\geq 1}^m.$$

4. For every  $s \geq s_0$  and every  $M \in \mathbb{N}$ , we set

$$|a(\vec{U}_p)|_{m,s,M} := \max_{|\beta| \leq M} \sup_{|\xi| \geq \frac{1}{2}} |\xi|^{-m+|\beta|} \left\| \partial_\xi^\beta a(\vec{U}_p; \bullet, \xi) \right\|_{s,I}. \quad (4.5)$$

In particular, the tame estimate Eq. (4.5) defining  $\Gamma_{\geq q}^m$  reads

$$|a(\vec{U}_p)|_{m,s,M} \lesssim_{s,M} \|\vec{U}_p\|_{s_0+\mu_M^a, I}^{q-1} \|\vec{U}_p\|_{s+\mu_M^a, I}.$$

For a background-independent symbol  $a_0 \in \tilde{\Gamma}^m$ , we set

$$|a_0|_{m,M} := \max_{|\beta| \leq M} \sup_{|\xi| \geq \frac{1}{2}} |\xi|^{-m+|\beta|} \left| \partial_\xi^\beta a_0(\xi) \right|.$$

Accordingly, if

$$a = a_0 + a_{\geq 1} \in \Gamma_{\geq 0}^m, \quad a_0 \in \tilde{\Gamma}^m, \quad a_{\geq 1} \in \Gamma_{\geq 1}^m, \quad (4.6)$$

we use the convention

$$|a(\vec{U}_p)|_{m,s,M} := |a_0|_{m,M} + |a_{\geq 1}(\vec{U}_p)|_{m,s,M}.$$

5. For  $N, N_1, N_2 \in \mathbb{N}^*$ , the notations  $\left(\Gamma_{\geq q}^m\right)^{N_1 \times N_2}$  and  $\left(\Gamma_{\geq q}^m\right)^N$  are componentwise abbreviations, in accordance with the scalar convention above.

6. A symbol  $a(\vec{U}_p; x, \xi) \in \Gamma_{\geq q}^m$  is *pointwise real* if the map

$$(x, \xi) \mapsto a(\vec{U}_p; x, \xi)$$

is real-valued for every real-valued  $\vec{U}_p$ . It is *real-to-real* if

$$a(\vec{U}_p; x, -\xi) = \overline{a(\vec{U}_p; x, \xi)}.$$

**Remark 4.8** (Elementary rules for the vanishing order). Let  $q_1, q_2 \in \mathbb{N}$  and  $m_1, m_2 \in \mathbb{R}$ .

1. If  $m_1 \leq m_2$  and  $q_1 \geq q_2$ , then

$$\Gamma_{\geq q_1}^{m_1} \subset \Gamma_{\geq q_2}^{m_2}.$$

2. If  $a \in \Gamma_{\geq q_1}^{m_1}$  and  $b \in \Gamma_{\geq q_2}^{m_2}$ , then, using the Fourier multiplier–nonlinear decomposition Eq. (4.6),

$$ab \in \Gamma_{\geq q_1+q_2}^{m_1+m_2}.$$

3. If  $a_0 \in \tilde{\Gamma}^{m_1}$  and  $b \in \Gamma_{\geq q}^{m_2}$ , then

$$a_0 b \in \Gamma_{\geq q}^{m_1+m_2}.$$

4. Sums satisfy

$$\Gamma_{\geq q_1}^{m_1} + \Gamma_{\geq q_2}^{m_2} \subseteq \Gamma_{\geq \min\{q_1, q_2\}}^{\max\{m_1, m_2\}}.$$

**Definition 4.9** (Functions). For  $q \geq 1$  we set

$$\mathcal{F}_{\geq q} := \left\{ f \in \Gamma_{\geq q}^0 : \partial_{\xi} f = 0 \right\}, \quad \mathcal{F} := \mathcal{F}_{\geq 1}.$$

We denote by

$$\mathcal{F}_{\geq 0} := \mathbb{C} + \mathcal{F}_{\geq 1}.$$

The superscript  $\mathbb{R}$  denotes the pointwise real-valued subclass; thus  $\mathcal{F}_{\geq q}^{\mathbb{R}}$ ,  $\mathcal{F}^{\mathbb{R}}$ , and  $\mathcal{F}_{\geq 0}^{\mathbb{R}}$  have the analogous meaning.

**Quantizations and para-composition rules.** Let us recall the Weyl quantization formula of a symbol

$$\text{Op}^W(a)[u](x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} \hat{a}\left(\xi - \varsigma, \frac{\xi + \varsigma}{2}\right) \hat{u}(\varsigma) \, d\varsigma d\xi.$$

Let  $\epsilon \in (0, \frac{1}{4})$  and consider a smooth, radial cutoff  $\chi \in \mathcal{C}^\infty(\mathbb{R}^d; [0, 1])$  such that

$$\chi(\xi) = \begin{cases} 1 & \text{if } |\xi| \leq 1.1, \\ 0 & \text{if } |\xi| \geq 1.9, \end{cases} \quad \chi_\epsilon(\xi) := \chi\left(\frac{\xi}{\epsilon}\right). \quad (4.7)$$

The function  $(\varsigma, \xi) \mapsto \chi_\epsilon(\varsigma/|\xi|)$ , extended by zero at  $\xi = 0$ .

Given a symbol  $a$  in  $\Gamma_{\geq 0}^m$  we define the *regularized* symbol

$$a_{\chi_\epsilon}(x, \xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix \cdot \varsigma} \chi_\epsilon\left(\frac{\varsigma}{|\xi|}\right) \hat{a}(\varsigma, \xi) \, d\varsigma = \chi_\epsilon(D/|\xi|) a(x, \xi), \quad \xi \neq 0,$$

and set  $a_{\chi_\epsilon}(x, 0) := 0$ .

**Notation 4.10** (High-frequency extension and low-frequency remainder). Let  $m \in \mathbb{R}$ ,  $a \in \Gamma_{\geq 0}^m$  smooth in  $\xi \neq 0$ , and let  $\chi$  be the even cutoff fixed in Eq. (4.7). We decompose

$$a = a^\sharp + a^\flat, \quad a^\sharp(U; x, \xi) := (1 - \chi(\xi)) a(U; x, \xi), \quad a^\flat(U; x, \xi) := \chi(\xi) a(U; x, \xi).$$

- The symbol  $a^\sharp$  is the fixed high-frequency extension of  $a$ . It is independent of the chosen extension  $a^{\text{ext}}$ .
- The symbol  $a^\flat$  is the corresponding low-frequency remainder. For a complete homogeneous Fourier multiplier, the untruncated homogeneous expression remains the multiplier being defined on  $\xi \neq 0$ ;  $^\sharp$  is used only for localized symbolic reductions, with the low-frequency part treated separately.

**Definition 4.11** (Bony-Weyl quantization). Let  $\epsilon \in (0, \frac{1}{4})$ ,  $m \in \mathbb{R}$ ,  $a \in \Gamma_{\geq 0}^m$ , and  $u \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$ . We define the *Bony-Weyl paradifferential operator*  $\text{Op}^{\text{BW}}(a) := \text{Op}^W(a_{\chi_\epsilon})$ , i.e.

$$\text{Op}^{\text{BW}}(a)[u](x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \chi_\epsilon\left(\frac{2(\xi - \varsigma)}{|\xi + \varsigma|}\right) \hat{a}\left(\xi - \varsigma, \frac{\xi + \varsigma}{2}\right) \hat{u}(\varsigma) \, d\varsigma \right) e^{ix \cdot \xi} d\xi. \quad (4.8)$$

If  $A = [a_{ij}]_{1 \leq i \leq N_1, 1 \leq j \leq N_2}$  is a finite matrix of symbols in  $\Gamma_{\geq 0}^m$ , then  $\text{Op}^{\text{BW}}(A)$  is defined componentwise by

$$\text{Op}^{\text{BW}}(A) := [\text{Op}^{\text{BW}}(a_{ij})]_{1 \leq i \leq N_1, 1 \leq j \leq N_2}.$$

Notice that the diagonal  $\{\xi = -\varsigma\}$  is a set of measure zero in  $\mathbb{R}^{2d}$ . Moreover, there is no issue of integrability near this diagonal, because the integrand of Eq. (4.8) vanishes in a neighborhood of it. Moreover, if  $\chi_\epsilon\left(\frac{2(\xi - \varsigma)}{|\xi + \varsigma|}\right) \neq 0$  then  $|\xi - \varsigma| \leq \epsilon|\varsigma + \xi|$ , and therefore, for  $\epsilon \in (0, \frac{1}{4})$ ,

$$\frac{1 - \epsilon}{1 + \epsilon} |\varsigma| \leq |\xi| \leq \frac{1 + \epsilon}{1 - \epsilon} |\varsigma|, \quad \forall \xi, \varsigma \in \mathbb{R}^d. \quad (4.9)$$

**Lemma 4.12** (Schur test). *Let*

$$K : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{C}$$

*be measurable and assume that*

$$A_K := \sup_{\zeta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |K(\xi, \zeta)| d\xi < +\infty, \quad B_K := \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} |K(\xi, \zeta)| d\zeta < +\infty. \quad (4.10)$$

*Then the integral operator*

$$T_K f(\xi) := \int_{\mathbb{R}^d} K(\xi, \zeta) f(\zeta) d\zeta$$

*extends to an operator in  $\mathcal{L}(L^2(\mathbb{R}^d))$ , and*

$$\|T_K\|_{\mathcal{L}(L^2)} \leq \sqrt{A_K B_K}.$$

*Proof.* For  $f \in \mathcal{S}(\mathbb{R}^d)$ , Cauchy–Schwarz with respect to the measure  $|K(\xi, \zeta)| d\zeta$  gives

$$|T_K f(\xi)|^2 \leq \left( \int_{\mathbb{R}^d} |K(\xi, \zeta)| d\zeta \right) \left( \int_{\mathbb{R}^d} |K(\xi, \zeta)| |f(\zeta)|^2 d\zeta \right).$$

Integrating in  $\xi$ , using Eq. (4.10) and Fubini's theorem, we obtain

$$\|T_K f\|_{L^2}^2 \leq B_K \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |K(\xi, \zeta)| |f(\zeta)|^2 d\zeta d\xi \leq A_K B_K \|f\|_{L^2}^2.$$

The conclusion follows by density.  $\square$

The following lemma is immediate from the definition of Bony-Weyl quantization.

**Lemma 4.13** (Adjoint of Bony-Weyl quantization). *Let  $m \in \mathbb{R}$  and  $a \in \Gamma_{\geq 0}^m$ , then the adjoint operator  $\text{Op}^{\text{BW}}(a)^*$  is given by*

$$\text{Op}^{\text{BW}}(a)^* = \text{Op}^{\text{BW}}(\bar{a}).$$

*Proof.* Cf. [25].  $\square$

**Remark 4.14.** Let  $a \in \Gamma_{\geq 1}^m$ , the definition of the operator  $\text{Op}^{\text{BW}}(a)$  can be naturally extended to tempered distributions by duality, using Lemma 4.13:

$$\langle \text{Op}^{\text{BW}}(a) [\varphi'] \mid \varphi \rangle := \langle \varphi' \mid \text{Op}^{\text{BW}}(\bar{a}) \varphi \rangle, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}), \quad \varphi' \in \mathcal{S}'(\mathbb{R}^d; \mathbb{C})$$

thus proving that constants belong to  $\ker(\text{Op}^{\text{BW}}(a))$ , in fact

$$\begin{aligned} \langle \text{Op}^{\text{BW}}(a) [C] \mid \varphi \rangle &= \langle C \mid \text{Op}^{\text{BW}}(\bar{a}) [\varphi] \rangle \\ &= \langle \hat{C} \mid \mathcal{F}(\text{Op}^{\text{BW}}(\bar{a}) [\varphi]) \rangle \\ &= C \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \delta_0(\xi) \hat{a}_{\chi_c} \left( \xi - \zeta, \frac{\xi + \zeta}{2} \right) \overline{\hat{\varphi}(\zeta)} d\zeta d\xi \\ &= C \int_{\mathbb{R}^d} \underbrace{\hat{a}_{\chi_c} \left( -\zeta, \frac{\zeta}{2} \right)}_{=0} \overline{\hat{\varphi}(\zeta)} d\zeta = 0. \end{aligned}$$

◆

**Theorem 4.15** (Action of paradifferential operators). *Let  $m \in \mathbb{R}$  and  $a \in \Gamma_{\geq 0}^m$ . There exists  $N_d \in \mathbb{N}$ , depending only on  $d$ , such that  $\text{Op}^{\text{BW}}(a(U; x, \xi))$  extends to a bounded operator in*

$$\mathcal{L}(H^s(\mathbb{R}^d; \mathbb{C}); H^{s-m}(\mathbb{R}^d; \mathbb{C}))$$

*for every  $s \in \mathbb{R}$ , and*

$$\|\text{Op}^{\text{BW}}(a(U))\|_{\mathcal{L}(H^s(\mathbb{R}^d; \mathbb{C}); H^{s-m}(\mathbb{R}^d; \mathbb{C}))} \lesssim |a(U)|_{m, s_0, N_d} + \|\text{Op}^{\text{BW}}(a^b(U))\|_{\mathcal{L}(H^s(\mathbb{R}^d; \mathbb{C}); H^{s-m}(\mathbb{R}^d; \mathbb{C}))}, \quad (4.11)$$

*where  $s_0$  is as in Definition 4.7.*

This is the standard whole-space Bony–Weyl boundedness theorem; see, for instance, [28, Theorem 2.4] for the periodic counterpart.

**Remark 4.16.** The seminorm in Eq. (4.11) controls only the region  $|\xi| \geq 1/2$ ; the low-frequency component  $a^\flat$  has therefore to be estimated separately. This distinction is essential for symbols that are singular or merely defined away from  $\xi = 0$ .  $\blacklozenge$

We recall also the following classical results.

**Remark 4.17.** Notice that for  $q \in \mathbb{N}^*$  and  $m, \rho \in \mathbb{R}$ ,

1. If  $a \in \Gamma_{\geq q}^m$  then  $\text{Op}^{\text{BW}}(a(U; x, \xi)) U \in \mathcal{F}_{\geq q}$ ;
2. if  $r_\rho \in \mathcal{R}_{\geq q}^\rho$  then  $r_\rho(U) U \in \mathcal{F}_{\geq q}$ ;
3. if  $m \in \tilde{\Gamma}^m$ , then  $\text{Op}^{\text{BW}}(m) = m(D)$ ;
4. Theorem 4.15 implies that if  $a \in \Gamma_{\geq q}^m$ , then  $\text{Op}^{\text{BW}}(a) \in \mathcal{R}_{\geq q}^{-m}$ .

**Lemma 4.18** (Bony paraproduct decomposition). *Let  $r_1 + r_2 > 0$  and  $u \in H^{r_1}(\mathbb{R}^d; \mathbb{C})$ ,  $v \in H^{r_2}(\mathbb{R}^d; \mathbb{C})$ . Then*

$$uv = \text{Op}^{\text{BW}}(u) v + \text{Op}^{\text{BW}}(v) u + R^{\text{BW}}[u, v]$$

where  $R^{\text{BW}}[u, v]$  is a symmetric bilinear map

$$H^{r_1}(\mathbb{R}^d; \mathbb{C}) \times H^{r_2}(\mathbb{R}^d; \mathbb{C}) \longrightarrow H^{r_1+r_2-d/2}(\mathbb{R}^d; \mathbb{C}),$$

and

$$\|R^{\text{BW}}[u, v]\|_{r_1+r_2-d/2} \lesssim \|u\|_{r_1} \|v\|_{r_2}.$$

*Proof.* See Lemma 2.7 of [28] for the periodic setting.  $\square$

**Remark 4.19.** For every  $\rho \geq 0$  and  $s_0 > d/2 + \rho$ , the operator  $R(u)h := R^{\text{BW}}[u, h]$  satisfies

$$\|R(u)h\|_{H^s} \lesssim_s \|u\|_{H^{s_0}} \|h\|_{H^{s-\rho}}, \quad s \geq s_0.$$

Indeed, apply Lemma 4.18 with  $r_1 = s_0$  and  $r_2 = s - \rho$  and use  $s_0 + s - \rho - d/2 > s$ . Thus  $R \in \mathcal{R}_{\geq 1}^\rho$ .  $\blacklozenge$

**Lemma 4.20** (Bony parilinearization formula). *Let  $\rho \geq 0$  and let  $f \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$  satisfy  $f(0) = 0$ . There exist  $s_0 > d/2 + \rho$ ,  $r_0 > 0$ , and  $r_f \in \mathcal{R}_{\geq 1}^\rho$  such that, for every  $s \geq s_0$  and  $u \in B_{s_0, \mathbb{R}}(I; r_0) \cap \mathcal{C}(I; H^s)$ ,*

$$f(u) = \text{Op}^{\text{BW}}(f'(u)) u + r_f(u) u.$$

*If  $f^{(j)}(0) = 0$  for  $0 \leq j < q$ , with  $q \geq 2$ , then  $r_f$  may be chosen in  $\mathcal{R}_{\geq q-1}^\rho$ .*

*Proof.* Use the dyadic parilinearization in [84] and the change of quantization in [25]. Keeping one maximal-frequency factor as the argument gives

$$\|r_f(u)h\|_{H^s} \lesssim_{s, \rho} \|u\|_{H^{s_0}} \|h\|_{H^{s-\rho}}, \quad s \geq s_0,$$

after increasing  $s_0$  according to  $\rho$ . If  $f$  vanishes to order  $q \geq 2$ , Taylor's formula leaves  $q - 1$  background factors and gives the asserted vanishing order.  $\square$

**Definition 4.21** (Poisson bracket). • Let  $a, b \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C})$ , we define the *Poisson bracket* as

$$\{a, b\} := \nabla_\xi a \cdot \nabla_x b - \nabla_x a \cdot \nabla_\xi b = J_\xi a \nabla_x b - J_x a \nabla_\xi b; \quad (4.12)$$

- Let  $N_1, N_2, N_3 \in \mathbb{N}^*$ , and

$$A \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N_1 \times N_2}), \quad B \in \mathcal{C}^1(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N_2 \times N_3}).$$

We use the notation

$$\{A, B\} := \left[ \sum_{j=1}^{N_2} \{a_{ij}, b_{jk}\} \right]_{\substack{i=1, \dots, N_1 \\ k=1, \dots, N_3}} \in \mathcal{C}^0(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{C}^{N_1 \times N_3}). \quad (4.13)$$

**Definition 4.22** (Asymptotic expansion of composition symbol). Let  $m_1, m_2 \in \mathbb{R}$ ,  $\rho \geq 0$ , and

$$\sigma(D_x, D_\xi, D_y, D_\zeta) := D_\xi \cdot D_y - D_x \cdot D_\zeta,$$

where  $D_x = \frac{1}{i} \partial_x$  and  $D_\xi, D_y, D_\zeta$  are similarly defined. Let  $a \in \Gamma_{\geq 0}^{m_1}$  and  $b \in \Gamma_{\geq 0}^{m_2}$ . Choose a common admissible base index  $s_0$ , the finite losses required by the displayed expansion, and a radius below the two radii supplied by Definition 4.7. For  $s > s_0 + \rho$  and  $U$  in the corresponding ball, we define

$$(a \#_\rho b)(U; x, \xi) := \sum_{k=0}^{\lfloor \rho \rfloor} \frac{1}{k!} \left( \frac{i}{2} \sigma(D_x, D_\xi, D_y, D_\zeta) \right)^k [a(U; x, \xi) b(U; y, \zeta)] \Big|_{x=y, \xi=\zeta}. \quad (4.14)$$

**Remark 4.23.** 1. For  $\lfloor \rho \rfloor = 1$ , we have that

$$a \#_\rho b = ab + \frac{1}{2i} \{a, b\}.$$

2. If  $a \in \Gamma_{\geq q_1}^{m_1}$  and  $b \in \Gamma_{\geq q_2}^{m_2}$  then

$$\{a, b\} \in \Gamma_{\geq q_1 + q_2}^{m_1 + m_2 - 1}.$$

◆

**Composition rules.** The following result is classical, we refer the reader to [28, Theorem 2.5]

**Proposition 4.24** (Composition with vanishing order). Let  $m_1, m_2 \in \mathbb{R}$ ,  $\rho \geq 0$ ,  $q_1, q_2 \in \mathbb{N}$ , and let  $a \in \Gamma_{\geq q_1}^{m_1}$ ,  $b \in \Gamma_{\geq q_2}^{m_2}$ . Assume moreover that the low-frequency contributions in Eq. (4.11) associated with  $a$ ,  $b$ , and  $a \#_\rho b$  are bounded. Define

$$r_\rho[a, b] := \text{Op}^{\text{BW}}(a) \circ \text{Op}^{\text{BW}}(b) - \text{Op}^{\text{BW}}(a \#_\rho b).$$

Then  $a \#_\rho b \in \Gamma_{\geq q_1 + q_2}^{m_1 + m_2}$  and

$$r_\rho[a, b] \in \mathcal{R}_{\geq q_1 + q_2}^{\rho - m_1 - m_2}.$$

When  $q_i = 0$ , the assertion is understood after splitting the corresponding symbol into its flat and nonlinear parts.

**Lemma 4.25.** Let  $\rho, \rho', m \in \mathbb{R}$ , let  $q_a, q_f, q_r, q'_r \in \mathbb{N}$ , and let  $a \in \Gamma_{\geq q_a}^m$ ,  $f \in \mathcal{F}_{\geq q_f}$ ,  $r_\rho \in \mathcal{R}_{\geq q_r}^\rho$ , and  $r_{\rho'} \in \mathcal{R}_{\geq q'_r}^{\rho'}$ . Then the following tame rules apply, after choosing a common, sufficiently large baseline and reducing the radius. For full paradifferential operators, the low-frequency bound in Eq. (4.11) is assumed with vanishing order  $q_a$ :

- i  $\text{Op}^{\text{BW}}(a) \circ r_\rho, r_\rho \circ \text{Op}^{\text{BW}}(a) \in \mathcal{R}_{\geq q_a + q_r}^{\rho - m};$
- ii  $r_\rho \circ r_{\rho'} \in \mathcal{R}_{\geq q_r + q'_r}^{\rho + \rho'};$
- iii Assume that  $a(\vec{U}_p, v; x, \xi) \in \Gamma_{\geq q_a}^m$ ,  $f(\vec{U}_p) \in \mathcal{F}_{\geq 1}$ , then, after possibly reducing the radius of definition, the symbol  $\tilde{a}(\vec{U}_p; x, \xi) := a(\vec{U}_p, f(\vec{U}_p); x, \xi)$  satisfies  $\tilde{a} \in \Gamma_{\geq q_a}^m$ .
- iv Assume that  $r_\rho(\vec{U}_p, v) \in \mathcal{R}_{\geq q_r}^\rho$ ,  $f(\vec{U}_p) \in \mathcal{F}_{\geq 1}$ , then, after possibly reducing the radius of definition, the operator  $\tilde{r}_\rho(\vec{U}_p) := r_\rho(\vec{U}_p, f(\vec{U}_p))$  satisfies  $\tilde{r}_\rho \in \mathcal{R}_{\geq q_r}^\rho$ .

*Proof.* Choose a common baseline  $S$  large enough for all fixed regularity indices below. By Theorem 4.15 and Eq. (4.3),

$$\begin{aligned} \|\text{Op}^{\text{BW}}(a) r_\varrho(U) h\|_{H^s} &\lesssim_s \|U\|_S^{q_a} \|r_\varrho(U) h\|_{H^{s+m}} \lesssim_s \|U\|_S^{q_a+q_r} \|h\|_{H^{s+m-\varrho}}, \\ \|r_\varrho(U) \text{Op}^{\text{BW}}(a) h\|_{H^s} &\lesssim_s \|U\|_S^{q_r} \|\text{Op}^{\text{BW}}(a) h\|_{H^{s-\varrho}} \lesssim_s \|U\|_S^{q_a+q_r} \|h\|_{H^{s+m-\varrho}}, \\ \|r_\varrho(U) r_{\varrho'}(U) h\|_{H^s} &\lesssim_s \|U\|_S^{q_r} \|r_{\varrho'}(U) h\|_{H^{s-\varrho}} \lesssim_s \|U\|_S^{q_r+q_{r'}} \|h\|_{H^{s-\varrho-\varrho'}}. \end{aligned}$$

This proves Items i and ii. For the substitutions, Definitions 4.7 and 4.9 give  $\|f(U)\|_{s_0} \lesssim \|U\|_{s_0+\mu}$  for a fixed finite loss  $\mu$ . Increase  $S$  and reduce the radius so that  $(U, f(U))$  belongs to the required low-norm ball. Substitution in Eqs. (4.3) and (4.4) gives Items iii and iv.  $\square$

## 4.2 $z$ -dependent paradifferential calculus

The purpose of this subsection is to collect the  $z$ -dependent paradifferential tools used in the parilinearization of the singular integrals. We first introduce the scalar kernel classes and the corresponding operator-valued smoothing classes, together with their elementary composition and integration rules. We then prove the kernel-to-symbol reduction for  $z$ -dependent Bony–Weyl operators and its compensated endpoint version. The remaining paragraphs deal with the two special mechanisms used in the parilinearization: far-field tails, which give order-zero paraproducts modulo smoothing remainders, and localized homogeneous convolution models, whose cutoff can be removed modulo smoothing Fourier multipliers. Finally, we record the local Taylor and finite-difference estimates needed to verify that the concrete kernels arising from  $B_0$  and  $q$  belong to the abstract classes introduced above.

### Kernel classes.

**Definition 4.26** (Kernel classes). Let  $n \in \mathbb{R}$ ,  $q \in \mathbb{N}^*$ ,  $p \in \mathbb{N}^*$ , and let  $\vec{U}_p := (u_1, \dots, u_p)^\top$ .

1. The class  $K\mathcal{F}_{\text{loc}, \geq q}^n$  consists of scalar kernels  $k = k(\vec{U}_p; x, z)$  for which there exists  $s_0^k > d/2$  such that the following holds: for every  $N \in \mathbb{N}$ , there exists a loss  $\mu_N^k \geq 0$  such that, for every  $s \geq s_0^k$ , there exists a radius  $r_{s,N}^k > 0$ . For every  $0 < r \leq r_{s,N}^k$  and every

$$\vec{U}_p \in \left( B_{s_0^k + \mu_N^k, \mathbb{R}}(I; r) \cap \mathcal{C}\left(I; H^{s+\mu_N^k}(\mathbb{R}^d; \mathbb{R})\right) \right)^p,$$

the map  $z \mapsto k(\vec{U}_p; \bullet, z)$  is measurable on  $\mathbb{R}^d$  and belongs to  $\mathcal{C}^N(\mathbb{R}^d \setminus \{0\}; H^s(\mathbb{R}^d; \mathbb{C}))$ . Moreover, for every  $\alpha \in \mathbb{N}^d$  with  $|\alpha| \leq N$  and every  $0 < |z| \leq 2$ , one has

$$\|\partial_z^\alpha k(\vec{U}_p; \bullet, z)\|_{s,I} \lesssim_{s,\alpha,N} \|\vec{U}_p\|_{s_0^k + \mu_N^k, I}^{q-1} \|\vec{U}_p\|_{s+\mu_N^k, I} \|z\|^{n-|\alpha|}. \quad (4.15)$$

2. The class  $K\mathcal{F}_{\geq q}^n$  consists of the kernels  $k \in K\mathcal{F}_{\text{loc}, \geq q}^n$  satisfying

$$\text{supp}_z k(\vec{U}_p; x, \cdot) \subseteq \{z \in \mathbb{R}^d : |z| \leq 2\}.$$

Under the same assumptions on  $s$ ,  $r$ , and  $\vec{U}_p$  as above, for every  $\alpha \in \mathbb{N}^d$  with  $|\alpha| \leq N$  and every  $z \neq 0$ , one has

$$\|\partial_z^\alpha k(\vec{U}_p; \bullet, z)\|_{s,I} \lesssim_{s,\alpha,N} \|\vec{U}_p\|_{s_0^k + \mu_N^k, I}^{q-1} \|\vec{U}_p\|_{s+\mu_N^k, I} \|z\|^{n-|\alpha|} \mathbb{1}_{\{0 < |z| \leq 2\}}. \quad (4.16)$$

3. The class  $K\mathcal{F}_{\text{tail}, \geq q}$  consists of scalar maps

$$k_{\text{tail}} : B_{s_0, \mathbb{R}}(I; r_0) \longrightarrow L_{\text{loc}}^\infty(\mathbb{R}_x^d \times \mathbb{R}_z^d; \mathbb{C}), \quad u \longmapsto k_{\text{tail}}(u; x, z),$$

for which there exist  $s_0 > d/2$ ,  $r_0 > 0$ ,  $m > 0$ ,  $r_g > 0$ ,  $b \in \mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$  and  $g \in \mathcal{C}^\omega((-r_g, r_g); \mathbb{C})$ , such that

$$k_{\text{tail}}(u; x, z) := (1 - \chi(z)) b(z) g(\Delta_z u(x)),$$

where  $\chi$  is the cutoff function introduced in Eq. (4.7). The function  $b$  satisfies, for every  $\alpha \in \mathbb{N}^d$ ,

$$|\partial_z^\alpha b(z)| \lesssim_\alpha |z|^{-m-|\alpha|}, \quad |z| \geq 1, \quad (4.17)$$

while  $g$  vanishes to order at least  $q$  at the origin,

$$g^{(j)}(0) = 0, \quad 0 \leq j \leq q-1. \quad (4.18)$$

We further require  $m+q > d$ .

4. We use the conventions

$$K\mathcal{F}_{\text{loc}}^n := K\mathcal{F}_{\text{loc}, \geq 1}^n, \quad K\mathcal{F}^n := K\mathcal{F}_{\geq 1}^n, \quad K\mathcal{F}_{\text{tail}} := K\mathcal{F}_{\text{tail}, \geq 1}.$$

**Remark 4.27** (Cutoff of a local kernel). For every  $n \in \mathbb{R}$  and  $q \in \mathbb{N}^*$ ,

$$k \in K\mathcal{F}_{\text{loc}, \geq q}^n \implies \chi(z)k \in K\mathcal{F}_{\geq q}^n, \quad K\mathcal{F}_{\geq q}^n \subseteq K\mathcal{F}_{\text{loc}, \geq q}^n.$$

Indeed, the inclusion on the right is immediate. For the implication on the left, apply Leibniz' rule to  $\partial_z^\alpha (\chi(z)k)$ . The terms in which no derivative falls on  $\chi$  are estimated by Eq. (4.15), while the remaining terms are supported in the fixed annulus where all powers of  $|z|$  are harmless. The support property follows from the compact support of  $\chi$ . Thus local Taylor remainders become compactly supported kernels after multiplication by the fixed cutoff  $\chi(z)$ .  $\blacklozenge$

**Remark 4.28** (Integration of rapidly decreasing scalar kernels). Let  $n > -d$ ,  $q \in \mathbb{N}^*$ , and  $k \in K\mathcal{F}_{\geq q}^n$ . Then

$$v_k(\vec{U}_p; x) := \int_{\mathbb{R}^d} k(\vec{U}_p; x, z) dz \in \mathcal{F}_{\geq q}.$$

Moreover, for every  $s \geq s_0^k$  one has

$$\|v_k(\vec{U}_p; \bullet)\|_{s, I} \lesssim_s \|\vec{U}_p\|_{s_0^k + \mu, I}^{q-1} \|\vec{U}_p\|_{s+\mu, I}$$

for a finite loss  $\mu \geq 0$ . Indeed, this follows directly from Minkowski's integral inequality, Eq. (4.16), and  $\int_0^2 r^{n+d-1} dr < +\infty$ . The localized case is reduced to this one by Remark 4.27.  $\blacklozenge$

### Operator-valued smoothing kernel classes.

**Definition 4.29** (Kernel-smoothing operators). Let  $\varrho, n \in \mathbb{R}$ ,  $p \in \mathbb{N}^*$ ,  $q \in \mathbb{N}$ , and let  $\vec{U}_p := (u_1, \dots, u_p)^\top$ .

1. The class  $K\mathcal{R}_{\text{loc}, \geq q}^{\varrho, n}$  consists of  $z$ -dependent families of linear operators  $r_\varrho(\vec{U}_p; z)$  for which there exists  $s_0^R > d/2$  such that, for every  $s \geq s_0^R$ , there exists a radius  $r_s^R > 0$  with the following property.

For every  $0 < r \leq r_s^R$ , every

$$\vec{U}_p \in \left(B_{s_0^R, \mathbb{R}}(I; r)\right)^p,$$

and every  $h \in H^{s-\varrho}(\mathbb{R}^d; \mathbb{C})$ , the map  $z \mapsto r_\varrho(\vec{U}_p; z)h$  is measurable and, for  $0 < |z| \leq 2$ , satisfies

$$\|r_\varrho(\vec{U}_p; z)h\|_{s, I} \lesssim_{s, \varrho} \|\vec{U}_p\|_{s_0^R, I}^q \|h\|_{s-\varrho} |z|^n. \quad (4.19)$$

2. The class  $K\mathcal{R}_{\geq q}^{\varrho, n}$  consists of the families  $r_\varrho \in K\mathcal{R}_{\text{loc}, \geq q}^{\varrho, n}$  satisfying  $\text{supp}_z r_\varrho(\vec{U}_p; \cdot) \subseteq \{z \in \mathbb{R}^d : |z| \leq 2\}$ , and, for every  $z \neq 0$ ,

$$\|r_\varrho(\vec{U}_p; z)h\|_{s, I} \lesssim_{s, \varrho} \|\vec{U}_p\|_{s_0^R, I}^q \|h\|_{s-\varrho} |z|^n \mathbb{1}_{\{0 < |z| \leq 2\}}, \quad (4.20)$$

under the same assumptions on  $s$ ,  $\vec{U}_p$ , and  $h$  as above.

3. For  $q = 0$ , the background factor in the preceding estimates is understood as 1. We use the conventions

$$K\mathcal{R}_{\text{loc}}^{\varrho, n} := K\mathcal{R}_{\text{loc}, \geq 1}^{\varrho, n}, \quad K\mathcal{R}^{\varrho, n} := K\mathcal{R}_{\geq 1}^{\varrho, n}.$$

**Remark 4.30** (Separately estimated parameters). When a separately estimated parameter appears in brackets, as in  $a[\omega]$ ,  $k[\omega]$ , or  $r[\omega]$ , the notation

$$\mathcal{L}(H^{s_0}; Y)$$

is understood according to the structure of the target class  $Y$ .

If

$$Y \in \{\Gamma^m, \mathcal{F}, K\mathcal{F}_{\text{loc}}^n, K\mathcal{F}^n\},$$

the notation records linear dependence on the parameter together with the corresponding tame Sobolev-scale estimates. For instance, if  $a_{[m, \geq q]}[\omega](U)$  is linear in  $\omega$  and has vanishing order at least  $q$  in the sense of Remark 4.31, then, schematically, for every  $s \geq s_0$  and  $M \in \mathbb{N}$ ,

$$\begin{aligned} \left| a_{[m, \geq q]}[\omega](U) \right|_{m, s, M} &\lesssim_{s, M} \|U\|_{H^{s_0 + \mu_M}}^{q-1} \|\omega\|_{H^{s + \mu_M}} \\ &\quad + \mathbb{1}_{\{q \geq 2\}} \|U\|_{H^{s_0 + \mu_M}}^{q-2} \|U\|_{H^{s + \mu_M}} \|\omega\|_{H^{s_0 + \mu_M}}, \end{aligned}$$

for some finite loss  $\mu_M \geq 0$  independent of  $s$ . The analogous interpretation is used for functions and kernel functions.

If instead

$$Y \in \{\mathcal{R}^\varrho, K\mathcal{R}_{\text{loc}}^{\varrho, n}, K\mathcal{R}^{\varrho, n}\},$$

the notation refers to a fixed-baseline estimate. For instance, a remainder  $r_{[\varrho, \geq q]}[\omega](U)$  linear in  $\omega$  satisfies, for some  $S$  independent of  $s$ ,

$$\left\| r_{[\varrho, \geq q]}[\omega](U) h \right\|_{H^s} \lesssim_{s, \varrho} \|U\|_{H^S}^{q-1} \|\omega\|_{H^S} \|h\|_{H^{s-\varrho}}.$$

In particular, no Sobolev norm of the parameter whose index grows with the output index  $s$  is allowed in the smoothing estimates.

The convention above concerns only the Sobolev regularity of the separately estimated parameter; its contribution to the vanishing order is specified in Remark 4.31.  $\blacklozenge$

**Remark 4.31** (Vanishing order for linear arguments). When an object in any of the classes introduced above depends linearly on a separately estimated Sobolev argument, this argument contributes one to its vanishing order. Thus, if  $T[\omega](U)$  is linear in  $\omega$  and vanishes to order at least  $q-1$  with respect to  $U$ , we say that  $T[\omega](U)$  has vanishing order at least  $q$ . The same convention is used for functions, symbols, smoothing operators, kernel functions, and kernel-smoothing operators.  $\blacklozenge$

**Remark 4.32** (Cutoff of a local smoothing kernel). For every  $\varrho, n \in \mathbb{R}$  and  $q \in \mathbb{N}$ ,

$$R \in K\mathcal{R}_{\text{loc}, \geq q}^{\varrho, n} \implies \chi(z) R \in K\mathcal{R}_{\geq q}^{\varrho, n}, \quad K\mathcal{R}_{\geq q}^{\varrho, n} \subseteq K\mathcal{R}_{\text{loc}, \geq q}^{\varrho, n}.$$

Indeed, multiplication by  $\chi$  preserves the tame estimate in Eq. (4.19) and makes the family compactly supported in  $z$ . The inclusion is immediate.  $\blacklozenge$

**Lemma 4.33** (Composition rules for  $z$ -dependent operators). Let  $q_1, q_2 \in \mathbb{N}^*$ , let  $m, m', n, n' \in \mathbb{R}$ , and let  $\varrho, \varrho' \geq 0$ .

i If

$$k \in K\mathcal{F}_{\text{loc}, \geq q_1}^m, \quad k' \in K\mathcal{F}_{\text{loc}, \geq q_2}^{m'},$$

then, for every  $\varrho > 0$ ,

$$\text{Op}^{\text{BW}}(kk'(\vec{U}_p; x, z)) - \text{Op}^{\text{BW}}(k(\vec{U}_p; x, z)) \circ \text{Op}^{\text{BW}}(k'(\vec{U}_p; x, z)) \in K\mathcal{R}_{\text{loc}, \geq q_1 + q_2}^{\varrho, m+m'}. \quad (4.21)$$

ii If

$$k \in K\mathcal{F}_{\text{loc}, \geq q_1}^m, \quad r_\varrho \in K\mathcal{R}_{\text{loc}, \geq q_2}^{\varrho, n},$$

then

$$\text{Op}^{\text{BW}}(k(\vec{U}_p; x, z)) \circ r_\varrho(\vec{U}_p; z) \in K\mathcal{R}_{\text{loc}, \geq q_1 + q_2}^{\varrho, m+n}, \quad r_\varrho(\vec{U}_p; z) \circ \text{Op}^{\text{BW}}(k(\vec{U}_p; x, z)) \in K\mathcal{R}_{\text{loc}, \geq q_1 + q_2}^{\varrho, m+n}.$$

iii If

$$r_\varrho \in K\mathcal{R}_{\text{loc}, \geq q_1}^{\varrho, n}, \quad r_{\varrho'} \in K\mathcal{R}_{\text{loc}, \geq q_2}^{\varrho', n'},$$

then

$$r_\varrho(\vec{U}_p; z) \circ r_{\varrho'}(\vec{U}_p; z) \in K\mathcal{R}_{\text{loc}, \geq q_1 + q_2}^{\varrho + \varrho', n + n'}.$$

The same conclusions hold with the local classes replaced by their compactly supported counterparts.

*Proof.* For Item i, apply Proposition 4.24 for each  $z \neq 0$ . Choose a fixed  $S$  large enough for the prescribed gain  $\varrho$ . By Eq. (4.15), increasing the baseline to absorb the finite kernel losses gives

$$\|k(U; \bullet, z)\|_{H^S} \|k'(U; \bullet, z)\|_{H^S} \lesssim \|U\|_{s_0}^{q_1 + q_2} |z|^{m + m'}.$$

This proves Eq. (4.21). The order-zero paraproduct bound and Eq. (4.19) give Item ii. Applying the two kernel-smoothing estimates successively gives Item iii; the smoothing, local, and vanishing orders add. Compact support is preserved in each case.  $\square$

**Lemma 4.34** (Integration of kernel-smoothing maps). *Let  $\varrho \in \mathbb{R}$ ,  $q \in \mathbb{N}$ , and let  $r_\varrho(\vec{U}_p; z)$  be a  $z$ -dependent family of operators. Assume that, for some  $n > -d$ , either*

$$r_\varrho \in K\mathcal{R}_{\geq q}^{\varrho, n} \quad \text{or} \quad r_\varrho = \chi(z) r_0, \quad r_0 \in K\mathcal{R}_{\text{loc}, \geq q}^{\varrho, n}.$$

Then

$$\int_{\mathbb{R}^d} r_\varrho(\vec{U}_p; z) dz \in \mathcal{R}_{\geq q}^\varrho, \quad \int_{\mathbb{R}^d} r_\varrho(\vec{U}_p; z) [h(\cdot - z)] dz = \tilde{r}_\varrho(\vec{U}_p) h, \quad \tilde{r}_\varrho \in \mathcal{R}_{\geq q}^\varrho.$$

*Proof.* By Remark 4.32, it is enough to consider  $r \in K\mathcal{R}_{\geq q}^{\varrho, n}$ . Let  $s_0$  be the baseline index in Definition 4.29. Minkowski's inequality and Eq. (4.20) give

$$\left\| \int_{\mathbb{R}^d} r_\varrho(\vec{U}_p; z) h dz \right\|_{s, I} \lesssim_{s, \varrho} \|\vec{U}_p\|_{s_0, I}^q \|h\|_{H^{s-\varrho}} \int_{|z| \leq 2} |z|^n dz.$$

The last integral is finite because  $n > -d$ , and this is exactly the tame estimate defining  $\mathcal{R}_{\geq q}^\varrho$ .

For the translated operator, the same argument applies, since  $\|h(\cdot - z)\|_{H^s} = \|h\|_{H^s}$  for every  $s \in \mathbb{R}$ . The vanishing order is unchanged by integration.  $\square$

### Integration of $z$ -dependent paradifferential operators.

**Proposition 4.35** (Kernel-to-symbol reduction for compactly supported kernels). *Let  $n > -d$ ,  $q \in \mathbb{N}^*$ ,  $\varrho > 0$ , and let  $k \in K\mathcal{F}_{\geq q}^n$  be a scalar kernel. Let  $\sigma \in \mathbb{R}$  and  $\omega \in H^\sigma(\mathbb{R}^d; \mathbb{C})$ , and define*

$$(\tau_k^0(u)\omega)(x) := \int_{\mathbb{R}^d} \text{Op}^{\text{BW}}(k(u; \cdot, z)) [\omega(\cdot - z)](x) dz.$$

Then there exists  $a_k \in \Gamma_{\geq q}^{-(n+d)}$  such that

$$\tau_k^0(u)\omega = \text{Op}^{\text{BW}}(a_k(u; x, \xi))\omega.$$

If  $k$  is real-valued for real-valued  $u$ , then

$$a_k(u; x, -\xi) = \overline{a_k(u; x, \xi)}. \quad (4.22)$$

*Proof.* By density it is enough to argue for smooth  $u$  and  $\omega$ .

**Step 1** (Exact Bony–Weyl representation). Define the centered symbol

$$a_k(u; x, \xi) := \int_{\mathbb{R}^d} k\left(u; x + \frac{z}{2}, z\right) e^{-iz \cdot \xi} dz. \quad (4.23)$$

Since  $n > -d$  and  $k$  is compactly supported in  $z$ , the integral is well-defined. Taking the Fourier transform in  $x$  gives

$$\widehat{a}_k(u; \zeta, \xi) = \int_{\mathbb{R}^d} \widehat{k}(u; \zeta, z) e^{-i(\xi - \frac{\zeta}{2}) \cdot z} dz. \quad (4.24)$$

On the other hand, by the explicit Bony–Weyl formula Eq. (4.8),

$$\mathcal{F}_{x \rightarrow v}(\tau_k^0(u) \omega)(v) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \chi_\varepsilon\left(\frac{2(v - v')}{|v + v'|}\right) \int_{\mathbb{R}^d} \widehat{k}(u; v - v', z) e^{-iv' \cdot z} dz \widehat{\omega}(v') dv'. \quad (4.25)$$

Setting  $\zeta := v - v'$ , and  $\xi := \frac{v+v'}{2}$  one has  $v' = \xi - \zeta/2$ . Hence Eq. (4.24) and Eq. (4.25) coincide exactly with the Fourier representation of  $\text{Op}^{\text{BW}}(a_k(u))$ . Therefore

$$\tau_k^0(u) = \text{Op}^{\text{BW}}(a_k(u; x, \xi)).$$

It remains only to prove that  $a_k \in \Gamma_{\geq q}^{-(n+d)}$ .

**Step 2** (Symbol estimate). Fix  $s \geq s_0^k$  and  $M \in \mathbb{N}$ . Choose  $N_* \in \mathbb{N}$  such that

$$N_* > n + M + d, \quad (4.26)$$

and set

$$\tilde{\mu}_M^k := \mu_{N_*}^k + N_*. \quad (4.27)$$

Introduce the centered kernel

$$K_u(x, z) := k\left(u; x + \frac{z}{2}, z\right). \quad (4.28)$$

For every multi-index  $\alpha$  with  $|\alpha| \leq N_*$ , the chain rule gives

$$\partial_z^\alpha K_u(x, z) = \sum_{\gamma \leq \alpha} c_{\alpha, \gamma} (\partial_x^{\alpha - \gamma} \partial_z^\gamma k)\left(u; x + \frac{z}{2}, z\right), \quad c_{\alpha, \gamma} \in \mathbb{R}.$$

By translation invariance of Sobolev norms and Eq. (4.16),

$$\|\partial_z^\alpha K_u(\cdot, z)\|_{H^s} \lesssim_{s, N_*} \|u\|_{H^{s_0^k + \tilde{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \tilde{\mu}_M^k}} |z|^{n - |\alpha|} \mathbb{1}_{\{0 < |z| \leq 2\}}. \quad (4.29)$$

Let now  $\beta \in \mathbb{N}^d$ ,  $|\beta| \leq M$ . From Eq. (4.23),

$$\partial_\xi^\beta a_k(u; x, \xi) = (-i)^{|\beta|} \int_{\mathbb{R}^d} z^\beta K_u(x, z) e^{-iz \cdot \xi} dz. \quad (4.30)$$

For  $1/2 \leq |\xi| \leq 2$ , the estimate

$$\|\partial_\xi^\beta a_k(u; \cdot, \xi)\|_{H^s} \lesssim_{s, M} \|u\|_{H^{s_0^k + \tilde{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \tilde{\mu}_M^k}} \quad (4.31)$$

follows directly from Eq. (4.29) with  $\alpha = 0$ , since  $n > -d$ .

Assume henceforth that  $|\xi| \geq 2$ . Let  $\chi_1 := \chi$  and  $\chi_2 := 1 - \chi$ . We split Eq. (4.30) as

$$\partial_\xi^\beta a_k = a_{k, \beta}^0 + a_{k, \beta}^\infty,$$

where

$$\begin{aligned} a_{k, \beta}^0(u; x, \xi) &:= (-i)^{|\beta|} \int_{\mathbb{R}^d} \chi_1(|\xi| z) z^\beta K_u(x, z) e^{-iz \cdot \xi} dz, \\ a_{k, \beta}^\infty(u; x, \xi) &:= (-i)^{|\beta|} \int_{\mathbb{R}^d} \chi_2(|\xi| z) z^\beta K_u(x, z) e^{-iz \cdot \xi} dz. \end{aligned}$$

**Substep 2.1** (Close field). On the support of  $\chi_1(|\xi| z)$  one has

$$|z| \lesssim |\xi|^{-1}.$$

Therefore, by Minkowski's inequality and Eq. (4.29),

$$\begin{aligned} \|a_{k, \beta}^0(u; \cdot, \xi)\|_{H^s} &\lesssim \|u\|_{H^{s_0^k + \tilde{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \tilde{\mu}_M^k}} \int_0^{c|\xi|^{-1}} r^{n + |\beta| + d - 1} dr \\ &\lesssim \|u\|_{H^{s_0^k + \tilde{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \tilde{\mu}_M^k}} |\xi|^{-(n+d) - |\beta|}. \end{aligned} \quad (4.32)$$

The radial integral is finite precisely because  $n > -d$ .

**Substep 2.2 (Far field).** On the support of  $\chi_2(|\xi|z)$  one has

$$|z| \gtrsim |\xi|^{-1}.$$

Recalling that  $\hat{\xi} := \frac{\xi}{|\xi|}$ , observe that

$$e^{-iz \cdot \xi} = (-i|\xi|)^{-N_*} \partial_{\hat{\xi}}^{N_*} e^{-iz \cdot \xi}.$$

Therefore, integration by parts  $N_*$  times yields

$$a_{k,\beta}^\infty(u; x, \xi) = c_{\beta, N_*} |\xi|^{-N_*} \int_{\mathbb{R}^d} \partial_{\hat{\xi}}^{N_*} \left( \chi_2(|\xi|z) z^\beta K_u(x, z) \right) e^{-iz \cdot \xi} dz. \quad (4.33)$$

There are no boundary contributions: the cutoff  $\chi_2(|\xi|z)$  removes a neighbourhood of  $z = 0$ , while  $K_u$  is compactly supported in  $\{|z| \leq 2\}$ .

Expanding the derivative in Eq. (4.33), every term contains derivatives of the cutoff, of  $z^\beta$ , and of  $K_u$ . Derivatives of the cutoff are supported where  $|z| \sim |\xi|^{-1}$ , and hence satisfy

$$|\partial_z^\gamma \chi_2(|\xi|z)| \lesssim_\gamma |z|^{-|\gamma|}.$$

Combining this bound with Eq. (4.29) gives

$$\left\| \partial_{\hat{\xi}}^{N_*} \left( \chi_2(|\xi|z) z^\beta K_u(\cdot, z) \right) \right\|_{H^s} \lesssim_{s, M, N_*} \|u\|_{H^{s_0^k + \bar{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \bar{\mu}_M^k}} |z|^{n+|\beta|-N_*} \mathbb{1}_{\{|z|^{-1} \leq |z| \leq 2\}}.$$

Consequently,

$$\begin{aligned} \left\| a_{k,\beta}^\infty(u; \cdot, \xi) \right\|_{H^s} &\lesssim \|u\|_{H^{s_0^k + \bar{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \bar{\mu}_M^k}} |\xi|^{-N_*} \int_{|z|^{-1}}^2 r^{n+|\beta|-N_*+d-1} dr \\ &\lesssim \|u\|_{H^{s_0^k + \bar{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \bar{\mu}_M^k}} |\xi|^{-(n+d)-|\beta|}, \end{aligned} \quad (4.34)$$

where in the last inequality we used

$$N_* > n + M + d \geq n + |\beta| + d.$$

Combining Eqs. (4.31), (4.32) and (4.34), we obtain, for every  $|\beta| \leq M$  and every  $|\xi| \geq 1/2$ ,

$$\left\| \partial_{\hat{\xi}}^\beta a_k(u; \cdot, \xi) \right\|_{H^s} \lesssim_{s, M} \|u\|_{H^{s_0^k + \bar{\mu}_M^k}}^{q-1} \|u\|_{H^{s + \bar{\mu}_M^k}} |\xi|^{-(n+d)-|\beta|}.$$

Hence

$$a_k \in \Gamma_{\geq q}^{-(n+d)}.$$

**Step 3 (Reality).** Finally, if  $k$  is real-valued for real-valued  $u$ , then Eq. (4.23) gives

$$a_k(u; x, -\xi) = \overline{a_k(u; x, \xi)}.$$

□

**Remark 4.36** (Kernel and symbol regularity). The indices  $N$  in Definition 4.26 and  $M$  in Definition 4.7 are finite-seminorm indices and should not be interpreted as maximal differentiability orders. More precisely, there are two distinct mechanisms in Proposition 4.35.

First, since  $n > -d$  and the kernel is compactly supported in  $z$ , Eq. (4.15) with  $\alpha = 0$  implies that the centered kernel in Eq. (4.28) is integrable in  $z$  with values in  $H^s$ . Its compact support also implies that all its  $z$ -moments are integrable. Hence Eq. (4.30) is well defined for every multi-index  $\beta$ , and the symbol in Eq. (4.23) is  $\mathcal{C}^\infty$  in  $\xi$ . No  $z$ -derivatives of the kernel are needed for this conclusion.

The  $z$ -regularity enters instead in the quantitative symbol estimates. To control the symbol seminorm of index  $M$ , the proof chooses a finite number  $N_*$  of  $z$ -derivatives according to Eq. (4.26) and integrates by parts in the far region. This yields the decay required for all frequency derivatives  $|\beta| \leq M$ . Because the Bony–Weyl symbol involves the centered kernel  $k(u; x + z/2, z)$ , differentiation in  $z$  also produces  $x$ -derivatives, which accounts for the loss in Eq. (4.27).

Thus every finite symbol seminorm requires only finitely many kernel seminorms, with losses and radii allowed to depend on their respective indices. This is precisely the reason for the quantifiers in Definitions 4.7 and 4.26:  $\mathcal{C}^\infty$  regularity in  $\xi$  is compatible with finite Sobolev regularity of the background, while higher quantitative symbol seminorms may require correspondingly higher Sobolev regularity. ♦

**Far-field tails.** For  $q \in \mathbb{N}^*$  and  $k \in K\mathcal{F}_{\text{tail}, \geq q}$ , set

$$\tau_k^\infty(u) \omega(x) := \int_{\mathbb{R}^d} k(u; x, z) \omega(x - z) dz.$$

**Proposition 4.37** (Far-field tail parilinearization). *Let  $q \in \mathbb{N}^*$ ,  $k \in K\mathcal{F}_{\text{tail}, \geq q}$ , and  $\varrho \geq 0$ . There exists  $s_0 > 0$  such that, for every  $s \geq s_0$ , there exists  $r_s > 0$  for which, for every  $u \in B_{s_0, \mathbb{R}}(I; r_s) \cap \mathcal{C}(I; H^s)$  and  $\omega \in H^s$ ,*

$$\tau_k^\infty(u) \omega = \text{Op}^{\text{BW}} \left( f_{k, \geq q}^\infty[\omega](u; x) \right) u + r_{k, [\varrho, \geq q]}^\infty(u) \omega + r_{k, [\varrho, \geq q]}^\infty[\omega](u) u. \quad (4.35)$$

The objects in Eq. (4.35) are as follows.

- The function  $f_{k, \geq q}^\infty$  satisfies

$$f_{k, \geq q}^\infty[\cdot](u) - f_{k, \geq q}^\infty[\cdot](0) \in \mathcal{L}\left(H^{s_0}; \mathcal{F}_{\geq \max\{1, q-1\}}\right)$$

and is defined by

$$f_{k, \geq q}^\infty[\omega](u; x) := \int_{\mathbb{R}^d} (1 - \chi(z)) \frac{b(z)}{|z|} g'(\Delta_z u(x)) \omega(x - z) dz.$$

- $r_{k, [\varrho, \geq q]}^\infty \in \mathcal{R}_{\geq q}^\varrho$ ,
- $r_{k, [\varrho, \geq q]}^\infty[\cdot] \in \mathcal{L}\left(H^{s_0}; \mathcal{R}_{\geq q-1}^\varrho\right).$

If  $k$  is real-valued for real-valued  $u$ , all the objects above can be chosen real-to-real.

*Proof.* Fix  $\varrho \geq 0$  and choose  $s_0$  sufficiently large, in particular

$$s_0 > \frac{d}{2} + \varrho.$$

By the definition of  $K\mathcal{F}_{\text{tail}, \geq q}$ , there exist  $m > 0$ ,  $b$ , and  $g$  satisfying Eqs. (4.17) and (4.18), with  $m + q > d$ , such that

$$k(u; x, z) = (1 - \chi(z)) b(z) g(\Delta_z u(x)).$$

For  $u \in B_{s_0, \mathbb{R}}(I; r_0)$  with  $r_0 > 0$  sufficiently small, the Taylor expansion

$$g(X) = \sum_{j=q}^{+\infty} g_j X^j$$

converges uniformly for  $X = \Delta_z u(x)$ ,  $z \in \text{supp}(1 - \chi)$ . By Eq. (4.1),

$$(\Delta_z u(x))^j = |z|^{-j} \sum_{a=0}^j (-1)^a \binom{j}{a} u^{j-a}(x) u^a(x - z).$$

Therefore

$$\tau_k^\infty(u) \omega = \sum_{j=q}^{+\infty} g_j \sum_{a=0}^j (-1)^a \binom{j}{a} u^{j-a} [c_j * (u^a \omega)], \quad (4.36)$$

where

$$c_j(z) := (1 - \chi(z)) b(z) |z|^{-j}.$$

For every  $M \in \mathbb{N}$ , Leibniz' rule and Eq. (4.17) give

$$|\partial_z^\alpha c_j(z)| \lesssim_M (1+j)^M |z|^{-m-j-|\alpha|}, \quad |\alpha| \leq M, \quad |z| \geq 1.9.$$

The derivatives falling on  $\chi$  are supported in the fixed annulus  $1.1 \leq |z| \leq 1.9$ . Since  $m + j \geq m + q > d$ , it follows that

$$\|c_j\|_{W^{M,1}} \lesssim_M (1+j)^M. \quad (4.37)$$

For every  $\nu \geq 0$ , taking  $M > \nu$  and using Eq. (4.37) give

$$\sup_{\xi \in \mathbb{R}^d} \langle \xi \rangle^\nu |\widehat{c}_j(\xi)| \lesssim_{\nu, M} (1+j)^M,$$

and consequently

$$\|c_j * f\|_{H^{r+\nu}} \lesssim_{r, \nu, M} (1+j)^M \|f\|_{H^r}, \quad r \in \mathbb{R}. \quad (4.38)$$

In particular, convolution with  $c_j$  gains an arbitrary number of derivatives.

For  $j \geq q$  and  $0 \leq a \leq j$ , set

$$F_{j,a}(u, \omega) := c_j * (u^a \omega), \quad p := j - a.$$

By Eq. (4.38), the algebra property of  $H^{s_0-\varrho}$ , and  $s_0 - \varrho > d/2$ , for every fixed  $N \in \mathbb{R}$  one has, after increasing  $M$ ,

$$\|F_{j,a}(u, h)\|_{H^N} \lesssim_{N, \varrho, M} C^j (1+j)^M \|u\|_{H^{s_0}}^a \|h\|_{H^{s_0-\varrho}}. \quad (4.39)$$

If  $p = 0$ , Eq. (4.39) directly gives

$$F_{j,j}(u, \omega) = r_{j,j,\varrho}(u) \omega,$$

with

$$\|r_{j,j,\varrho}(u) h\|_{H^s} \lesssim_{s, \varrho} C^j (1+j)^{M_s} \|u\|_{H^{s_0}}^j \|h\|_{H^{s-\varrho}}, \quad s \geq s_0. \quad (4.40)$$

Let now  $p \geq 1$ . By Lemma 4.18,

$$\begin{aligned} u^p F_{j,a}(u, \omega) &= \text{Op}^{\text{BW}}(u^p) F_{j,a}(u, \omega) + \text{Op}^{\text{BW}}(F_{j,a}(u, \omega)) u^p \\ &\quad + R^{\text{BW}}[u^p, F_{j,a}(u, \omega)]. \end{aligned} \quad (4.41)$$

The first and third terms in Eq. (4.41) define an operator  $r_{j,a,\varrho}(u)$  satisfying

$$\begin{aligned} r_{j,a,\varrho}(u) h &:= \text{Op}^{\text{BW}}(u^p) F_{j,a}(u, h) + R^{\text{BW}}[u^p, F_{j,a}(u, h)], \\ \|r_{j,a,\varrho}(u) h\|_{H^s} &\lesssim_{s, \varrho} C^j (1+j)^{M_s} \|u\|_{H^{s_0}}^j \|h\|_{H^{s-\varrho}}. \end{aligned}$$

Indeed, the paraproduct estimate only uses  $\|u^p\|_{H^{s_0}}$ , whereas in the Bony remainder the additional regularity is placed on  $F_{j,a}$  and follows from Eq. (4.39).

For the second term in Eq. (4.41), Lemma 4.20, applied to  $v \mapsto v^p$ , gives

$$u^p = \text{Op}^{\text{BW}}(p u^{p-1}) u + r_{p,\varrho}(u) u,$$

where, by increasing  $s_0$  with respect to  $\varrho$  if necessary,

$$\|r_{p,\varrho}(u) h\|_{H^s} \lesssim_{s, \varrho} C^p \|u\|_{H^{s_0}}^{p-1} \|h\|_{H^{s-\varrho}}. \quad (4.42)$$

Hence

$$\begin{aligned} \text{Op}^{\text{BW}}(F_{j,a}(u, \omega)) u^p &= \text{Op}^{\text{BW}}(F_{j,a}(u, \omega)) \text{Op}^{\text{BW}}(p u^{p-1}) u \\ &\quad + \text{Op}^{\text{BW}}(F_{j,a}(u, \omega)) r_{p,\varrho}(u) u. \end{aligned} \quad (4.43)$$

For  $p = 1$ , take  $r_{p,\varrho} = 0$ . For  $p \geq 2$ , the estimate Eq. (4.42) follows by induction: apply Lemma 4.18 to  $u^{p-1} u$ , then use Remark 4.19 and Proposition 4.24. Applying Proposition 4.24 to the functions  $F_{j,a}(u, \omega)$  and  $p u^{p-1}$  gives

$$\text{Op}^{\text{BW}}(F_{j,a}(u, \omega)) \text{Op}^{\text{BW}}(p u^{p-1}) = \text{Op}^{\text{BW}}(p u^{p-1} F_{j,a}(u, \omega)) + r_{j,a,\varrho}^{\text{comp}}[\omega](u),$$

where

$$\|r_{j,a,\varrho}^{\text{comp}}[\omega](u) h\|_{H^s} \lesssim_{s, \varrho} C^j (1+j)^{M_s} \|u\|_{H^{s_0}}^{j-1} \|\omega\|_{H^{s_0}} \|h\|_{H^{s-\varrho}}.$$

Together with Eqs. (4.39) and (4.42), the second term in Eq. (4.43) satisfies the same estimate. Thus, setting

$$\tilde{r}_{j,a,\varrho}[\omega](u) := \text{Op}^{\text{BW}}(F_{j,a}(u, \omega)) r_{p,\varrho}(u) + r_{j,a,\varrho}^{\text{comp}}[\omega](u),$$

we obtain

$$\begin{aligned} u^{j-a} F_{j,a}(u, \omega) &= \text{Op}^{\text{BW}} \left( (j-a) u^{j-a-1} F_{j,a}(u, \omega) \right) u \\ &\quad + r_{j,a,\varrho}(u) \omega + \tilde{r}_{j,a,\varrho}[\omega](u) u, \end{aligned} \quad (4.44)$$

and

$$\begin{aligned} \|r_{j,a,\varrho}(u) h\|_{H^s} &\lesssim_{s,\varrho} C^j (1+j)^{M_s} \|u\|_{H^{s_0}}^j \|h\|_{H^{s-e}}, \\ \|\tilde{r}_{j,a,\varrho}[\omega](u) h\|_{H^s} &\lesssim_{s,\varrho} C^j (1+j)^{M_s} \|u\|_{H^{s_0}}^{j-1} \|\omega\|_{H^{s_0}} \|h\|_{H^{s-e}}. \end{aligned} \quad (4.45)$$

It remains to identify the first term in Eq. (4.44). By Eq. (4.36), its coefficient is

$$\sum_{j=q}^{+\infty} g_j \sum_{a=0}^{j-1} (-1)^a \binom{j}{a} (j-a) u^{j-a-1} [c_j * (u^a \omega)].$$

Using

$$(j-a) \binom{j}{a} = j \binom{j-1}{a}$$

and the definition of  $c_j$ , this is equal to

$$f_{k,\geq q}^\infty[\omega](u; x) = \int_{\mathbb{R}^d} (1 - \chi(z)) \frac{b(z)}{|z|} g'(\Delta_z u(x)) \omega(x-z) dz.$$

The terms with  $j \geq \max\{q, 2\}$  give the asserted  $\mathcal{F}$  membership, while the possible  $j = 1$  term is independent of  $u$ . Hence  $f_{k,\geq q}^\infty[\omega](u)$  has total vanishing order  $q$  by Remark 4.31. Finally, set

$$\begin{aligned} r_{k,[\varrho,\geq q]}^\infty(u) &:= \sum_{j=q}^{+\infty} g_j \sum_{a=0}^j (-1)^a \binom{j}{a} r_{j,a,\varrho}(u), \\ r_{k,[\varrho,\geq q]}^\infty[\omega](u) &:= \sum_{j=q}^{+\infty} g_j \sum_{a=0}^{j-1} (-1)^a \binom{j}{a} \tilde{r}_{j,a,\varrho}[\omega](u). \end{aligned}$$

By Eqs. (4.40) and (4.45) and the analyticity of  $g$ , for each  $s \geq s_0$  the two series converge after reducing  $r_s$ . Their vanishing orders in  $u$  are  $q$  and  $q-1$ , respectively; the second remainder is linear in  $\omega$  and hence has total vanishing order  $q$  by Remark 4.31. Substituting into Eq. (4.36) yields the decomposition in the statement.

If  $k$  is real-valued for real-valued  $u$ , all the terms in the above construction preserve real-valued functions, which proves the real-to-real property.  $\square$

### Homogeneous angular multipliers

**Proposition 4.38** (Homogeneous angular multipliers on  $\mathbb{R}^d$ ). *Let  $f \in \mathcal{S}(\mathbb{R}^d; \mathbb{C})$ . The following statements hold.*

1. *Let  $a \in C^\infty(\mathbb{S}^{d-1}; \mathbb{R})$  be odd. Then its spherical mean vanishes and there exists a real, odd Fourier multiplier  $\mathfrak{m}_a^{(0)}$ , smooth for  $\xi \neq 0$  and homogeneous of degree zero, explicitly given by*

$$\mathfrak{m}_a^{(0)}(\xi) := -\frac{\pi}{2} \int_{\mathbb{S}^{d-1}} \text{sgn}(\vartheta \cdot \xi) a(\vartheta) d\sigma(\vartheta), \quad (4.46)$$

such that

$$\text{p.v.} \int_{\mathbb{R}^d} \frac{a(\hat{z})}{|z|^d} f(\bullet - z) dz = \mathfrak{m}_a^{(0)}(D) f. \quad (4.47)$$

Moreover,  $\mathfrak{m}_a^{(0)}(D)$  is real-to-real.

2. Let  $a \in C^\infty(\mathbb{S}^{d-1}; \mathbb{R})$  be even and  $d \geq 2$ . Then there exists a real, even Fourier multiplier  $\mathfrak{m}_a^{(-1)} \in \tilde{\Gamma}^{-1}$ , smooth for  $\xi \neq 0$  and homogeneous of degree  $-1$ , explicitly given by

$$\begin{aligned} \mathfrak{m}_a^{(-1)}(\xi) &:= \pi \int_{\mathbb{S}^{d-1}} \delta(\vartheta \cdot \xi) a(\vartheta) d\sigma(\vartheta) \\ &= \frac{\pi}{|\xi|} \int_{\mathbb{S}^{d-1} \cap \xi^\perp} a(\vartheta) d\sigma_{d-2}(\vartheta), \end{aligned} \quad (4.48)$$

such that

$$\int_{\mathbb{R}^d} \frac{a(\hat{z})}{|z|^{d-1}} f(\bullet - z) dz = \mathfrak{m}_a^{(-1)}(D) f. \quad (4.49)$$

Moreover,  $\mathfrak{m}_a^{(-1)}(D)$  is real-to-real.

If  $d = 1$  and  $a$  is even on  $\mathbb{S}^0 = \{-1, 1\}$ , then  $a(\hat{z})$  is constant on  $\mathbb{R} \setminus \{0\}$  and

$$\text{supp } \mathcal{F}_{z \rightarrow \xi} [a(\hat{z})] \subseteq \{0\}. \quad (4.50)$$

In particular, its complete high-frequency homogeneous multiplier vanishes.

3. Let  $a \in C^\infty(\mathbb{S}^{d-1}; \mathbb{R})$  be even. Then there exists a real, even Fourier multiplier  $\mathfrak{m}_a^{(1)} \in \tilde{\Gamma}^1$ , smooth for  $\xi \neq 0$  and homogeneous of degree one, explicitly given by

$$\mathfrak{m}_a^{(1)}(\xi) := \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\vartheta \cdot \xi| a(\vartheta) d\sigma(\vartheta), \quad (4.51)$$

such that

$$\frac{1}{2} \int_{\mathbb{R}^d} \frac{a(\hat{z})}{|z|^{d+1}} \delta_z^{(2)} f dz = \mathfrak{m}_a^{(1)}(D) f. \quad (4.52)$$

Moreover,  $\mathfrak{m}_a^{(1)}(D)$  is real-to-real.

Moreover, for every  $M \in \mathbb{N}$ , in each of the cases above, we have

$$\max_{|\beta| \leq M} \sup_{|\xi| \geq 1/2} |\xi|^{-j+|\beta|} \left| \partial_\xi^\beta \mathfrak{m}_a^{(j)}(\xi) \right| \lesssim_M \|a\|_{C^M(\mathbb{S}^{d-1})}, \quad (4.53)$$

where  $j \in \{-1, 0, 1\}$ , with  $j = -1$  restricted to  $d \geq 2$ .

*Proof.*

**Part 1** (Homogeneous multiplier identities). All Fourier identities below are understood in the sense of tempered distributions. We use throughout Eq. (A.1).

**Subpart 1.1** (Proof of Eq. (4.47)). By oddness,

$$\int_{\mathbb{S}^{d-1}} a d\sigma = 0.$$

In polar coordinates  $z = r\vartheta$ , pairing  $\vartheta$  with  $-\vartheta$  and using the first identity in Eq. (A.1), we obtain

$$\begin{aligned} \text{p.v.} \int_{\mathbb{R}^d} \frac{a(\hat{z})}{|z|^d} e^{-iz \cdot \xi} dz &= -i \int_{\mathbb{S}^{d-1}} a(\vartheta) \int_0^\infty \frac{\sin(r\vartheta \cdot \xi)}{r} dr d\sigma(\vartheta) \\ &= i \mathfrak{m}_a^{(0)}(\xi), \end{aligned}$$

where the last equality is Eq. (4.46).

**Subpart 1.2** (Proof of the first identity in Eq. (4.48)). Polar coordinates and the second identity in Eq. (A.1) give

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{a(\hat{z})}{|z|^{d-1}} e^{-iz \cdot \xi} dz &= \int_{\mathbb{S}^{d-1}} a(\vartheta) \left( \int_0^\infty e^{-ir\vartheta \cdot \xi} dr \right) d\sigma(\vartheta) \\ &= \int_{\mathbb{S}^{d-1}} a(\vartheta) \left( \pi \delta(\vartheta \cdot \xi) - \text{i.p.v.} \frac{1}{\vartheta \cdot \xi} \right) d\sigma(\vartheta) \\ &= \pi \int_{\mathbb{S}^{d-1}} \delta(\vartheta \cdot \xi) a(\vartheta) d\sigma(\vartheta). \end{aligned} \quad (4.54)$$

The principal-value contribution vanishes under  $\vartheta \mapsto -\vartheta$ , since  $a$  is even.

**Subpart 1.3** (Proof of the second identity in Eq. (4.48) and of Eq. (4.49)). For  $d \geq 2$ , it remains to identify geometrically the delta contribution in Eq. (4.54). Fix  $\xi \neq 0$  and define

$$g_\xi(\vartheta) := \vartheta \cdot \xi, \quad \vartheta \in \mathbb{S}^{d-1}.$$

Then

$$g_\xi^{-1}(0) = \mathbb{S}^{d-1} \cap \langle \xi \rangle^\perp. \quad (4.55)$$

Moreover, since the spherical gradient is the orthogonal projection of the Euclidean gradient onto  $T_\vartheta \mathbb{S}^{d-1}$ ,

$$\nabla_{\mathbb{S}^{d-1}} g_\xi(\vartheta) = \xi - (\vartheta \cdot \xi) \vartheta = (\text{Id} - \vartheta \otimes \vartheta) \xi. \quad (4.56)$$

Let  $\bar{\vartheta} \in g_\xi^{-1}(0)$ , by Eqs. (4.55) and (4.56) we have that  $\bar{\vartheta} \cdot \xi = 0$  and  $\nabla_{\mathbb{S}^{d-1}} g_\xi(\bar{\vartheta}) = \xi$ , this implies that 0 is a regular value of  $g_\xi$  and, on  $g_\xi^{-1}(0)$  holds  $|\nabla_{\mathbb{S}^{d-1}} g_\xi(\vartheta)| = |\xi|$ . To identify the action of the delta distribution, let  $\varphi \in C_c^\infty(\mathbb{R})$ . For  $d \geq 2$ , fix  $\xi \neq 0$ . We write  $\vartheta$  via the decomposition  $\langle \hat{\xi} \rangle \oplus \langle \hat{\xi} \rangle^\perp$  as

$$\vartheta = \mathbb{P}_{\hat{\xi}} \vartheta + \mathbb{P}_{\hat{\xi}}^\perp \vartheta = \hat{\xi} \otimes \hat{\xi} \vartheta + (\text{Id} - \hat{\xi} \otimes \hat{\xi}) \vartheta = t \hat{\xi} - (\vartheta - t \hat{\xi}).$$

Explicit computations show that  $|\vartheta - t \hat{\xi}| = 1 - t^2$  thus defining  $\omega := \frac{\vartheta - t \hat{\xi}}{1 - t^2}$  we obtain the decomposition

$$\vartheta = t \hat{\xi} + \sqrt{1 - t^2} \omega, \quad t \in (-1, 1), \quad \omega \in \mathbb{S}^{d-1} \cap \xi^\perp.$$

Notice that, writing

$$\vartheta = \cos \phi \hat{\xi} + \sin \phi \omega, \quad \omega \in \mathbb{S}^{d-1} \cap \xi^\perp,$$

the standard spherical measure is

$$d\sigma(\vartheta) = \sin^{d-2} \phi d\phi d\sigma_{d-2}(\omega).$$

Since  $t = \cos \phi$ , so that  $dt = -\sin \phi d\phi$  and  $\sin \phi = \sqrt{1 - t^2}$ , this becomes

$$d\sigma(\vartheta) = (1 - t^2)^{\frac{d-3}{2}} dt d\sigma_{d-2}(\omega).$$

which we combine with  $\vartheta \cdot \xi = |\xi| t$  in order to obtain that

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} \delta(\vartheta \cdot \xi) a(\vartheta) d\sigma(\vartheta) &= \int_{-1}^1 \delta(|\xi| t) (1 - t^2)^{\frac{d-3}{2}} \int_{\mathbb{S}^{d-1} \cap \xi^\perp} a(t \hat{\xi} + \sqrt{1 - t^2} \omega) d\sigma_{d-2}(\omega) dt \\ &= \frac{1}{|\xi|} \int_{\mathbb{S}^{d-1} \cap \xi^\perp} a(\omega) d\sigma_{d-2}(\omega), \end{aligned}$$

where in the last equality we used

$$\delta(|\xi| t) = \frac{1}{|\xi|} \delta(t).$$

Together with Eq. (4.54), it also gives Eq. (4.49).

**Subpart 1.4** (Proof of Eq. (4.50)). Let  $d = 1$ . Since  $\mathbb{S}^0 = \{-1, 1\}$  and  $a$  is even, there is  $c_a \in \mathbb{R}$  such that

$$a(\hat{z}) = c_a, \quad z \in \mathbb{R} \setminus \{0\}.$$

The value at  $z = 0$  is irrelevant as a tempered distribution, hence  $a(\hat{z})$  is the constant distribution  $c_a$ . Therefore

$$\text{supp } \mathcal{F}_{z \rightarrow \xi} [a(\hat{z})] \subseteq \{0\}.$$

**Subpart 1.5** (Proof of Eq. (4.52)). By Eq. (4.66),

$$\mathcal{F} \left[ \frac{1}{2} \delta_z^{(2)} f \right] (\xi) = (1 - \cos(z \cdot \xi)) \hat{f}(\xi).$$

Hence, by polar coordinates and the third identity in Eq. (A.1),

$$\begin{aligned} \int_{\mathbb{R}^d} \frac{a(\hat{z})}{|z|^{d+1}} (1 - \cos(z \cdot \xi)) dz &= \int_{\mathbb{S}^{d-1}} a(\vartheta) \int_0^\infty \frac{1 - \cos(r \vartheta \cdot \xi)}{r^2} dr d\sigma(\vartheta) \\ &= \mathfrak{m}_a^{(1)}(\xi), \end{aligned}$$

where the last equality is Eq. (4.51).

**Part 2** (Symbol properties and seminorms). The formulas in Eqs. (4.46), (4.48) and (4.51) give the stated homogeneity, reality and parity properties. It remains to prove smoothness away from the origin and the corresponding symbol estimates.

By homogeneity,

$$\dot{m}_a^{(j)}(\xi) = |\xi|^j \dot{m}_a^{(j)}(\hat{\xi}), \quad j \in \{-1, 0, 1\},$$

where  $j = -1$  is restricted to  $d \geq 2$ .

If  $d = 1$ , counting measure on  $\mathbb{S}^0 = \{-1, 1\}$  gives

$$\dot{m}_a^{(0)}(\xi) = -\pi a(1) \operatorname{sgn}(\xi), \quad \dot{m}_a^{(1)}(\xi) = \pi a(1) |\xi|.$$

Hence smoothness on  $\mathbb{R} \setminus \{0\}$  and Eq. (4.53) are immediate. We may therefore assume  $d \geq 2$ .

We first record how derivatives in  $\xi$  reduce to derivatives in the angular variable. For  $k \in \{1, \dots, d\}$ , let  $e_k$  denote the  $k$ -th standard basis vector of  $\mathbb{R}^d$ , and define the smooth tangential vector field

$$X_k(\hat{\xi}) := e_k - \hat{\xi}_k \hat{\xi} \in T_{\hat{\xi}} \mathbb{S}^{d-1}, \quad \hat{\xi}_k := \hat{\xi} \cdot e_k.$$

Since

$$\partial_{\xi_k} |\xi| = \hat{\xi}_k, \quad \partial_{\xi_k} \hat{\xi} = \frac{e_k - \hat{\xi}_k \hat{\xi}}{|\xi|} = \frac{X_k(\hat{\xi})}{|\xi|},$$

the chain rule gives, for every  $b \in C^\infty(\mathbb{S}^{d-1})$ ,

$$\partial_{\xi_k} \left[ |\xi|^j b(\hat{\xi}) \right] = |\xi|^{j-1} (j \hat{\xi}_k b(\hat{\xi}) + X_k b(\hat{\xi})). \quad (4.57)$$

Iterating Eq. (4.57) shows that, for every multi-index  $\beta$ , there exists a tangential differential operator  $P_\beta^j$  on  $\mathbb{S}^{d-1}$ , of order at most  $|\beta|$  and with smooth coefficients, such that

$$\partial_\xi^\beta \dot{m}_a^{(j)}(\xi) = |\xi|^{j-|\beta|} P_\beta^j \left[ \dot{m}_a^{(j)}|_{\mathbb{S}^{d-1}} \right](\hat{\xi}). \quad (4.58)$$

More explicitly, on any coordinate neighborhood  $U \subset \mathbb{S}^{d-1}$ ,

$$P_\beta^j = \sum_{|\gamma| \leq |\beta|} c_{\beta, \gamma}^{j, U}(\hat{\xi}) \partial_{\hat{\xi}}^\gamma, \quad c_{\beta, \gamma}^{j, U} \in C^\infty(U). \quad (4.59)$$

Thus Eqs. (4.58) and (4.59) reduce the required estimates in  $\xi$  to estimates for derivatives of the restriction of  $\dot{m}_a^{(j)}$  to the unit sphere. We now establish these angular estimates.

Fix coordinate neighborhoods  $U \Subset V \subset \mathbb{S}^{d-1}$  and a smooth family of rotations

$$V \ni \hat{\xi} \longmapsto Q_{\hat{\xi}} \in \operatorname{SO}(d), \quad Q_{\hat{\xi}} e_d = \hat{\xi}.$$

Such a family exists locally by smoothly completing  $v$  to a positively oriented orthonormal basis. Since  $\overline{U} \subset V$ , all its coordinate derivatives are uniformly bounded on  $U$ .

For  $\phi \in \mathbb{S}^{d-1}$ , set

$$\phi_d := \phi \cdot e_d.$$

The change of variables  $\vartheta = Q_{\hat{\xi}} \phi$  preserves surface measure and satisfies

$$\vartheta \cdot \hat{\xi} = (Q_{\hat{\xi}} \phi) \cdot (Q_{\hat{\xi}} e_d) = \phi_d.$$

Consequently, Eqs. (4.46), (4.48) and (4.51) give

$$\begin{aligned} \dot{m}_a^{(0)}(\hat{\xi}) &= -\frac{\pi}{2} \int_{\mathbb{S}^{d-1}} \operatorname{sgn}(\phi_d) a(Q_{\hat{\xi}} \phi) d\sigma(\phi), \\ \dot{m}_a^{(-1)}(\hat{\xi}) &= \pi \int_{\mathbb{S}^{d-1} \cap e_d^\perp} a(Q_{\hat{\xi}} \phi) d\sigma_{d-2}(\phi), \\ \dot{m}_a^{(1)}(\hat{\xi}) &= \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\phi_d| a(Q_{\hat{\xi}} \phi) d\sigma(\phi). \end{aligned} \quad (4.60)$$

The purpose of the rotation is that the integration domains and the possibly nonsmooth weights are now independent of  $\hat{\xi}$ . All dependence on  $\hat{\xi}$  is contained in the smooth factor  $a(Q_{\hat{\xi}}\phi)$ .

Let  $\gamma$  be a multi-index in the local coordinates of  $V$ . By the chain rule and the smoothness of  $\hat{\xi} \mapsto Q_{\hat{\xi}}$ ,

$$\sup_{\substack{\hat{\xi} \in U \\ \phi \in \mathbb{S}^{d-1}}} \left| \partial_{\hat{\xi}}^{\gamma} \left[ a(Q_{\hat{\xi}}\phi) \right] \right| \lesssim_{\gamma, U, V} \|a\|_{C^{|\gamma|}(\mathbb{S}^{d-1})}.$$

Since the domains in Eq. (4.60) have finite measure and their weights are bounded, differentiation under the integral sign is therefore justified. We obtain

$$\sup_{\hat{\xi} \in U} \left| \partial_{\hat{\xi}}^{\gamma} \mathfrak{m}_a^{(j)}(\nu) \right| \lesssim_{\gamma, U, V} \|a\|_{C^{|\gamma|}(\mathbb{S}^{d-1})}, \quad j \in \{-1, 0, 1\}. \quad (4.61)$$

In particular, the restriction of each multiplier to  $U$  is smooth. A finite collection of such neighborhoods covers  $\mathbb{S}^{d-1}$ , and hence Eq. (4.61) holds uniformly on the sphere.

We may now substitute the angular bounds into the derivative formula obtained at the beginning of this proofpart. Indeed, Eqs. (4.58), (4.59) and (4.61) give, for every multi-index  $\beta$ ,

$$\left| \partial_{\xi}^{\beta} \mathfrak{m}_a^{(j)}(\xi) \right| \lesssim_{\beta} |\xi|^{j-|\beta|} \|a\|_{C^{|\beta|}(\mathbb{S}^{d-1})}.$$

Therefore, for every  $M \in \mathbb{N}$ ,

$$\max_{|\beta| \leq M} \sup_{|\xi| \geq 1/2} |\xi|^{-j+|\beta|} \left| \partial_{\xi}^{\beta} \mathfrak{m}_a^{(j)}(\xi) \right| \lesssim_M \|a\|_{C^M(\mathbb{S}^{d-1})},$$

which proves Eq. (4.53). By Definition 4.6,

$$\mathfrak{m}_a^{(j)} \in \tilde{\Gamma}^j.$$

Finally, reality and parity give

$$\mathfrak{im}_a^{(0)}(-\xi) = \overline{\mathfrak{im}_a^{(0)}(\xi)},$$

whereas, for  $j \in \{-1, 1\}$ ,

$$\mathfrak{m}_a^{(j)}(-\xi) = \overline{\mathfrak{m}_a^{(j)}(\xi)},$$

with  $j = -1$  restricted to  $d \geq 2$ . These are precisely the Fourier reality conditions, and therefore prove the stated real-to-real properties. □

**Homogeneous convolution models and cutoff removal.** The following setting of the models corresponds to the critical regime of the graph integral operators, which will be used in the proof of Theorem 5.7.

**Definition 4.39** (Localized homogeneous convolution models). Let  $\lambda \in \{-d, -d+1\}$  and let  $h \in C^{\infty}(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$  be homogeneous of degree  $\lambda$ . If  $\lambda = -d$ , we assume the cancellation condition

$$\int_{\mathbb{S}^{d-1}} h(\vartheta) d\sigma(\vartheta) = 0. \quad (4.62)$$

We associate with  $h$  the tempered distribution

$$\mathfrak{h} := \begin{cases} \text{p.v.} h, & \lambda = -d, \\ h, & \lambda = -d+1. \end{cases}$$

We define

$$t_h^{\text{loc}} \omega := (\chi \mathfrak{h}) * \omega,$$

and the corresponding complete homogeneous multiplier by

$$\mathfrak{m}_h := (2\pi)^{d/2} \mathcal{F}_{z \rightarrow \xi} \mathfrak{h}. \quad (4.63)$$

**Remark 4.40.** The cancellation conditions Eq. (4.62) ensures that the convolution operator

$$T_h f(x) := \text{p.v.} \int_{\mathbb{R}^d} h(y) f(x-y) dy$$

is Calderon-Zygmund operator, which is  $L^p$  bounded for any  $1 < p < \infty$ .  $\blacklozenge$

**Lemma 4.41** (Localized homogeneous models are Fourier multipliers modulo smoothing). *Let  $\lambda \in \{-d, -d+1\}$  and let  $h$  satisfy Definition 4.39. Then,*

$$\dot{m}_h \in \tilde{\Gamma}^{-(\lambda+d)}$$

and

$$t_h^{\text{loc}} - \dot{m}_h(D) \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M}. \quad (4.64)$$

Moreover, if  $h$  is real-valued, then both terms in Eq. (4.64) are real-to-real.

*Proof.* Let

$$m_{\chi h} := (2\pi)^{d/2} \mathcal{F}_{z \rightarrow \xi}(\chi(z) h).$$

Since  $\chi h$  is a compactly supported distribution, convolution with  $\chi h$  is the Fourier multiplier  $m_{\chi h}(D)$ . Thus

$$t_h^{\text{loc}} = m_{\chi h}(D). \quad (4.65)$$

The cancellation condition in Eq. (4.62) guarantees that, when  $\lambda = -d$ , the principal value p.v. $h$  is a homogeneous tempered distribution of degree  $-d$ . When  $\lambda = -d+1$ , the function  $h$  is locally integrable at the origin and defines a homogeneous tempered distribution of degree  $-d+1$ . Hence, in both cases, the Fourier transform  $\mathcal{F}h$  is smooth on  $\mathbb{R}^d \setminus \{0\}$ , homogeneous of degree  $-(\lambda+d)$ , and therefore satisfies the symbol estimates

$$\left| \partial_\xi^\beta \dot{m}_h(\xi) \right| \lesssim_\beta |\xi|^{-(\lambda+d)-|\beta|}, \quad |\xi| \geq \frac{1}{2}.$$

This proves  $\dot{m}_h \in \tilde{\Gamma}^{-(\lambda+d)}$ .

Now, by construction Eqs. (4.63) and (4.65), one has

$$t_h^{\text{loc}} - \dot{m}_h = m_{\chi h} - \dot{m}_h = (2\pi)^{d/2} \mathcal{F}_{z \rightarrow \xi}[(\chi(z) - 1)h(z)].$$

Note that the factor  $\chi(z) - 1$  vanishes near  $z = 0$ . Thus

$$g(z) := (\chi(z) - 1)h(z)$$

is smooth on  $\mathbb{R}^d$  and is equal to  $-h(z)$  for large  $|z|$ . Fix a multi-index  $\beta$  and  $M \in \mathbb{N}$ . Since  $g$  vanishes in a neighborhood of the origin, there is no integrability issue there. Moreover, for  $|z|$  sufficiently large one has  $g(z) = -h(z)$ . Hence for any  $L \in \mathbb{N}$ , by the homogeneity of  $h$ ,

$$(-\Delta_z)^L \left[ (-iz)^\beta g(z) \right] = \mathcal{O}\left(|z|^{\lambda+|\beta|-2L}\right), \quad |z| \rightarrow +\infty.$$

Therefore

$$(-\Delta_z)^L \left[ (-iz)^\beta g(z) \right] \in L^1(\mathbb{R}^d; \mathbb{C})$$

provided  $2L > d + \lambda + |\beta|$ . Since  $\lambda \in \{-d, -d+1\}$ , such an  $L$  can be chosen arbitrarily large. Indeed, by homogeneity the last expression is  $\mathcal{O}\left(|z|^{\lambda+|\beta|-2L}\right)$  as  $|z| \rightarrow +\infty$ , and it is smooth near the origin. Therefore, for  $|\xi| \geq 2$ ,

$$\left| \partial_\xi^\beta \mathcal{F}_{z \rightarrow \xi} g(\xi) \right| \leq |\xi|^{-2L} \left\| (-\Delta_z)^L \left[ (-iz)^\beta g \right] \right\|_{L^1} \lesssim_{M, \beta, h} |\xi|^{-M-|\beta|},$$

after choosing  $L$  so large that  $2L \geq M + |\beta|$ . Hence,

$$t_h^{\text{loc}} - \dot{m}_h(D) \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M}.$$

Finally, if  $h$  is real-valued, then the distributions  $h$ ,  $\chi h$ , and  $g$  are real. Their Fourier multipliers satisfy the usual symmetry

$$\overline{m(\xi)} = m(-\xi),$$

and therefore the corresponding Fourier multiplier operators map real-valued functions to real-valued functions.  $\square$

**Remark 4.42** (Composition with smoothing multipliers). Let  $q \in \mathbb{N}^*$ ,  $m \in \mathbb{R}$ , and let  $a \in \Gamma_{\geq q}^m$  be independent of  $z$ . If  $\mathfrak{s} \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M} \cap L^\infty(|\xi| \leq 1; \mathbb{C})$ , then, for every prescribed  $\varrho \geq 0$ ,

$$\text{Op}^{\text{BW}}(a(U; x, \xi)) \circ \mathfrak{s}(D) \in \mathcal{R}_{\geq q}^\varrho.$$

The same statement holds for a separately estimated linear parameter. If

$$a[\cdot] \in \mathcal{L}\left(H^{s_0}; \Gamma_{\geq q}^m\right),$$

then

$$\omega \longmapsto \text{Op}^{\text{BW}}(a[\omega](U; x, \xi)) \circ \mathfrak{s}(D) \in \mathcal{L}\left(H^{s_0}; \mathcal{R}_{\geq q}^\varrho\right).$$

The same conclusion holds componentwise for vector-valued smoothing multipliers. This is the standard Bony–Weyl composition estimate with one factor of arbitrarily negative order.  $\blacklozenge$

**Definition 4.43** (Localized compensated homogeneous convolution models). Let  $h \in C^\infty(\mathbb{R}^d \setminus \{0\}; \mathbb{C})$  be even and homogeneous of degree  $-d-1$ :

$$h(rz) = r^{-d-1} h(z), \quad h(-z) = h(z), \quad r > 0.$$

For a scalar function  $f$ , set

$$\delta_z^{(2)} f := \delta_z f + \delta_{-z} f = 2f - f(\bullet - z) - f(\bullet + z). \quad (4.66)$$

We define the localized compensated model by

$$t_h^{\text{loc}, (2)} f := \frac{1}{2} \int_{\mathbb{R}^d} \chi(z) h(z) \delta_z^{(2)} f \, dz.$$

The corresponding global homogeneous multiplier is

$$\dot{\mathfrak{m}}_h^{(2)}(\xi) := \frac{1}{2} \int_{\mathbb{R}^d} h(z) \left(2 - e^{-i\xi \cdot z} - e^{i\xi \cdot z}\right) dz, \quad \xi \in \mathbb{R}^d \setminus \{0\}.$$

We set

$$c_h := - \int_{\mathbb{R}^d} (1 - \chi(z)) h(z) \, dz \in \mathbb{C}.$$

**Lemma 4.44** (Localized compensated homogeneous models are Fourier multipliers modulo smoothing). Let  $h$  satisfy Definition 4.43. Then,

$$\dot{\mathfrak{m}}_h^{(2)} \in \tilde{\Gamma}^1$$

and

$$t_h^{\text{loc}, (2)} - \dot{\mathfrak{m}}_h^{(2)}(D) - c_h \cdot \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M}. \quad (4.67)$$

If  $h$  is real-valued, then all terms in Eq. (4.67) are real-to-real.

*Proof.* We first discuss the global multiplier. Since  $2 - e^{-i\xi \cdot z} - e^{i\xi \cdot z} = 2(1 - \cos(\xi \cdot z))$ , the kernel multiplied by the second difference is  $O(|z|^{1-d})$  near  $z = 0$ , and the polar measure therefore produces a bounded radial integrand. At infinity the polar radial weight is  $O(r^{-2})$ . Hence the integral defining  $\dot{\mathfrak{m}}_h^{(2)}$  is absolutely convergent. Writing  $z = r\vartheta$ , with  $\vartheta \in \mathbb{S}^{d-1}$ , gives

$$\dot{\mathfrak{m}}_h^{(2)}(\xi) = \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} h(\vartheta) |\vartheta \cdot \xi| \, d\sigma(\vartheta). \quad (4.68)$$

Here we use the third identity in Eq. (A.1). Formula Eq. (4.68) shows that  $\dot{\mathfrak{m}}_h^{(2)}$  is homogeneous of degree 1. If  $d = 1$ , its smoothness away from the origin and the estimates below follow directly from the formula on the two-point sphere  $\mathbb{S}^0$ . If  $d \geq 2$ , fix a coordinate patch on  $\mathbb{S}^{d-1}$  and a smooth family of rotations  $Q_\nu \in SO(d)$  satisfying  $Q_\nu e_d = \nu$ . After the change of variables  $\vartheta = Q_\nu \phi$ , all angular derivatives fall on the smooth amplitude  $h(Q_\nu \phi)$ . A finite cover of the sphere therefore gives

$$\left| \partial_\xi^\beta \dot{\mathfrak{m}}_h(\xi) \right| \lesssim_\beta |\xi|^{1-|\beta|}, \quad |\xi| \geq \frac{1}{2},$$

after the fixed high-frequency truncation. Thus,  $\mathfrak{m}_h^{(2)} \in \tilde{\Gamma}^1$ .

Let  $\mathfrak{m}_{\chi h}$  denote the multiplier of the localized model:

$$\mathfrak{m}_{\chi h}^{(2)}(\xi) := \frac{1}{2} \int_{\mathbb{R}^d} \chi(z) h(z) \left( 2 - e^{-i\xi \cdot z} - e^{i\xi \cdot z} \right) dz.$$

For Schwartz function  $f$ , the Fourier transform of  $\delta_z^{(2)} f$  is the multiplication by  $2 - e^{-i\xi \cdot z} - e^{i\xi \cdot z}$ . Hence,

$$t_h^{\text{loc},(2)} = \mathfrak{m}_{\chi h}^{(2)}(D).$$

Set  $g(z) := (1 - \chi(z)) h(z)$ . Then  $g$  is smooth on  $\mathbb{R}^d$  and is a classical symbol of degree  $-d - 1$  as  $|z| \rightarrow +\infty$ . By subtracting the global multiplier from the localized one, we obtain

$$\begin{aligned} t_h^{\text{loc},(2)} - \mathfrak{m}_h^{(2)}(\xi) &= \mathfrak{m}_{\chi h}^{(2)}(\xi) - \mathfrak{m}_h^{(2)}(\xi) = - \int_{\mathbb{R}^d} g(z) dz + \frac{1}{2} \int_{\mathbb{R}^d} g(z) \left( e^{-i\xi \cdot z} + e^{i\xi \cdot z} \right) dz \\ &= c_h + \mathfrak{s}_h^{\text{tail}}(\xi). \end{aligned}$$

We claim that  $\mathfrak{s}_h^{\text{tail}}$  is rapidly decaying at high frequency. Let  $\beta \in \mathbb{N}_0^d$  and  $M \in \mathbb{N}$ . Choose  $L$  large enough, such that

$$(-\Delta_z)^L \left[ z^\beta g(z) \right] \in L^1(\mathbb{R}^d; \mathbb{C}).$$

This is possible because  $g$  has degree  $-d - 1$  at infinity and is smooth near the origin. Integrating by parts gives, for  $|\xi| \geq 2$ ,

$$\left| \partial_\xi^\beta \int_{\mathbb{R}^d} g(z) e^{\pm i\xi \cdot z} dz \right| \lesssim_{\beta, L} |\xi|^{-2L} \left\| (-\Delta_z)^L \left[ z^\beta g \right] \right\|_{L^1}.$$

Taking  $L$  with  $2L \geq M + |\beta|$  yields

$$\left| \partial_\xi^\beta \mathfrak{s}_h^{\text{tail}}(\xi) \right| \lesssim_{\beta, M} |\xi|^{-M - |\beta|}, \quad |\xi| \geq 2.$$

This proves that

$$\mathfrak{s}_h^{\text{tail}} \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M}.$$

If  $h$  is real-valued, then  $c_h$  is real and the multipliers satisfy

$$\overline{\mathfrak{m}(\xi)} = \mathfrak{m}(-\xi).$$

Thus the associated Fourier multiplier operators map real-valued functions to real-valued functions.  $\square$

### Local Taylor remainders for composed nonlinearities.

**Lemma 4.45** (Local Taylor remainder). *Let  $r_g > 0$  and  $g \in C^\omega(B_{\mathbb{R}}(0, r_g); \mathbb{R})$ . Define*

$$k_{[2]}(\eta; x, z) := g(\Delta_z \eta(x)) - g(\nabla_x \eta(x) \cdot \hat{z}) + \frac{|z|}{2} g'(\nabla_x \eta(x) \cdot \hat{z}) (\hat{z}^\top \nabla_x^2 \eta(x) \hat{z}).$$

Then

$$k_{[2]} \in K\mathcal{F}_{\text{loc}, \geq 1}^2. \quad (4.69)$$

If, in addition,  $g'(0) = 0$ , then

$$k_{[2]} \in K\mathcal{F}_{\text{loc}, \geq 2}^2, \quad \chi(z) k_{[2]} \in K\mathcal{F}_{\geq 2}^2. \quad (4.70)$$

*Proof.* The statement is purely local in the convolution variable. Choose  $s_+ > d/2 + 2$  and then  $r_0 > 0$  so small that every  $\eta \in B_{s_+, \mathbb{R}}(I; r_0)$  satisfies

$$\|\nabla_x \eta\|_{L^\infty} + \sup_{0 < |z| \leq 2} \|\Delta_z \eta\|_{L^\infty} < \frac{r_g}{4}.$$

Thus all arguments of  $g$  remain in a compact subset of its analytic interval.

Set

$$v(x, z) := \nabla_x \eta(x) \cdot \hat{z}, \quad h(x, z) := \hat{z}^\top \nabla_x^2 \eta(x) \hat{z}.$$

The second order expansion of the difference quotient reads

$$\Delta_z \eta(x) = v(x, z) - \frac{|z|}{2} h(x, z) + e(x, z).$$

For every  $s \geq s_+$  and every multi-index  $\alpha$ ,

$$\begin{aligned} \|\partial_z^\alpha v(\cdot, z)\|_{H^s} &\lesssim_{s, \alpha} |z|^{-|\alpha|} \|\eta\|_{H^{s+1+|\alpha|}}, \\ \|\partial_z^\alpha h(\cdot, z)\|_{H^s} &\lesssim_{s, \alpha} |z|^{-|\alpha|} \|\eta\|_{H^{s+2+|\alpha|}}, \quad 0 < |z| \leq 2. \\ \|\partial_z^\alpha e(\cdot, z)\|_{H^s} &\lesssim_{s, \alpha} |z|^{2-|\alpha|} \|\eta\|_{H^{s+3+|\alpha|}}, \end{aligned} \quad (4.71)$$

Indeed, derivatives of  $\hat{z}$  have homogeneity  $-|\alpha|$ , while  $e$  is  $|z|^2$  times an averaged third derivative of  $\eta$  along the segment  $x - \theta z$ ; differentiating in  $z$  either lowers the homogeneous degree or differentiates the translated factor, producing the finite loss displayed above.

Let

$$\vartheta(x, z) := \Delta_z \eta(x) - v(x, z) = -\frac{|z|}{2} h(x, z) + e(x, z).$$

Taylor's formula for  $g$  at  $v$  gives

$$g(v + \vartheta) = g(v) + g'(v) \vartheta + \sum_{\ell \geq 2} \frac{g^{(\ell)}(v)}{\ell!} \vartheta^\ell. \quad (4.72)$$

Hence

$$k_{[2]} = g'(v) e + \sum_{\ell \geq 2} \frac{g^{(\ell)}(v)}{\ell!} \vartheta^\ell. \quad (4.73)$$

The analytic-composition estimates and the tame product estimates, combined with Eq. (4.71), imply that for every  $|\alpha| \leq N$ ,

$$\|\partial_z^\alpha k_{[2]}(\eta; \cdot, z)\|_{H^s} \lesssim_{s, N} |z|^{2-|\alpha|} \|\eta\|_{H^{s+\mu_N}}, \quad 0 < |z| \leq 2,$$

for a finite loss  $\mu_N$ .

The vanishing order follows by separating

$$g'(v) = g'(0) + v \int_0^1 g''(\theta v) d\theta.$$

If  $g'(0) \neq 0$ , the term  $g'(0) e$  is linear in  $\eta$ , and this gives the order  $\geq 1$ . If  $g'(0) = 0$ , the first term in Eq. (4.73) contains the additional factor  $v$ , while the series starts quadratically in  $\vartheta$ ; therefore

$$\|\partial_z^\alpha k_{[2]}(\eta; \cdot, z)\|_{H^s} \lesssim_{s, N} \|\eta\|_{H^{s+\mu_N}} \|\eta\|_{H^{s+\mu_N}} |z|^{2-|\alpha|}.$$

This proves the local claims in Eq. (4.69) and Eq. (4.70). Their rapidly decreasing versions follow from Remark 4.27.  $\square$

### Finite-difference Sobolev estimates.

**Lemma 4.46.** *For every  $s \in \mathbb{R}$  one has*

$$\|\delta_z \eta\|_{H^s} \lesssim |z| \|\eta\|_{H^{s+1}}, \quad \|\Delta_z \eta - \nabla_x \eta \cdot \hat{z}\|_{H^s} \lesssim |z| \|\eta\|_{H^{s+2}}. \quad (4.74)$$

*Proof.* The estimates are non-homogeneous. By Plancherel,

$$\|\delta_z \eta\|_{H^s}^2 = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{\eta}(\xi)|^2 |1 - e^{-i\xi \cdot z}|^2 d\xi.$$

Since  $|1 - e^{-iX}| \lesssim \min\{1, |X|\} \lesssim |X|$ , we get

$$\|\delta_z \eta\|_{H^s}^2 \lesssim |z|^2 \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\xi|^2 |\hat{\eta}(\xi)|^2 d\xi \lesssim |z|^2 \|\eta\|_{H^{s+1}}^2.$$

Similarly,

$$\|\Delta_z \eta - \nabla_x \eta \cdot \hat{z}\|_{H^s}^2 = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\hat{\eta}(\xi)|^2 \left| \frac{1 - e^{-i\xi \cdot z} - i\xi \cdot z}{|z|} \right|^2 d\xi.$$

Using  $|1 - e^{-iX} - iX| \lesssim \min\{|X|, |X|^2\} \lesssim |X|^2$  we infer

$$\|\Delta_z \eta - \nabla_x \eta \cdot \hat{z}\|_{H^s}^2 \lesssim |z|^2 \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} |\xi|^4 |\hat{\eta}(\xi)|^2 d\xi \lesssim |z|^2 \|\eta\|_{H^{s+2}}^2.$$

Taking square roots proves Eq. (4.74).  $\square$

**Remark 4.47** (Dimension convention). All the results of this section are stated on  $\mathbb{R}^d$  for an arbitrary  $d \geq 1$ . Unless explicitly stated otherwise, in the remainder of the manuscript we set  $d = 2$ ; the only exception is Section 5.1, where the dimension remains arbitrary.  $\blacklozenge$

## 5 Parilinearization

This section is devoted to the parilinearization of the system of interest in this study. It represents the core of the manuscript.

**Notation 5.1.** 1. In the present section  $1 \ll \varrho \ll s_0$  and the explicit value of  $s_0$  may vary from line to line.

2. Let  $m, n \in \mathbb{R}$ ,  $p \in \mathbb{N}^*$ , and  $\vec{U}_p \in (B_{s_0, \mathbb{R}}(I; r_0))^p$ .

(a) For  $\ell \in \mathbb{N}$ , we denote with

$$a_{[m, \geq \ell]}(\vec{U}_p; x, \xi), \quad A_{[m, \geq \ell]}(\vec{U}_p; x, \xi), \quad \mathbf{A}_{[m, \geq \ell]}(\vec{U}_p; x, \xi), \quad \mathfrak{A}_{[m, \geq \ell]}(\vec{U}_p; x, \xi)$$

a generic element in

$$\Gamma_{\geq \ell}^m, \quad (\Gamma_{\geq \ell}^m)^2, \quad (\Gamma_{\geq \ell}^m)^{2 \times 2}, \quad (\Gamma_{\geq \ell}^m)^{2 \times 3},$$

respectively, whose explicit expression may vary from line to line. When  $\ell = 1$  we omit the homogeneity subscript. When the corresponding symbols are pointwise real componentwise, we add the superscript  $\mathbb{R}$ , writing

$$a_{[m, \geq \ell]}^{\mathbb{R}}, \quad A_{[m, \geq \ell]}^{\mathbb{R}}, \quad \mathbf{A}_{[m, \geq \ell]}^{\mathbb{R}}, \quad \mathfrak{A}_{[m, \geq \ell]}^{\mathbb{R}},$$

respectively. This superscript records pointwise reality and not the real-to-real symmetry of a symbol. Of these we define the subclasses

i. *Linear symbols* defined as

$$\begin{aligned} A_{[m, \geq \ell]}^{\text{lin}}(\eta, \omega) &:= \begin{bmatrix} r_{[m, \geq \ell]}[\omega](\eta) \\ r_{[m, \geq \ell]}(\eta) \end{bmatrix}, \quad \mathbf{A}_{[m, \geq \ell]}^{\text{lin}}(\eta, \omega) := \begin{bmatrix} A_{[m, \geq \ell]}[\omega](\eta) & A_{[m, \geq \ell]}(\eta) \end{bmatrix}, \\ \mathbf{a}_{[m, \geq \ell]}^{\text{lin}}(\eta, \Omega) &:= \begin{bmatrix} r_{[m, \geq \ell]}[\Omega](\eta) \\ A_{[m, \geq \ell]}(\eta) \end{bmatrix}, \quad \mathfrak{A}_{[m, \geq \ell]}^{\text{lin}}(\eta, \Omega) := \begin{bmatrix} A_{[m, \geq \ell]}[\Omega](\eta) & \mathbf{A}_{[m, \geq \ell]}(\eta) \end{bmatrix}. \end{aligned}$$

ii. *Bilinear symbols* defined as

$$\begin{aligned} \mathbf{a}_{[m, \geq \ell]}^{\text{bil}}(\eta, \omega_1, \omega_2) &:= \begin{bmatrix} a_{[m, \geq \ell]}[\omega_1, \omega_2](\eta) \\ a_{[m, \geq \ell]}[\omega_2](\eta) \\ a_{[m, \geq \ell]}[\omega_1](\eta) \end{bmatrix}, \\ \mathfrak{A}_{[m, \geq \ell]}^{\text{bil}}(\eta, \omega_1, \omega_2) &:= \begin{bmatrix} A_{[m, \geq \ell]}[\omega_1, \omega_2](\eta) & A_{[m, \geq \ell]}[\omega_2](\eta) & A_{[m, \geq \ell]}[\omega_1](\eta) \end{bmatrix}. \end{aligned}$$

The above definition are such that the application

$$\omega \mapsto \text{Op}^{\text{BW}}\left(A_{[m, \geq \ell]}^{\text{lin}}(\eta, \omega)\right) \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix} \in \mathcal{L}(H^s; H^{s-m})$$

for all  $s \geq s_0$ , the same holds for all the other terms with due modifications.

(b) For  $\ell \in \mathbb{N}^*$ , we denote with

$$k_{[n, \geq \ell]}(\vec{U}_p; x, z), \quad K_{[n, \geq \ell]}(\vec{U}_p; x, z), \quad \mathbf{K}_{[n, \geq \ell]}(\vec{U}_p; x, z), \quad \mathfrak{K}_{[n, \geq \ell]}(\vec{U}_p; x, z)$$

a generic element of

$$K\mathcal{F}_{\geq \ell}^n, \quad (K\mathcal{F}_{\geq \ell}^n)^2, \quad (K\mathcal{F}_{\geq \ell}^n)^{2 \times 2}, \quad (K\mathcal{F}_{\geq \ell}^n)^{2 \times 3},$$

respectively, whose explicit expression may vary from line to line. When  $\ell = 1$  we omit the homogeneity subscript. The forms

$$k_{[n, \geq \ell]}^{\text{loc}}, \quad K_{[n, \geq \ell]}^{\text{loc}}, \quad \mathbf{K}_{[n, \geq \ell]}^{\text{loc}}, \quad \mathfrak{K}_{[n, \geq \ell]}^{\text{loc}}$$

denote generic elements of  $K\mathcal{F}_{\text{loc}, \geq \ell}^n$  and its corresponding vector and matrix classes. For  $n < 0$  with  $\ell - n > d$ , we write

$$k_{[n, \geq \ell]}^{\text{tail}}, \quad K_{[n, \geq \ell]}^{\text{tail}}, \quad \mathbf{K}_{[n, \geq \ell]}^{\text{tail}}, \quad \mathfrak{K}_{[n, \geq \ell]}^{\text{tail}}$$

for generic elements of the existing class  $K\mathcal{F}_{\text{tail}, \geq \ell}$  and its corresponding vector and matrix classes, whose profiles in Definition 4.26 satisfy Eq. (4.17) with  $m = -n$ , componentwise. For tail kernels,  $n$  records the order of the profile  $b$ , not the decay of the full kernel.

When these kernels are real-valued componentwise for real backgrounds and real separately estimated arguments, we add the superscript  $\mathbb{R}$ , writing

$$k_{[n, \geq \ell]}^{\mathbb{R}}, \quad K_{[n, \geq \ell]}^{\mathbb{R}}, \quad \mathbf{K}_{[n, \geq \ell]}^{\mathbb{R}}, \quad \mathfrak{K}_{[n, \geq \ell]}^{\mathbb{R}},$$

respectively. The corresponding local forms are

$$k_{[n, \geq \ell]}^{\text{loc}, \mathbb{R}}, \quad K_{[n, \geq \ell]}^{\text{loc}, \mathbb{R}}, \quad \mathbf{K}_{[n, \geq \ell]}^{\text{loc}, \mathbb{R}}, \quad \mathfrak{K}_{[n, \geq \ell]}^{\text{loc}, \mathbb{R}},$$

and the tail forms are

$$k_{[n, \geq \ell]}^{\text{tail}, \mathbb{R}}, \quad K_{[n, \geq \ell]}^{\text{tail}, \mathbb{R}}, \quad \mathbf{K}_{[n, \geq \ell]}^{\text{tail}, \mathbb{R}}, \quad \mathfrak{K}_{[n, \geq \ell]}^{\text{tail}, \mathbb{R}}.$$

(c) For  $\ell \in \mathbb{N}^*$ , we denote with

$$r_{[\varrho, \geq \ell]}(\vec{U}_p), \quad R_{[\varrho, \geq \ell]}(\vec{U}_p), \quad \mathbf{R}_{[\varrho, \geq \ell]}(\vec{U}_p), \quad \mathfrak{R}_{[\varrho, \geq \ell]}(\vec{U}_p)$$

a generic element of

$$\mathcal{R}_{\geq \ell}^\varrho, \quad (\mathcal{R}_{\geq \ell}^\varrho)^2, \quad (\mathcal{R}_{\geq \ell}^\varrho)^{2 \times 2}, \quad (\mathcal{R}_{\geq \ell}^\varrho)^{2 \times 3},$$

respectively, whose explicit expression may vary from line to line. When the corresponding smoothing operators are real-to-real componentwise, we add the superscript  $\mathbb{R}$ , writing

$$r_{[\varrho, \geq \ell]}^{\mathbb{R}}, \quad R_{[\varrho, \geq \ell]}^{\mathbb{R}}, \quad \mathbf{R}_{[\varrho, \geq \ell]}^{\mathbb{R}}, \quad \mathfrak{R}_{[\varrho, \geq \ell]}^{\mathbb{R}},$$

respectively.

We denote with

$$r_{[\varrho, n, \geq \ell]}(\vec{U}_p; z), \quad R_{[\varrho, n, \geq \ell]}(\vec{U}_p; z), \quad \mathbf{R}_{[\varrho, n, \geq \ell]}(\vec{U}_p; z), \quad \mathfrak{R}_{[\varrho, n, \geq \ell]}(\vec{U}_p; z)$$

a generic element of

$$K\mathcal{R}_{\geq \ell}^{\varrho, n}, \quad (K\mathcal{R}_{\geq \ell}^{\varrho, n})^2, \quad (K\mathcal{R}_{\geq \ell}^{\varrho, n})^{2 \times 2}, \quad (K\mathcal{R}_{\geq \ell}^{\varrho, n})^{2 \times 3},$$

respectively, whose explicit expression may vary from line to line. Aggregate notation with one threshold is used only when every component has the displayed class membership. Of these we define the subclasses

i. *Linear remainders* defined as

$$\begin{aligned} R_{[\varrho, \geq \ell]}^{\text{lin}}(\eta, \omega) &:= \begin{bmatrix} r_{[\varrho, \geq \ell]}[\omega](\eta) \\ r_{[\varrho, \geq \ell]}(\eta) \end{bmatrix}, & \mathbf{R}_{[\varrho, \geq \ell]}^{\text{lin}}(\eta, \omega) &:= \begin{bmatrix} R_{[\varrho, \geq \ell]}[\omega](\eta) & R_{[\varrho, \geq \ell]}(\eta) \end{bmatrix}, \\ \mathbf{r}_{[\varrho, \geq \ell]}^{\text{lin}}(\eta, \Omega) &:= \begin{bmatrix} r_{[\varrho, \geq \ell]}[\Omega](\eta) \\ R_{[\varrho, \geq \ell]}(\eta) \end{bmatrix}, & \mathfrak{R}_{[\varrho, \geq \ell]}^{\text{lin}}(\eta, \Omega) &:= \begin{bmatrix} R_{[\varrho, \geq \ell]}[\Omega](\eta) & \mathbf{R}_{[\varrho, \geq \ell]}(\eta) \end{bmatrix}. \end{aligned}$$

ii. *Bilinear remainders* defined as

$$\begin{aligned} \mathbf{r}_{[\varrho, \geq \ell]}^{\text{bil}}(\eta, \omega_1, \omega_2) &:= \begin{bmatrix} r_{[\varrho, \geq \ell]}[\omega_1, \omega_2](\eta) \\ r_{[\varrho, \geq \ell]}[\omega_2](\eta) \\ r_{[\varrho, \geq \ell]}[\omega_1](\eta) \end{bmatrix}, \\ \mathfrak{R}_{[\varrho, \geq \ell]}^{\text{bil}}(\eta, \omega_1, \omega_2) &:= \begin{bmatrix} R_{[\varrho, \geq \ell]}[\omega_1, \omega_2](\eta) & R_{[\varrho, \geq \ell]}[\omega_2](\eta) & R_{[\varrho, \geq \ell]}[\omega_1](\eta) \end{bmatrix}. \end{aligned}$$

The real-to-real versions

$$\mathbf{R}_{[\varrho, \geq \ell]}^{\text{lin}, \mathbb{R}}, \quad \mathbf{R}_{[\varrho, \geq \ell]}^{\text{lin}, \mathbb{R}}, \quad \mathbf{r}_{[\varrho, \geq \ell]}^{\text{lin}, \mathbb{R}}, \quad \mathfrak{R}_{[\varrho, \geq \ell]}^{\text{lin}, \mathbb{R}}, \quad \mathbf{r}_{[\varrho, \geq \ell]}^{\text{bil}, \mathbb{R}}, \quad \mathfrak{R}_{[\varrho, \geq \ell]}^{\text{bil}, \mathbb{R}}$$

are defined by the same formulas with each constituent

$$r_{[\varrho, \geq \ell]}, \quad R_{[\varrho, \geq \ell]}, \quad \mathbf{R}_{[\varrho, \geq \ell]}$$

replaced by its  $\mathbb{R}$ -superscripted version. For constituents with bracketed arguments, the real-to-real property holds for every real choice of those arguments.

The above definitions are such that the application

$$\omega \mapsto R_{[\varrho, \geq \ell]}^{\text{lin}}(\eta, \omega) \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix} \in \mathcal{L}(H^s; H^{s+\varrho})$$

for all  $s \geq s_0$ , the same holds for all the other terms with due modifications.

- (d) For  $\ell \in \mathbb{N}^*$ , we denote with  $a_{[m, \geq \ell]}(\mathfrak{a}|\vec{U}_p; x, \xi)$ ,  $r_{[\varrho, \geq \ell]}(\mathfrak{a}|\vec{U}_p)$  and so on families of, respectively symbols and smoothing operators, that are real analytic w.r.t. the Atwood number  $\mathfrak{a} \in [-2, 2]$ . An analogous notation is used for vector- and matrix-valued symbols and smoothing operators.
- (e) Given  $A \in (\Gamma_{\geq 1}^m)^2$  and a vector-valued function  $U$ , we use the compact notation

$$\text{Op}^{\text{BW}}(A) \cdot [U] := \text{Op}^{\text{BW}}(A) \cdot U := \sum_{j=1}^2 \text{Op}^{\text{BW}}(a_j) [u_j].$$

We shall consider  $\Omega \approx \nabla^\perp \psi$ . Therefore, the quasilinear terms are of order 0 in  $\Omega$  and 1 in  $\eta$ .

The main goal of this section is to prove the following theorem.

**Theorem 5.2** (Paralinearization of the canonical CDE). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle \rho^\pm \rangle > 0$ ,  $\mathfrak{g} \in \mathbb{R}$ , and  $\bar{\Omega} \in \mathbb{R}^2$ . There exists  $\varrho_0 > 0$  such that, for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that, if*

$$\eta, \tilde{\psi} \in B_{s_0, \mathbb{R}}(I; r_0)$$

*solve the canonical CDE Eq. (2.9), then  $(\eta, \tilde{\psi})$  solves the paradifferential system*

$$\begin{aligned} \begin{bmatrix} \eta_t \\ \tilde{\psi}_t \end{bmatrix} &= \text{Op}^{\text{BW}}(\mathbf{Q}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) + \mathbf{B}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) + i(\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \xi) \text{Id}_{\mathbb{C}^2}) \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \\ &+ \text{Op}^{\text{BW}} \left( \begin{bmatrix} A_{[0, \geq 1]}^\eta(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) & A_{[-1, \geq 1]}^\psi(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) \end{bmatrix} \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} + \mathbf{R}_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \right). \end{aligned} \quad (5.1)$$

*When the lower-order term in the second line is viewed as a two-by-two block, it has the order structure*

$$\begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \end{bmatrix}. \quad (5.2)$$

*Here the objects appearing in Eq. (5.1) are defined as follows.*

- $\mathfrak{a}$  is the Atwood number, defined in Eq. (1.5).

- $\mathbf{Q} \in (\Gamma_{\geq 0}^2)^{2 \times 2}$  is the quasilinear symbol explicitly defined as

$$\mathbf{Q}(\rho^\pm, \mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := \begin{bmatrix} 0 & \lambda_1(\eta; x, \xi) + \mathfrak{a}\lambda_0(\eta; x, \xi) \\ -\mathfrak{g}\mathfrak{a} + a_2^{\pi_0, \eta}(\rho^\pm|\eta; x, \xi) + \lambda_1(\eta; x, \xi) q_0^{\tilde{\pi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) & 0 \end{bmatrix}. \quad (5.3)$$

More explicitly:

- $\lambda_1 \in \Gamma_{\geq 0}^1$  is real-valued and is explicitly defined as

$$\lambda_1(\eta; x, \xi) := \frac{|\xi|}{2} \sqrt{1 + \left| \nabla^\perp \eta \cdot \frac{\xi}{|\xi|} \right|^2}. \quad (5.4)$$

- $\lambda_0 \in \Gamma_{\geq 1}^0$  is real valued and defined as

$$\lambda_0(\eta; x, \xi) := \left( \frac{|\xi|}{4\sqrt{2} \lambda_1(\eta; x, \xi)} \right)^2 (\hat{\xi}^\perp)^\top \mathbf{H}_x \eta \hat{\xi}^\perp, \quad (5.5)$$

where  $\lambda_1$  is defiend in Eq. (5.4);

- $a_2^{\pi_0, \eta} \in \Gamma_{\geq 0}^2$  is real-valued and is explicitly defined as

$$a_2^{\pi_0, \eta}(\rho^\pm|\eta; x, \xi) := -\frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} \frac{|\xi|^2 + |\nabla^\perp \eta \cdot \xi|^2}{(1 + |\nabla \eta|^2)^{\frac{3}{2}}} = -\frac{4\sigma}{\langle\langle \rho^\pm \rangle\rangle} \frac{\lambda_1^2(\eta; x, \xi)}{(1 + |\nabla \eta(x)|^2)^{\frac{3}{2}}}. \quad (5.6)$$

- $q_0^{\tilde{\pi}} \in \Gamma_{\geq 0}^0$  is real-valued, even in  $\xi$  and is explicitly defined as

$$q_0^{\tilde{\pi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := (4 - \mathfrak{a}^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega(\bar{\Omega}|\eta, \tilde{\psi}; x) \right)^2, \quad (5.7)$$

where  $A_0^{\mathcal{B}_0, \omega}$  is the vector symbol defined as

$$A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) := \nabla_\xi \phi_1^{\mathcal{B}_0}(\eta; x, \xi), \quad \phi_1^{\mathcal{B}_0}(\eta; x, \xi) := -\frac{\lambda_1(\eta; x, \xi)}{1 + |\nabla \eta(x)|^2}; \quad (5.8)$$

- $\mathbf{B} \in (\Gamma_{\geq 1}^1)^{2 \times 2}$  has the Alinhac block form

$$\mathbf{B}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := - \begin{bmatrix} -b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) & 0 \\ (b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x))^2 & b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) \end{bmatrix} \lambda_1(\eta; x, \xi). \quad (5.9)$$

Here

$$b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) := \mathfrak{a} \nu_3^\star [\Omega(\bar{\Omega}|\eta, \tilde{\psi})](\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega(\bar{\Omega}|\eta, \tilde{\psi}; x)}{1 + |\nabla \eta|^2}, \quad (5.10)$$

and the canonical vorticity variable is

$$\Omega(\bar{\Omega}|\eta, \tilde{\psi}; x) := \bar{\Omega}[\bar{\Omega}](\eta; x) + S(\mathfrak{a}|\eta) \tilde{\psi}(x), \quad (5.11)$$

where  $S(\mathfrak{a}|\eta)$  is defined in Eq. (2.11) and  $\bar{\Omega}[\bar{\Omega}](\eta)$  is defined in Eq. (2.10);

- The transport velocity is

$$\begin{aligned} \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) &:= V^\star [\Omega(\bar{\Omega}|\eta, \tilde{\psi})](\eta; x) \\ &+ \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} \mathbf{J}(\Omega(\bar{\Omega}|\eta, \tilde{\psi}; x) + (\nabla \eta \cdot \Omega(\bar{\Omega}|\eta, \tilde{\psi}; x)) \nabla \eta). \end{aligned} \quad (5.12)$$

where  $\mathbf{v}^\star = \begin{bmatrix} V^\star \\ \nu_3^\star \end{bmatrix}$  is defined in Eq. (2.14). In particular, the transport coefficient belongs to  $(\mathcal{F}_{\geq 0}^\mathbb{R})^2$  and splits as

$$\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) = \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp + \mathcal{V}_{\text{can}}^{\text{nl}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}), \quad \mathcal{V}_{\text{can}}^{\text{nl}} \in (\mathcal{F}_{\geq 1}^\mathbb{R})^2.$$

Moreover,  $\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x)$  is real-valued and independent of  $\xi$ .

- The lower-order symbols satisfy

$$A_{[0,\geq 1]}^\eta \in (\Gamma_{\geq 1}^0)^2, \quad A_{[-1,\geq 1]}^\psi \in (\Gamma_{\geq 1}^{-1})^2,$$

or equivalently the associated two-by-two block has the order structure stated in Eq. (5.2). They denote generic symbols as per Notation 5.1.

- $R_{[\ell,\geq 1]}^{\text{lin},\mathbb{R}} \in (\mathcal{R}_{\geq 1}^\ell)^{2 \times 2}$  denotes a generic smoothing remainder as per Notation 5.1.

**Remark 5.3** (On the equivalent formulas for  $\lambda_1$ ). Let

$$\tilde{\lambda}_1(x, \xi) := \sqrt{\left(1 + |\nabla \eta(x)|^2\right) |\xi|^2 - (\nabla \eta(x) \cdot \xi)^2},$$

which is the principal symbol in the expansion of the DN operator in [8]. Factoring  $|\xi|$  gives

$$\tilde{\lambda}_1(x, \xi) = |\xi| \sqrt{1 + |\nabla \eta|^2 - (\nabla \eta \cdot \hat{\xi})^2}.$$

Notice that  $(\hat{\xi}, \hat{\xi}^\perp)$  is an orthonormal basis. Hence,

$$|\nabla \eta|^2 = (\nabla \eta \cdot \hat{\xi})^2 + (\nabla \eta \cdot \hat{\xi}^\perp)^2 = (\nabla \eta \cdot \hat{\xi})^2 + (\nabla^\perp \eta \cdot \hat{\xi})^2,$$

so that

$$\tilde{\lambda}_1(x, \xi) = |\xi| \sqrt{1 + |\nabla^\perp \eta(x) \cdot \hat{\xi}|^2}.$$

Therefore,  $\lambda_1 = \tilde{\lambda}_1/2$ , and the factor 1/2 depends on the fact that here  $\psi$  is  $\psi/2$  of [8].  $\blacklozenge$

The proof of Theorem 5.2 is organized into four logical steps:

1. **Abstract structure-preserving parilinearization.** In Section 5.1, we establish the abstract parilinearization theorem for critical homogeneous graph-kernel operators on  $\mathbb{R}^d$ . The proof separates the local–tail and Bony reductions from the quasilinear and transported contributions, and identifies the corresponding homogeneous Fourier multipliers and lower-order symbols.
2. **Parilinearization of the singular integral operators.** In Section 5.2, and more specifically in Section 5.2, we apply Theorem 5.7 to the primitive singular integral operators  $\mathcal{B}_0(\eta)$  and  $q(\eta)$ . We then identify their principal symbols and combine the resulting parilinearizations in the form needed for the subsequent analysis.
3. **Parilinearization of the evolution operators.** In Section 5.3, we use the preceding singular-integral expansions to parilinearize the operators entering the canonical evolution equations. Starting from  $\nu^\star$ , we treat successively  $T(\eta)$  and  $\tau_0(\eta)$ , the change of variable  $S(\mathfrak{a}|\eta)$ , the Dirichlet–Neumann-type operators  $\mathcal{J}^\star(\eta)$  and  $\mathcal{J}(\mathfrak{a}|\eta)$ , and the Bernoulli contributions  $\pi^\star$ ,  $\pi$ , and  $\tilde{\pi}$ .
4. **Synthesis.** Finally, we assemble these expansions in the canonical variables  $(\eta, \tilde{\psi})$  and obtain the complete parilinear system Eq. (5.1), thereby concluding the proof of Theorem 5.2.

## 5.1 Abstract structure-preserving parilinearization on $\mathbb{R}^d$

**Dimension and scalar convention.** Throughout this subsection,  $d \geq 1$ . In accordance with Notation 1.1, scalar functions, operators, and symbols are denoted by lower-case letters; only  $x, \xi \in \mathbb{R}^d$  retain the lower-case notation reserved there for spatial and frequency variables. We state the result in the scalar case; vector- and matrix-valued versions follow componentwise. The Fourier normalization and the measure on  $\mathbb{S}^{d-1}$  are fixed in Notation 4.1 below.

**Definition 5.4** (Admissible homogeneous pair). Let  $0 < n \leq d$  and  $\ell \in \mathbb{N}^\star$ . Thus  $-n$  is the degree of homogeneity of the kernel  $|z|^{-n}$ , and the critical endpoint corresponds to  $n = d$ . An *admissible pair of order  $-n$  and vanishing order  $\ell$*  is a pair

$$(h, k) \in C^\infty(\mathbb{S}^{d-1}; \mathbb{R}) \times C^\omega((-r_k, r_k); \mathbb{R})$$

with the following properties:

- the function

$$\tilde{k}(X) := k(X) - k(0) \quad (5.13)$$

vanishes to order at least  $\ell$  at the origin, namely

$$\tilde{k}^{(j)}(0) = 0, \quad 0 \leq j \leq \ell - 1. \quad (5.14)$$

- In the critical case  $n = d$ , there exists  $\varepsilon \in \{-1, 1\}$  such that

$$h(-\vartheta) = \varepsilon h(\vartheta), \quad (5.15a)$$

$$k(-X) = -\varepsilon k(X). \quad (5.15b)$$

**Remark 5.5.** In the case  $n = d$ , for every  $p \in \mathbb{R}^d$  we have that

$$\begin{aligned} h(\vartheta) k(p \cdot \vartheta) &\text{ is odd in } \vartheta, \\ h(\vartheta) k'(p \cdot \vartheta) &\text{ is even in } \vartheta, \\ h(\vartheta) k''(p \cdot \vartheta) &\text{ is odd in } \vartheta. \end{aligned} \quad (5.16)$$

◆

Fix once and for all a number  $r_{\text{an}} \in (0, r_k)$  strictly inside the radius of convergence at the origin of the power series of  $k$ , and fix  $r_* \in (0, r_{\text{an}})$ . All analytic expansions below are performed on  $|X| \leq r_* < r_{\text{an}}$ ; control of the full operator uses the same bound for every difference quotient.

For an admissible pair as per Definition 5.4, set for all  $n \leq d$

$$\begin{aligned} t_{h,k}^{(n)}(\eta)[\omega](x) &:= \text{p.v.} \int_{\mathbb{R}^d} \frac{h(\hat{z})}{|z|^n} k(\Delta_z \eta(x)) \omega(x-z) dz, \\ \Delta_z \eta(x) &:= \frac{\eta(x) - \eta(x-z)}{|z|}, \quad \hat{z} := \frac{z}{|z|}. \end{aligned} \quad (5.17)$$

**Lemma 5.6** (Skew-symmetric structure of admissible graph kernels in the critical case). *For every real  $\eta$  for which Eq. (5.17) is defined the operator  $t_{h,k}^{(d)}(\eta)$  is  $L^2(\mathbb{R}^d; \mathbb{R})$ -skew-symmetric.*

*Proof.* We prove that for every  $f, g \in \mathcal{S}(\mathbb{R}^d; \mathbb{R})$ , we have that

$$\left\langle t_{h,k}^{(d)}(\eta)[f] \mid g \right\rangle_{L^2} = - \left\langle f \mid t_{h,k}^{(d)}(\eta)[g] \right\rangle_{L^2}, \quad (5.18)$$

the generic assertion thus follows by density. For  $0 < \varepsilon < R < +\infty$ , let

$$t_{h,k;\varepsilon,R}^{(d)}(\eta)[f](x) := \int_{\varepsilon < |x-y| < R} \mathcal{K}_\eta(x, y) f(y) dy,$$

where

$$\mathcal{K}_\eta(x, y) := \frac{h(\widehat{x-y})}{|x-y|^d} k\left(\frac{\eta(x) - \eta(y)}{|x-y|}\right).$$

The truncation removes the singularity at  $x = y$ , so that the following computations are justified by Fubini's theorem. We first verify that  $\mathcal{K}_\eta$  is skew-symmetric. Exchanging  $x$  and  $y$  gives

$$\widehat{y-x} = -\widehat{x-y}, \quad \frac{\eta(y) - \eta(x)}{|y-x|} = -\frac{\eta(x) - \eta(y)}{|x-y|}.$$

Hence, by Eqs. (5.15a) and (5.15b), we have

$$\begin{aligned} \mathcal{K}_\eta(y, x) &= \frac{h(\widehat{-x-y})}{|x-y|^d} k\left(-\frac{\eta(x) - \eta(y)}{|x-y|}\right) \\ &= \frac{\varepsilon h(\widehat{x-y})}{|x-y|^d} (-\varepsilon) k\left(\frac{\eta(x) - \eta(y)}{|x-y|}\right) \\ &= -\mathcal{K}_\eta(x, y). \end{aligned} \quad (5.19)$$

Since the truncated domain

$$\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : \epsilon < |x - y| < R\}$$

is invariant under the exchange  $x \leftrightarrow y$ , using Eq. (5.19), we obtain

$$\begin{aligned} \left\langle t_{h,k;\epsilon,R}^{(d)}(\eta)[f] \mid g \right\rangle_{L^2} &= \iint_{\epsilon < |x-y| < R} \mathcal{K}_\eta(x, y) f(y) g(x) dy dx \\ &= \iint_{\epsilon < |x-y| < R} \mathcal{K}_\eta(y, x) f(x) g(y) dx dy \\ &= - \iint_{\epsilon < |x-y| < R} \mathcal{K}_\eta(x, y) f(x) g(y) dy dx \\ &= - \left\langle f \mid t_{h,k;\epsilon,R}^{(d)}(\eta)[g] \right\rangle_{L^2}. \end{aligned}$$

Finally, by the definition of  $t_{h,k}^{(d)}(\eta)$  in Eq. (5.17), we let  $\epsilon \downarrow 0$  and  $R \uparrow +\infty$ . Passing to the principal-value limit in the preceding identity yields Eq. (5.18).  $\square$

**Theorem 5.7** (Abstract structure-preserving parilinearization). *Let  $(h, k)$  be an admissible homogeneous pair of critical order  $-d$  and vanishing order  $\ell$  in the sense of Definition 5.4. There exists  $\varrho_0 > 0$  such that, for every  $\varrho \geq \varrho_0$ , there exist  $s_0 > d/2 + 1$ ,  $\mu \geq 0$ , and  $r_0 > 0$  such that, for every  $s \geq s_0$ , every  $\eta \in H^{s+1+\mu}(\mathbb{R}^d; \mathbb{R})$ , and  $\omega \in H^{s+\mu}(\mathbb{R}^d; \mathbb{R})$  satisfying  $\|\eta\|_{H^{s_0+1+\mu}} < r_0$ , the operator defined in Eq. (5.17) admits the parilinearization*

$$t_{h,k}^{(d)}(\eta)[\omega] = \text{Op}^{\text{BW}} \left( A_1^{h,k}[\omega](\eta; x, \xi) + \left[ a_{[0, \geq \ell]}^{[0, \geq \ell]}[\omega](\eta; x, \xi) \right] \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix} + R_{[\varrho, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega) \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix} \right). \quad (5.20)$$

The objects in Eq. (5.20) are as follows.

- The vector symbol

$$A_1^{h,k}[\omega](\eta; x, \xi) := \begin{bmatrix} \omega(x) a_1^{h,k;\eta}(\nabla \eta(x), \xi) \\ i a_0^{h,k;\omega}(\nabla \eta(x), \xi) \end{bmatrix}$$

has the following components.

- $a_1^{h,k;\eta}(\nabla \eta, \xi) \in \Gamma_{\geq \ell-1}^1$  is the real-valued symbol

$$a_1^{h,k;\eta}(p, \xi) := \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\partial \cdot \xi| h(\partial) k'(p \cdot \partial) d\sigma(\partial). \quad (5.21)$$

It is even in  $\xi$ . In addition,  $a_1^{h,k;\eta}(0, \xi) = 0$  whenever  $\ell \geq 2$ ;

- $a_0^{h,k;\omega}(\nabla \eta, \xi) - a_0^{h,k;\omega}(0, \xi) \in \Gamma_{\geq \ell}^0$  is the real-valued symbol

$$a_0^{h,k;\omega}(p, \xi) := -\frac{\pi}{2} \int_{\mathbb{S}^{d-1}} \text{sgn}(\partial \cdot \xi) h(\partial) k(p \cdot \partial) d\sigma(\partial). \quad (5.22)$$

It is odd in  $\xi$ . In addition,  $a_0^{h,k;\omega}(0, \xi)$  is the symbol of the flat homogeneous contribution, whereas its nonlinear part vanishes to order at least  $\ell$  at  $\eta = 0$ ;

- the two components satisfy

$$a_1^{h,k;\eta}(p, \xi) = -\partial_p a_0^{h,k;\omega}(p, \xi). \quad (5.23)$$

- The lower-order vector symbol

$$\begin{bmatrix} a_{[0, \geq \ell]}^{[0, \geq \ell]}[\omega](\eta; x, \xi) \\ i a_{[-2, \geq \ell]}^{\mathbb{R}; \omega}(\eta; x, \xi) \end{bmatrix}$$

has the following components.

- $a_{[0, \geq \ell]}[\cdot] \in \mathcal{L}(H^{s_0}; \Gamma^0)$ , and for every real  $\omega$ ,

$$a_{[0, \geq \ell]}[\omega](\eta; x, -\xi) = \overline{a_{[0, \geq \ell]}[\omega](\eta; x, \xi)};$$

If  $\ell \geq 2$ , the range improves to  $\Gamma_{\geq \ell-1}^0$ ;

–  $a_{[-2, \geq \ell]}^{\mathbb{R}; \omega} \in \Gamma_{\geq \ell}^{-2}$  is real-valued and satisfies

$$a_{[-2, \geq \ell]}^{\mathbb{R}; \omega}(\eta; x, -\xi) = -a_{[-2, \geq \ell]}^{\mathbb{R}; \omega}(\eta; x, \xi). \quad (5.24)$$

- The smoothing vector  $R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega)$  is as per Notation 5.1; both components are real-to-real in the sense of Definition 4.5.

**Remark 5.8** (Skew-adjointness and the vanishing subprincipal symbol). Two distinct structural facts enter the critical parilinearization.

First, the exact skew-adjointness of the critical quasilinear branch implies that its complete Bony–Weyl symbol can be chosen purely imaginary. This is proved in Lemma 5.11 and yields the symbol in Eq. (5.20).

Second, the absence of an order- $(-1)$  term is a stronger statement and does not follow from skew-adjointness alone: a symbol of the form  $ib_{-1}$ , with  $b_{-1}$  real and odd in  $\xi$ , would still define a skew-adjoint Bony–Weyl operator. For  $d \geq 2$ , the proof below establishes the stronger cancellation explicitly. The first Bony–Weyl correction generated by the principal homogeneous kernel is  $-a_{-1}^{h,k}$ , whereas the next Taylor coefficient of the graph difference produces  $+a_{-1}^{h,k}$ . Hence

$$-a_{-1}^{h,k} + a_{-1}^{h,k} = 0,$$

as shown in Eqs. (5.50) and (5.56). For  $d = 1$ , the degree-zero principal multiplier is constant on each half-line, so its derivatives in  $\xi$  vanish away from the origin; the corresponding localized order-minus-one model also has vanishing high-frequency multiplier. Thus no independent order- $(-1)$  symbol occurs in any dimension.

The cancellation removes the subprincipal order, while skew-adjoint completion controls the reality type of the entire remaining symbol. These are complementary consequences of, respectively, the centered Weyl structure and the opposite parities in Eqs. (5.15a) and (5.15b).  $\blacklozenge$

### 5.1.1 Local–tail and Bony reduction

Let  $\chi$  be the even radial cutoff fixed in Eq. (4.7). For fixed  $z \neq 0$ , write  $\omega_z(x) := \omega(x - z)$ . Recall from Eq. (5.13) that  $\tilde{k} := k - k(0)$ , so that  $\tilde{k}$  vanishes to order at least  $\ell$  at the origin. Define the flat contribution without multiplying a possibly undefined principal value by zero:

$$t_{h,k}^{\text{flat}}[\omega] := \begin{cases} k(0) \text{ p.v. } \int_{\mathbb{R}^d} \frac{h(\hat{z})}{|z|^d} \omega(\bullet - z) dz, & k(0) \neq 0, \\ 0, & k(0) = 0. \end{cases} \quad (5.25)$$

If  $k(0) \neq 0$ , Eq. (5.15b) forces  $h$  to be odd, so the principal-value distribution in the first line is well-defined.

**Lemma 5.9** (Local-tail and paraproduct reduction). *With the same hypothesis of Theorem 5.7, one has*

$$\begin{aligned} t_{h,k}^{(d)}(\eta)[\omega] &= t_{h,k}^{\text{QL}}(\eta)[\omega] + t_{h,k}^{\text{tr}}(\eta)[\omega] + t_{h,k}^R(\eta)[\omega] \\ &\quad + \text{Op}^{\text{BW}}\left(f_{\geq \ell}^{\text{tail}, \mathbb{R}}[\omega](\eta; x)\right) \eta + R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega) \cdot \left[ \frac{\eta}{\omega} \right], \end{aligned} \quad (5.26)$$

where:

- the quasilinear branch, including the flat contribution, is

$$\begin{aligned} t_{h,k}^{\text{QL}}(\eta)[\omega] &:= t_{h,k}^{\text{flat}}[\omega] \\ &\quad + \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \text{Op}^{\text{BW}}(\tilde{k}(\Delta_z \eta)) \omega_z dz; \end{aligned} \quad (5.27)$$

- the transported branch is

$$t_{h,k}^{\text{tr}}(\eta)[\omega] := \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \text{Op}^{\text{BW}}(\omega_z) [\tilde{k}(\Delta_z \eta)] dz; \quad (5.28)$$

- the localized Bony remainder is

$$t_{h,k}^R(\eta)[\omega] := \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} R^{\text{BW}}[\tilde{k}(\Delta_z \eta), \omega_z] dz; \quad (5.29)$$

- the tail function is

$$f_{\geq \ell}^{\text{tail}, \mathbb{R}}[\omega](\eta; x) := \int_{\mathbb{R}^d} (1 - \chi(z)) \frac{h(\hat{z})}{|z|^{d+1}} k'(\Delta_z \eta(x)) \omega_z(x) dz; \quad (5.30)$$

- the smoothing vector  $R_{[\varrho, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega)$  is as per Notation 5.1.

*Proof.* Let  $r_0$  in Theorem 5.7 be such that  $0 < r_0 < r_*$ , so that  $k(\Delta_z \eta)$  is well defined. Using  $\tilde{k} := k - k(0)$ , we first split

$$\begin{aligned} t_{h,k}^{(d)}(\eta)[\omega] &= t_{h,k}^{\text{flat}}[\omega] \\ &+ \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \tilde{k}(\Delta_z \eta) \omega_z dz \\ &+ \int_{\mathbb{R}^d} (1 - \chi(z)) \frac{h(\hat{z})}{|z|^d} \tilde{k}(\Delta_z \eta) \omega_z dz. \end{aligned} \quad (5.31)$$

We first treat the last term. Set

$$b_h(z) := \frac{h(\hat{z})}{|z|^d}.$$

Since  $h \in C^\infty(\mathbb{S}^{d-1})$ , for every multi-index  $\alpha$ ,

$$|\partial_z^\alpha b_h(z)| \lesssim_{\alpha, h} |z|^{-d-|\alpha|}.$$

Thus  $b_h$  has far-field order  $m = d$ . Moreover,  $\tilde{k}$  vanishes to order at least  $\ell \geq 1$  by Eq. (5.14); hence  $m + \ell = d + \ell > d$ , which means

$$\int_1^\infty r^{d-1-m-\ell} dr = \int_1^\infty r^{-1-\ell} dr < \infty. \quad (5.32)$$

Therefore

$$k_{[-d, \geq \ell]}^{\text{tail}, \mathbb{R}}(\eta; x, z) := (1 - \chi(z)) \frac{h(\hat{z})}{|z|^d} \tilde{k}(\Delta_z \eta) \in K\mathcal{F}_{\text{tail}, \geq \ell}.$$

Applying Proposition 4.37, for every prescribed  $\varrho \geq 0$ ,

$$\int_{\mathbb{R}^d} k_{[-d, \geq \ell]}^{\text{tail}, \mathbb{R}}(\eta; x, z) \omega_z dz = \text{Op}^{\text{BW}}\left(f_{\geq \ell}^{\text{tail}, \mathbb{R}}[\omega](\eta; x)\right) \eta + R_{[\varrho]}^{\text{lin}, \mathbb{R}}(\eta, \omega) \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix}. \quad (5.33)$$

It remains to decompose the localized nonlinear term in Eq. (5.31). For each fixed  $z \neq 0$ , Lemma 4.18 gives

$$\tilde{k}(\Delta_z \eta) \omega_z = \text{Op}^{\text{BW}}(\tilde{k}(\Delta_z \eta)) \omega_z + \text{Op}^{\text{BW}}(\omega_z) [\tilde{k}(\Delta_z \eta)] + R^{\text{BW}}[\tilde{k}(\Delta_z \eta), \omega_z].$$

Multiplying by  $\chi(z) h(\hat{z}) / |z|^d$  and integrating in  $z$  gives, by Eqs. (5.27) to (5.29),

$$\text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \tilde{k}(\Delta_z \eta) \omega_z dz = t_{h,k}^{\text{QL}}(\eta)[\omega] - t_{h,k}^{\text{flat}}[\omega] + t_{h,k}^{\text{tr}}(\eta)[\omega] + t_{h,k}^R(\eta)[\omega].$$

Combining this identity with Eqs. (5.31) and (5.33) proves Eq. (5.26).  $\square$

**Lemma 5.10** (Localized Bony remainder). *For every prescribed  $\varrho \geq 0$ , the localized Bony remainder satisfies*

$$t_{h,k}^R(\eta)[\omega] = r_{[\varrho, \geq \ell]}^{\mathbb{R}}(\eta) \omega, \quad (5.34)$$

*Proof.* Set

$$\mathbf{v}[\eta](x, z) := \nabla \eta(x) \cdot \hat{z}, \quad \mathbf{h}[\eta](x, z) := \hat{z}^\top \mathbf{H} \eta(x) \hat{z}. \quad (5.35)$$

By bilinearity of the Bony remainder,  $t_{h,k}^R(\eta)[\omega]$  is the sum of

$$\begin{aligned} & \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} R^{\text{BW}}[\tilde{k}(\mathbf{v}[\eta]), \omega(x)] dz, \\ & \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} R^{\text{BW}}[\tilde{k}(\Delta_z \eta), \omega_z - \omega(x)] dz, \\ & \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} R^{\text{BW}}[\tilde{k}(\Delta_z \eta) - \tilde{k}(\mathbf{v}[\eta]), \omega(x)] dz. \end{aligned} \quad (5.36)$$

By Eqs. (5.15a) and (5.15b),

$$h(-\vartheta) = \varepsilon h(\vartheta), \quad \tilde{k}(-\mathbf{v}[\eta]) = -\varepsilon \tilde{k}(\mathbf{v}[\eta]).$$

Hence the first integrand in Eq. (5.36) is odd in  $z$ , and its integral over every symmetric annulus vanishes.

For the remaining two terms, the identities

$$\omega_z - \omega(x) = -|z| \int_0^1 \nabla \omega(x - tz) \cdot \hat{z} dt$$

and

$$\Delta_z \eta - \mathbf{v}[\eta] = -|z| \int_0^1 (1-t) \hat{z}^\top \mathbf{H} \eta(x - tz) \hat{z} dt$$

give one power of  $|z|$ . Moreover, by the mean value formula,

$$\tilde{k}(\Delta_z \eta) - \tilde{k}(\mathbf{v}[\eta]) = (\Delta_z \eta - \mathbf{v}[\eta]) \int_0^1 \tilde{k}'(\mathbf{v}[\eta] + t(\Delta_z \eta - \mathbf{v}[\eta])) dt,$$

and therefore, by Eq. (5.13) and Lemma 4.33,

$$\omega \longmapsto \omega_z - \omega(x) \in \mathcal{L}\left(H^{s+1}; K\mathcal{F}_{\text{loc}, \geq 1}^1\right), \quad \tilde{k}(\Delta_z \eta) - \tilde{k}(\mathbf{v}[\eta]) \in K\mathcal{F}_{\text{loc}, \geq \ell}^1.$$

Thus Remark 4.19 and Lemma 4.33, using one additional derivative in the term containing  $\omega_z - \omega$ , place both nonzero operator-valued kernels in

$$K\mathcal{R}_{\text{loc}, \geq \ell}^{\rho, 1-d}.$$

Since  $1 - d > -d$ , Lemma 4.34 yields

$$t_{h,k}^R(\eta)[\omega] = r_{[\rho, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad r_{[\rho, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\rho}.$$

The vanishing order follows from Eq. (5.13), and the operator is real-to-real because the kernels and the Bony remainder are real-to-real.  $\square$

### 5.1.2 Quasilinear contribution

**Lemma 5.11** (Quasilinear contribution). *For every prescribed  $\rho \geq 0$  and every dimension  $d \geq 1$ , the quasilinear branch in Eq. (5.27) satisfies*

$$t_{h,k}^{\text{QL}}(\eta)[\omega] = \text{i Op}^{\text{BW}}\left(a_0^{h,k;\omega}(\nabla \eta, \xi) + a_{[-2, \geq \ell]}^{\mathbb{R};\omega}(\eta; x, \xi)\right)\omega + r_{[\rho, \geq \ell]}^{\text{sk}, \mathbb{R}}(\eta)\omega. \quad (5.37)$$

The objects in this equation have the following properties.

- The symbol  $a_0^{h,k;\omega}$  is defined in Eq. (5.22); it is real-valued and odd in  $\xi$ . In particular, its nonlinear part belongs to  $\Gamma_{\geq \ell}^0$ ;
- $a_{[-2, \geq \ell]}^{\mathbb{R};\omega} \in \Gamma_{\geq \ell}^{-2}$  is real-valued and odd in  $\xi$ ;

- $r_{[\ell, \geq \ell]}^{\text{sk}, \mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\ell}$  is real-to-real and skew-adjoint.

*Proof.*

**Part 1** (Local Taylor decomposition). Remind  $v[\eta](x, z)$  and  $h[\eta](x, z)$  defined in Eq. (5.35). Taylor's formula along the segment  $t \mapsto x - tz$  gives

$$\begin{aligned} \Delta_z \eta &= v[\eta] - \frac{|z|}{2} h[\eta] + \tilde{k}_{[2, \geq 1]}^{\text{loc}, \mathbb{R}}, \\ \tilde{k}_{[2, \geq 1]}^{\text{loc}, \mathbb{R}}(\eta; x, z) &:= -|z| \int_0^1 (1-t) \hat{z}^\top [\mathbf{H}\eta(x-tz) - \mathbf{H}\eta(x)] \hat{z} dt. \end{aligned} \quad (5.38)$$

Equation (5.38) and Lemmas 4.45 and 4.46 imply  $\tilde{k}_{[2, \geq 1]}^{\text{loc}, \mathbb{R}} \in K\mathcal{F}_{\text{loc}, \geq 1}^2$ . Write

$$w[\eta] := \Delta_z \eta - v[\eta] = -\frac{|z|}{2} h[\eta] + \tilde{k}_{[2, \geq 1]}^{\text{loc}, \mathbb{R}}$$

and define

$$\begin{aligned} k_{[2, \geq \ell]}^{\text{loc}, \mathbb{R}}(\eta; x, z) &:= k(\Delta_z \eta(x)) - k(v[\eta](x, z)) + \frac{|z|}{2} k'(v[\eta](x, z)) h[\eta](x, z) \\ &= k'(v[\eta]) \tilde{k}_{[2, \geq 1]}^{\text{loc}, \mathbb{R}} + w[\eta]^2 \int_0^1 (1-t) k''(v[\eta] + tw[\eta]) dt. \end{aligned} \quad (5.39)$$

By Eq. (5.13), the composition  $k'(v[\eta])$  has background vanishing order at least  $\ell-1$ , while  $k''(v[\eta] + tw[\eta])$  has background vanishing order at least  $\max\{\ell-2, 0\}$ . Hence every term in the last line of Eq. (5.39) has local order two and background vanishing order at least  $\ell$ . The analytic composition and product rules in Lemmas 4.33 and 4.45 give

$$k_{[2, \geq \ell]}^{\text{loc}, \mathbb{R}} \in K\mathcal{F}_{\text{loc}, \geq \ell}^2, \quad \chi(z) \frac{h(\hat{z})}{|z|^d} k_{[2, \geq \ell]}^{\text{loc}, \mathbb{R}} \in K\mathcal{F}_{\geq \ell}^{-d+2}. \quad (5.40)$$

Accordingly,

$$t_{h,k}^{\text{QL}} = t_{h,k}^{\text{QL},0} + t_{h,k}^{\text{QL},-1} + t_{h,k}^{\text{QL},\leq -2}, \quad (5.41)$$

where

$$\begin{aligned} t_{h,k}^{\text{QL},0}(\eta)[\omega] &:= t_{h,k}^{\text{flat}}[\omega] \\ &\quad + \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \text{Op}^{\text{BW}}(\tilde{k}(v[\eta])) \omega_z dz, \end{aligned} \quad (5.42a)$$

$$t_{h,k}^{\text{QL},-1}(\eta)[\omega] := -\frac{1}{2} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{d-1}} \text{Op}^{\text{BW}}(k'(v[\eta]) h[\eta]) \omega_z dz, \quad (5.42b)$$

$$t_{h,k}^{\text{QL},\leq -2}(\eta)[\omega] := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \text{Op}^{\text{BW}}(k_{[2, \geq \ell]}^{\text{loc}, \mathbb{R}}) \omega_z dz.$$

We first analyze the leading and subprincipal branches, and then the lower Taylor remainder.

**Part 2** (Analysis of Eq. (5.42a)). For  $d \geq 2$ , we prove

$$t_{h,k}^{\text{QL},0}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(i a_0^{h,k;\omega}(\nabla \eta, \xi) - a_{-1}^{h,k}(\eta; x, \xi) + a_{[-2, \geq \ell]}(\eta; x, \xi)\right) \omega + r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta) \omega, \quad (5.43)$$

where

$$a_{[-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-2}, \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\ell}.$$

We first remove the localization in the homogeneous model and then identify the order 0 and order  $-1$  symbols in Eq. (5.43).

For every  $\alpha \in \mathbb{N}^d$  such that  $|\alpha| \geq \ell$  and  $\kappa_{|\alpha|} \neq 0$ , set

$$c_\alpha(\eta; x) := \kappa_{|\alpha|} \binom{|\alpha|}{\alpha} (\nabla \eta(x))^\alpha, \quad h_\alpha^{(0)}(z) := \frac{h(\hat{z}) \hat{z}^\alpha}{|z|^d}. \quad (5.44)$$

By Eq. (5.15b),  $\kappa_{|\alpha|} \neq 0$  implies  $(-1)^{|\alpha|} = -\varepsilon$ . Hence

$$h(-\vartheta)(-\vartheta)^\alpha = \varepsilon(-1)^{|\alpha|} h(\vartheta) \vartheta^\alpha = -h(\vartheta) \vartheta^\alpha.$$

Thus the angular amplitude of  $h_\alpha^{(0)}$  is odd and has zero spherical mean. In particular, the critical principal-value distribution in Eq. (5.44) satisfies the hypothesis of Item 1.

According to Definition 4.39, define

$$t_{h_\alpha^{(0)}}^{\text{loc}} \omega(x) := \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z}) \hat{z}^\alpha}{|z|^d} \omega(x-z) dz.$$

Expanding the operator  $t_{h,k}^{\text{QL},0}(\eta)$  defined in Eq. (5.42a) by means of the monomial decomposition in Eq. (5.44) gives

$$t_{h,k}^{\text{QL},0}(\eta)[\omega] = t_{h,k}^{\text{flat}}[\omega] + \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \geq \ell \\ \kappa_{|\alpha|} \neq 0}} \text{Op}^{\text{BW}}(c_\alpha(\eta; x)) \left[ t_{h_\alpha^{(0)}}^{\text{loc}} \omega \right]. \quad (5.45)$$

Using the Fourier multiplier  $\dot{m}_{h_\alpha^{(0)}}$  from Eq. (4.46), define

$$g_{h,k}^{\text{QL},0}(\eta)[\omega] := t_{h,k}^{\text{flat}}[\omega] + \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \geq \ell \\ \kappa_{|\alpha|} \neq 0}} \text{Op}^{\text{BW}}(c_\alpha(\eta; x)) \left[ \dot{m}_{h_\alpha^{(0)}}(D) \omega \right]. \quad (5.46)$$

Comparing Eq. (5.45) and Eq. (5.46), and applying Lemma 4.41 and Remark 4.42 term by term, and after summation for  $r_0$  sufficiently small, we obtain

$$t_{h,k}^{\text{QL},0}(\eta)[\omega] = g_{h,k}^{\text{QL},0}(\eta)[\omega] + r_{[\ell, \infty]}^{\mathbb{R}}(\eta) \omega, \quad r_{[\ell, \infty]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^\ell. \quad (5.47)$$

Thus Eq. (5.47) identifies the smoothing remainder in Eq. (5.43); it remains to identify the symbol of  $g_{h,k}^{\text{QL},0}(\eta)$  through order  $-1$ .

Define, for every  $d \geq 1$ , the first quasilinear subprincipal symbol by

$$a_{-1}^{h,k}(\eta; x, \xi) := \frac{1}{2} (\nabla_x \cdot \nabla_\xi) a_0^{h,k;\omega}(\nabla \eta(x), \xi). \quad (5.48)$$

For  $d \geq 2$ , it admits the explicit representation

$$a_{-1}^{h,k}(\eta; x, \xi) = -\frac{\pi}{2} \int_{\mathbb{S}^{d-1}} \delta(\vartheta \cdot \xi) h(\vartheta) k'(\nabla \eta(x) \cdot \vartheta) (\vartheta^\top \mathbf{H} \eta(x) \vartheta) d\sigma(\vartheta). \quad (5.49)$$

Indeed, in the sense of tempered distributions in  $\xi$ ,

$$\partial_{\xi_i} \text{sgn}(\vartheta \cdot \xi) = 2\vartheta_i \delta(\vartheta \cdot \xi).$$

Therefore, differentiating Eq. (5.22) and evaluating at  $p = \nabla \eta(x)$ , we obtain

$$(\nabla_x \cdot \nabla_\xi) a_0^{h,k;\omega}(\nabla \eta(x), \xi) = -\pi \int_{\mathbb{S}^{d-1}} \delta(\vartheta \cdot \xi) h(\vartheta) k'(\nabla \eta(x) \cdot \vartheta) (\vartheta^\top \mathbf{H} \eta(x) \vartheta) d\sigma(\vartheta),$$

which proves Eq. (5.49) by Eq. (5.48). For  $d = 1$ , Eq. (5.22) shows that  $a_0^{h,k;\omega}(p, \cdot)$  is constant on each half-line, and hence  $a_{-1}^{h,k} \equiv 0$ . Applying Proposition 4.24 to Eq. (5.46) and summing the monomial expansion gives, at orders 0 and  $-1$ ,

$$\begin{aligned} a_0^{h,k;\omega}(0, \xi) + \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \geq \ell \\ \kappa_{|\alpha|} \neq 0}} c_\alpha \dot{m}_\alpha^{(0)} &= a_0^{h,k;\omega}(\nabla \eta, \xi), \\ -\frac{1}{2} \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \geq \ell \\ \kappa_{|\alpha|} \neq 0}} \nabla_x c_\alpha \cdot \nabla_\xi \dot{m}_\alpha^{(0)} &= -\frac{1}{2} (\nabla_x \cdot \nabla_\xi) a_0^{h,k;\omega} = -a_{-1}^{h,k}. \end{aligned} \quad (5.50)$$

The first identity in Eq. (5.50), together with  $\mathfrak{m}_{h_\alpha^{(0)}} = \mathfrak{m}_\alpha^{(0)}$ , gives the principal term

$$\mathfrak{i}a_0^{h,k;\omega}(\nabla\eta, \xi)$$

in Eq. (5.43), while the second identity gives its subprincipal term

$$-a_{-1}^{h,k}(\eta; x, \xi).$$

The remaining symbolic contribution is  $a_{[-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-2}$ . Together with Eq. (5.47), this proves Eq. (5.43).

**Part 3** (Analysis of Eq. (5.42b)). For  $d \geq 2$ , we prove

$$t_{h,k}^{\text{QL}, -1}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(a_{-1}^{h,k}(\eta; x, \xi) + a_{[-2, \geq \ell]}(\eta; x, \xi)\right)\omega + r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad (5.51)$$

where

$$a_{[-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-2}, \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\ell}.$$

We first remove the localization and then identify the order  $-1$  symbol in Eq. (5.51).

For every  $\alpha \in \mathbb{N}^d$  such that  $|\alpha| + 1 \geq \ell$  and  $\kappa_{|\alpha|+1} \neq 0$ , and for  $1 \leq j, k \leq d$ , set

$$\begin{aligned} d_{\alpha,j,k}(\eta; x) &:= -\frac{1}{2}(|\alpha| + 1)\kappa_{|\alpha|+1} \binom{|\alpha|}{\alpha} (\nabla\eta(x))^\alpha \partial_{jk}\eta(x), \\ h_{\alpha,j,k}^{(-1)}(z) &:= \frac{h(\hat{z}) \hat{z}^\alpha \hat{z}_j \hat{z}_k}{|z|^{d-1}}. \end{aligned} \quad (5.52)$$

By Eq. (5.15b),  $\kappa_{|\alpha|+1} \neq 0$  implies  $(-1)^{|\alpha|} = \varepsilon$ . Hence

$$h(-\partial)(-\partial)^\alpha(-\partial_j)(-\partial_k) = h(\partial)\partial^\alpha\partial_j\partial_k.$$

Thus the angular amplitude of  $h_{\alpha,j,k}^{(-1)}$  is even and satisfies the hypothesis of Item 2.

According to Definition 4.39, define

$$t_{h_{\alpha,j,k}^{(-1)}}^{\text{loc}}\omega(x) := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z}) \hat{z}^\alpha \hat{z}_j \hat{z}_k}{|z|^{d-1}} \omega(x-z) dz.$$

Expanding the operator  $t_{h,k}^{\text{QL}, -1}(\eta)$  defined in Eq. (5.42b) by means of the monomial decomposition in Eq. (5.52) gives

$$t_{h,k}^{\text{QL}, -1}(\eta)[\omega] = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ \kappa_{|\alpha|+1} \neq 0}} \sum_{j,k=1}^d \text{Op}^{\text{BW}}(d_{\alpha,j,k}(\eta; x)) \left[ t_{h_{\alpha,j,k}^{(-1)}}^{\text{loc}}\omega \right]. \quad (5.53)$$

Using the Fourier multiplier  $\mathfrak{m}_{h_{\alpha,j,k}^{(-1)}}$  defined in Eq. (4.63), define

$$\mathcal{g}_{h,k}^{\text{QL}, -1}(\eta)[\omega] := \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ \kappa_{|\alpha|+1} \neq 0}} \sum_{j,k=1}^d \text{Op}^{\text{BW}}(d_{\alpha,j,k}(\eta; x)) \left[ \mathfrak{m}_{h_{\alpha,j,k}^{(-1)}}(D)\omega \right]. \quad (5.54)$$

Assume  $d \geq 2$ . Comparing Eq. (5.53) and Eq. (5.54), set, for every admissible  $\alpha, j, k$ ,

$$\mathfrak{s}_{\alpha,j,k}(\xi) := \mathfrak{m}_{\chi h_{\alpha,j,k}^{(-1)}}(\xi) - \mathfrak{m}_{h_{\alpha,j,k}^{(-1)}}(\xi).$$

By Lemma 4.41,

$$\mathfrak{s}_{\alpha,j,k} \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M}.$$

Moreover,  $\mathfrak{m}_{\chi h_{\alpha,j,k}^{(-1)}}$  is bounded, whereas  $\mathfrak{m}_{h_{\alpha,j,k}^{(-1)}}$  is homogeneous of degree  $-1$ . Hence

$$|\mathfrak{s}_{\alpha,j,k}(\xi)| \lesssim 1 + |\xi|^{-1}, \quad 0 < |\xi| \leq 1.$$

The high-frequency contribution  $\mathfrak{s}_{\alpha,j,k}^\sharp$  is therefore arbitrarily smoothing. It remains to consider  $\mathfrak{s}_{\alpha,j,k}^b(D)$ . Since

$$\left| \mathfrak{s}_{\alpha,j,k}^b(\xi) \right| \lesssim |\xi|^{-1}, \quad 0 < |\xi| \lesssim 1,$$

the support condition in Eqs. (4.8) and (4.9) and Cauchy–Schwarz give

$$\left| \mathfrak{s}_{\alpha,j,k}^b(\varsigma) \right| \int_{|\theta| \lesssim |\varsigma|} \left| \widehat{d_{\alpha,j,k}(\eta)}(\theta) \right| d\theta \lesssim |\varsigma|^{\frac{d}{2}-1} \|d_{\alpha,j,k}(\eta)\|_{L^2} \lesssim \|d_{\alpha,j,k}(\eta)\|_{L^2},$$

for  $d \geq 2$ . Schur’s lemma therefore gives the  $L^2$ -boundedness of the low-frequency contribution. Since, by Eq. (4.9), it is supported in a fixed low-frequency region, this contribution belongs to  $\mathcal{R}_{\geq \ell}^\ell$  for every  $\ell \geq 0$ . Summation therefore gives

$$t_{h,k}^{\text{QL},-1}(\eta)[\omega] = g_{h,k}^{\text{QL},-1}(\eta)[\omega] + r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^\ell. \quad (5.55)$$

Thus Eq. (5.55) identifies the smoothing remainder in Eq. (5.51); it remains to identify the symbol of  $g_{h,k}^{\text{QL},-1}(\eta)$  through order  $-1$ .

By Item 2, for every admissible  $\alpha, j, k$ , Item 2, for every admissible  $\alpha, j, k$ ,

$$\mathfrak{m}_{h_{\alpha,j,k}}^{(-1)} = \mathfrak{m}_{\alpha,j,k}^{(-1)}, \quad \mathfrak{m}_{\alpha,j,k}^{(-1)}(\xi) := \pi \int_{\mathbb{S}^{d-1}} \delta(\vartheta \cdot \xi) h(\vartheta) \vartheta^\alpha \vartheta_j \vartheta_k d\sigma.$$

Applying Proposition 4.24 to Eq. (5.54) and summing the monomial expansion gives

$$\sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ \kappa_{|\alpha|+1} \neq 0}} \sum_{j,k=1}^d d_{\alpha,j,k} \mathfrak{m}_{\alpha,j,k}^{(-1)} = -\frac{\pi}{2} \int_{\mathbb{S}^{d-1}} \delta(\vartheta \cdot \xi) h(\vartheta) k'(\nabla \eta \cdot \vartheta) (\vartheta^\top \mathbf{H} \eta \vartheta) d\sigma = a_{-1}^{h,k}, \quad (5.56)$$

where the last equality follows from Eq. (5.49). Thus Eq. (5.56) gives the order  $-1$  term

$$a_{-1}^{h,k}(\eta; x, \xi)$$

in Eq. (5.51).

Since  $\mathfrak{m}_{\alpha,j,k}^{(-1)}$  has order  $-1$ , all non-principal terms in the composition have order at most  $-2$ . Hence the remaining symbolic contribution in Eq. (5.51) is

$$a_{[-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-2}.$$

Together with Eq. (5.55), this proves Eq. (5.51).

**Part 4** (Exceptional one-dimensional case). Let  $d = 1$ . By Eq. (5.22),  $a_0^{h,k;\omega}(p, \xi)$  is constant on each of the two half-lines  $\xi > 0$  and  $\xi < 0$ . Hence

$$\partial_\xi^q a_0^{h,k;\omega}(p, \xi) = 0, \quad \xi \neq 0, \quad q \geq 1,$$

and therefore, by Eq. (5.48)  $a_{-1}^{h,k} \equiv 0$ . Applying Proposition 4.24 to Eq. (5.46), all non-principal composition terms are supported in the fixed transition annulus of the high-frequency extension and are therefore smoothing. Together with Eq. (5.47), this gives

$$t_{h,k}^{\text{QL},0}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(ia_0^{h,k;\omega}(\partial_x \eta, \xi)\right)\omega + r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^\ell. \quad (5.57)$$

For the order-minus-one branch, Eq. (5.52) reduces to

$$h_{\alpha,1,1}^{(-1)}(z) = h(\hat{z}) \hat{z}^\alpha.$$

Since  $\kappa_{\alpha+1} \neq 0$ , Eq. (5.15b) gives  $(-1)^\alpha = \varepsilon$ , and hence

$$h(-\vartheta)(-\vartheta)^\alpha = h(\vartheta)\vartheta^\alpha, \quad \vartheta \in \mathbb{S}^0.$$

Thus  $h_{\alpha,1,1}^{(-1)}$  is constant on  $\mathbb{R} \setminus \{0\}$ . By Eq. (4.50), its complete high-frequency multiplier vanishes. Therefore Eq. (5.55) gives

$$t_{h,k}^{\text{QL},-1}(\eta)[\omega] = r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^\ell. \quad (5.58)$$

**Part 5** (Lower Taylor remainder). Since  $-d + 2 > -d$ , the generalized Proposition 4.35 applied to Eq. (5.40) gives

$$t_{h,k}^{\text{QL},\leq-2}(\eta)[\omega] = \text{Op}^{\text{BW}}(a_{[-2,\geq\ell]}(\eta; x, \xi))\omega + r_{[\ell,\geq\ell]}^{\mathbb{R}}(\eta)\omega, \quad (5.59)$$

where

$$a_{[-2,\geq\ell]} \in \Gamma_{\geq\ell}^{-2}, \quad r_{[\ell,\geq\ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq\ell}^{\ell}.$$

Because the kernel in Eq. (5.40) is real, Proposition 4.35 also gives

$$a_{[-2,\geq\ell]}(\eta; x, -\xi) = \overline{a_{[-2,\geq\ell]}(\eta; x, \xi)},$$

and a real-to-real remainder.

**Part 6** (Preliminary synthesis). For  $d \geq 2$ , combining Eqs. (5.41), (5.43), (5.51) and (5.59) and cancelling the two opposite order- $(-1)$  contributions gives a principal symbol  $ia_0^{h,k;\omega}$  and generic terms of order at most  $-2$ . For  $d = 1$ , the same representation follows by combining Eqs. (5.41) and (5.57) to (5.59). Thus, after relabeling the generic symbols and remainders,

$$t_{h,k}^{\text{QL}}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(ia_0^{h,k;\omega}(\nabla\eta, \xi) + \tilde{a}_{[-2,\geq\ell]}(\eta; x, \xi)\right)\omega + \tilde{r}_{[\ell,\geq\ell]}^{\mathbb{R}}(\eta)\omega, \quad (5.60)$$

where

$$\tilde{a}_{[-2,\geq\ell]} \in \Gamma_{\geq\ell}^{-2}, \quad \tilde{r}_{[\ell,\geq\ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq\ell}^{\ell}.$$

Since the kernels and coefficients above are real-valued, the reality statements in Lemma 4.41 and Proposition 4.35 and the termwise symmetry of the Bony–Weyl expansion Eq. (4.14) allow the generic terms in Eq. (5.60) to be chosen so that

$$\tilde{a}_{[-2,\geq\ell]}(\eta; x, -\xi) = \overline{\tilde{a}_{[-2,\geq\ell]}(\eta; x, \xi)}, \quad (5.61)$$

and  $\tilde{r}_{[\ell,\geq\ell]}^{\mathbb{R}}$  is real-to-real.

**Part 7** (Skew-adjoint completion).

**Subpart 7.1** (Exact skew-adjointness). For  $z \neq 0$ , set

$$\begin{aligned} \tau_z \omega(x) &:= \omega(x - z), \\ \mathfrak{h}(z) &:= \chi(z) \frac{h(\hat{z})}{|z|^d}, \\ a_z(\eta; x) &:= \tilde{k}(\Delta_z \eta(x)). \end{aligned}$$

Evaluating Eq. (5.15b) at the origin gives  $(1 + \varepsilon)k(0) = 0$ . Consequently,

$$\tilde{k}(-X) = -\varepsilon \tilde{k}(X).$$

Since  $\chi$  is even,

$$\mathfrak{h}(-z) = \varepsilon \mathfrak{h}(z). \quad (5.62)$$

Moreover,

$$\begin{aligned} \Delta_{-z} \eta(x - z) &= \frac{\eta(x - z) - \eta(x)}{|z|} = -\Delta_z \eta(x), \\ a_{-z}(\eta; x - z) &= -\varepsilon a_z(\eta; x). \end{aligned} \quad (5.63)$$

The definition of Bony–Weyl quantization in Eq. (4.8) is covariant under translations. Since  $a_z$  is real-valued, Lemma 4.13 gives

$$(\text{Op}^{\text{BW}}(a_z) \tau_z)^{\star} = \tau_{-z} \text{Op}^{\text{BW}}(a_z) = \text{Op}^{\text{BW}}(a_z(\eta; \cdot + z)) \tau_{-z}. \quad (5.64)$$

For  $0 < \delta < R < +\infty$ , define the truncated localized branch by

$$t_{h,k;\delta,R}^{\text{QL,loc}}(\eta) := \int_{\delta < |z| < R} \mathfrak{h}(z) \text{Op}^{\text{BW}}(a_z) \tau_z dz.$$

Using Eq. (5.64), then changing  $z$  into  $-z$ , and finally applying Eqs. (5.62) and (5.63), we obtain

$$\begin{aligned}
\left(t_{h,k;\delta,R}^{\text{QL,loc}}(\eta)\right)^* &= \int_{\delta < |z| < R} \mathfrak{h}(z) \text{Op}^{\text{BW}}(a_z(\eta; \cdot + z)) \tau_{-z} dz \\
&= \int_{\delta < |z| < R} \mathfrak{h}(-z) \text{Op}^{\text{BW}}(a_{-z}(\eta; \cdot - z)) \tau_z dz \\
&= - \int_{\delta < |z| < R} \mathfrak{h}(z) \text{Op}^{\text{BW}}(a_z) \tau_z dz \\
&= -t_{h,k;\delta,R}^{\text{QL,loc}}(\eta).
\end{aligned}$$

Passing to the principal-value limit gives the same identity for the localized contribution in Eq. (5.27).

It remains to consider the flat contribution in Eq. (5.25). If  $k(0) = 0$ , it vanishes. If  $k(0) \neq 0$ , then Eq. (5.15b) forces  $\varepsilon = -1$ , and hence  $h$  is odd. Therefore Item 1 identifies the flat operator with a purely imaginary Fourier multiplier, so it is skew-adjoint. We have proved

$$\left(t_{h,k}^{\text{QL}}(\eta)\right)^* = -t_{h,k}^{\text{QL}}(\eta). \quad (5.65)$$

**Subpart 7.2** (Projection of the lower-order terms). We now combine the preliminary representation Eq. (5.60) with the exact skew-adjointness identity Eq. (5.65). Every smoothing remainder entering Eq. (5.60) is a finite sum of terms of the following three types: smoothing Fourier multipliers furnished by Lemma 4.41, Bony–Weyl composition remainders furnished by Proposition 4.24. In each case, adjunction only conjugates the symbol or exchanges the two Fourier variables of the kernel; the symbol bounds, support properties, tame estimates, and background vanishing orders are unchanged. Therefore

$$\tilde{r}_{[\varrho, \geq \ell]}^{\star, \mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\varrho}.$$

Since  $a_0^{h,k;\omega}$  is real-valued, Lemma 4.13 and Eq. (5.65) and Eq. (5.60) yield

$$\begin{aligned}
t_{h,k}^{\text{QL}}(\eta) &= \frac{1}{2} \left( t_{h,k}^{\text{QL}}(\eta) - \left( t_{h,k}^{\text{QL}}(\eta) \right)^* \right) \\
&= \text{Op}^{\text{BW}} \left( i a_0^{h,k;\omega}(\nabla \eta, \xi) + \frac{\tilde{a}_{[-2, \geq \ell]} - \overline{\tilde{a}_{[-2, \geq \ell]}}}{2} \right) + \frac{\tilde{r}_{[\varrho, \geq \ell]}^{\mathbb{R}} - \tilde{r}_{[\varrho, \geq \ell]}^{\star, \mathbb{R}}}{2}.
\end{aligned}$$

Define

$$a_{[-2, \geq \ell]}^{\mathbb{R}; \omega} := \frac{\tilde{a}_{[-2, \geq \ell]} - \overline{\tilde{a}_{[-2, \geq \ell]}}}{2i}, \quad r_{[\varrho, \geq \ell]}^{\text{sk}, \mathbb{R}} := \frac{\tilde{r}_{[\varrho, \geq \ell]}^{\mathbb{R}} - \tilde{r}_{[\varrho, \geq \ell]}^{\star, \mathbb{R}}}{2}. \quad (5.66)$$

Then  $a_{[-2, \geq \ell]}^{\mathbb{R}; \omega}$  is real-valued, belongs to  $\Gamma_{\geq \ell}^{-2}$ , and, by Eq. (5.61), satisfies

$$a_{[-2, \geq \ell]}^{\mathbb{R}; \omega}(\eta; x, -\xi) = -a_{[-2, \geq \ell]}^{\mathbb{R}; \omega}(\eta; x, \xi).$$

Moreover, since  $\tilde{r}_{[\varrho, \geq \ell]}^{\mathbb{R}}$  is real-to-real, so is  $\tilde{r}_{[\varrho, \geq \ell]}^{\star, \mathbb{R}}$ . Therefore  $r_{[\varrho, \geq \ell]}^{\text{sk}, \mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\varrho}$  is real-to-real and skew-adjoint. Substituting Eq. (5.66) into the preceding identity gives Eq. (5.37). □

### 5.1.3 Transported contribution

**Lemma 5.12** (Transported contribution). *For every prescribed  $\varrho \geq 0$ , the transported branch in Eq. (5.28) satisfies*

$$t_{h,k}^{\text{tr}}(\eta)[\omega] = \text{Op}^{\text{BW}} \left( \omega(x) a_1^{h,k;\eta}(\nabla \eta(x), \xi) + a_{[0, \geq \ell]}[\omega](\eta; x, \xi) \right) \eta + r_{[\varrho, \geq \ell]}^{\mathbb{R}}[\omega](\eta) \eta, \quad (5.67)$$

where

- $a_1^{h,k;\eta} \in \Gamma_{\geq \ell-1}^1$  is the real, even symbol defined in Eq. (5.21);

- $a_{[0,\geq\ell]}[\cdot] \in \mathcal{L}(H^{s_0}; \Gamma^0)$  and  $r_{[\ell,\geq\ell]}^{\mathbb{R}}[\cdot] \in \mathcal{L}(H^{s_0}; \mathcal{R}^\ell)$  are generic linear terms as per Notation 5.1. If  $\ell \geq 2$ , their ranges improve to  $\Gamma_{\geq\ell-1}^0$  and  $\mathcal{R}_{\geq\ell-1}^\ell$ , respectively.

Moreover,

$$a_{[0,\geq\ell]}[\omega](\eta; x, -\xi) = \overline{a_{[0,\geq\ell]}[\omega](\eta; x, \xi)}, \quad (5.68)$$

and the remainder is real-to-real.

*Proof.*

**Part 1** (Bony splitting). Applying Lemma 4.20 to  $\tilde{k}(\Delta_z \eta)$  gives

$$\tilde{k}(\Delta_z \eta) = \text{Op}^{\text{BW}}(k'(\Delta_z \eta)) \Delta_z \eta + q_k(\Delta_z \eta) [\Delta_z \eta],$$

where  $q_k(u) := R_{\tilde{k}}(u)$ . Substituting this identity into Eq. (5.28) yields

$$t_{h,k}^{\text{tr}}(\eta)[\omega] = t_{h,k}^{\text{tr,I}}(\eta)[\omega] + t_{h,k}^{\text{tr,II}}(\eta)[\omega], \quad (5.69)$$

where

$$t_{h,k}^{\text{tr,I}}(\eta)[\omega] := \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \text{Op}^{\text{BW}}(\omega_z) \text{Op}^{\text{BW}}(k'(\Delta_z \eta)) \Delta_z \eta dz, \quad (5.70a)$$

$$t_{h,k}^{\text{tr,II}}(\eta)[\omega] := \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \text{Op}^{\text{BW}}(\omega_z) q_k(\Delta_z \eta) [\Delta_z \eta] dz. \quad (5.70b)$$

**Part 2** (Analysis of Eq. (5.70a)). Let  $v[\eta]$  and  $h[\eta]$  be defined by Eq. (5.35). Since  $|v[\eta]| \leq \|\nabla \eta\|_{L^\infty} \lesssim \|\eta\|_{s_0} < r_0$ , the derivatives of  $k$  admit the absolutely convergent multinomial expansions

$$k'(v[\eta]) = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ k^{(|\alpha|+1)}(0) \neq 0}} \frac{k^{(|\alpha|+1)}(0)}{\alpha!} (\nabla \eta(x))^\alpha \hat{z}^\alpha, \quad (5.71a)$$

$$k''(v[\eta]) = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+2 \geq \ell \\ k^{(|\alpha|+2)}(0) \neq 0}} \frac{k^{(|\alpha|+2)}(0)}{\alpha!} (\nabla \eta(x))^\alpha \hat{z}^\alpha, \quad (5.71b)$$

where  $\alpha! := \prod_{j=1}^d \alpha_j!$  and the constraints on the summation indices follow from Eq. (5.14). Set

$$c_z[\omega](\eta) := \text{Op}^{\text{BW}}(\omega_z) \circ \text{Op}^{\text{BW}}(k'(\Delta_z \eta)) - \text{Op}^{\text{BW}}(\omega_z k'(\Delta_z \eta)). \quad (5.72)$$

Taylor's formula, Eq. (5.38), and Lemmas 4.33, 4.45 and 4.46 give

$$k'(\Delta_z \eta) = k'(v[\eta]) - \frac{|z|}{2} k''(v[\eta]) h[\eta] + k_{[2,\geq \max\{1,\ell-1\}]}^{\text{loc,tr},\mathbb{R}} \in K\mathcal{F}_{\text{loc},\geq \max\{1,\ell-1\}}^2, \quad k_{[2,\geq \max\{1,\ell-1\}]}^{\text{loc,tr},\mathbb{R}} \in K\mathcal{F}_{\text{loc},\geq \max\{1,\ell-1\}}^2, \\ \omega_z = \omega(x) - |z| \nabla \omega(x) \cdot \hat{z} + e_{[2]}[\omega], \quad e_{[2]} \in \mathcal{L}(H^{s_0}; K\mathcal{F}_{\text{loc}}^2),$$

whence

$$\omega_z k'(\Delta_z \eta) = \omega(x) k'(v[\eta]) - |z| \left[ \frac{1}{2} \omega(x) k''(v[\eta]) h[\eta] + (\nabla \omega(x) \cdot \hat{z}) k'(v[\eta]) \right] + e_{[2]}^{\text{tr}}[\omega], \quad (5.73)$$

where  $e_{[2]}^{\text{tr}}[\cdot] \in \mathcal{L}(H^{s_0}; K\mathcal{F}_{\text{loc},\geq\ell-1}^2)$  is defined by Eq. (5.73) itself. Accordingly, since  $\Delta_z \eta = \delta_z \eta / |z|$ ,

$$t_{h,k}^{\text{tr,I}}(\eta)[\omega] = t_{h,k}^{\text{tr,1,loc}}(\eta)[\omega] + t_{h,k}^{\text{tr,0,loc}}(\eta)[\omega] + t_{h,k}^{\text{tr,\leq-1,loc}}(\eta)[\omega] + t_{h,k}^{\text{tr,cp}}(\eta)[\omega]. \quad (5.74)$$

Here

$$t_{h,k}^{\text{tr,1,loc}}(\eta)[\omega] := \frac{1}{2} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{d+1}} \text{Op}^{\text{BW}}(\omega(x) k'(v[\eta])) \delta_z^{(2)} \eta dz, \quad (5.75a)$$

$$t_{h,k}^{\text{tr},0,\text{loc}}(\eta)[\omega] := \int_{\mathbb{R}^d} \text{Op}^{\text{BW}}(k^{(0)}[\omega]) \delta_z \eta \, dz, \quad (5.75b)$$

$$t_{h,k}^{\text{tr},\leq -1,\text{loc}}(\eta)[\omega] := \int_{\mathbb{R}^d} \text{Op}^{\text{BW}}(k^{(\leq -1)}[\omega]) \delta_z \eta \, dz, \quad (5.75c)$$

$$t_{h,k}^{\text{tr},\text{cp}}(\eta)[\omega] := \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} c_z[\omega](\eta) [\Delta_z \eta] \, dz, \quad (5.75d)$$

and

$$k^{(0)}[\omega](\eta; x, z) := -\chi(z) \frac{h(\hat{z})}{|z|^d} \left[ \frac{1}{2} \omega(x) k''(\nu[\eta]) h[\eta] + (\nabla \omega(x) \cdot \hat{z}) k'(\nu[\eta]) \right], \quad (5.76a)$$

$$k^{(\leq -1)}[\omega](\eta; x, z) := \chi(z) \frac{h(\hat{z})}{|z|^{d+1}} e_{[2]}^{\text{tr}}[\omega] \in K\mathcal{F}_{\geq \ell-1}^{1-d}. \quad (5.76b)$$

In Eq. (5.75a) we used that  $h(\vartheta) k'(p \cdot \vartheta)$  is even by Eq. (5.16): the change of variables  $z \mapsto -z$  replaces  $\delta_z \eta$  by  $\frac{1}{2} \delta_z^{(2)} \eta$  and turns the principal value into an absolutely convergent integral.

We analyze the terms in Eq. (5.74) in decreasing symbolic order.

**Subpart 2.1** (Analysis of Eq. (5.75a)). For  $|\nu[\eta]|$  sufficiently small we use Eq. (5.71a). For every index in this sum, set

$$q_\alpha[\omega](\eta; x) := \frac{k^{(|\alpha|+1)}(0)}{\alpha!} \omega(x) (\nabla \eta(x))^\alpha, \quad H_\alpha^{(1)}(z) := \frac{h(\hat{z}) \hat{z}^\alpha}{|z|^{d+1}}.$$

By Eq. (5.15b),  $k^{(|\alpha|+1)}(0) \neq 0$  implies  $(-1)^{|\alpha|} = \varepsilon$ ; hence  $H_\alpha^{(1)}$  is even. Define

$$p_\alpha^{1,\text{loc}}[f](x) := \frac{1}{2} \int_{\mathbb{R}^d} \chi(z) H_\alpha^{(1)}(z) \delta_z^{(2)} f(x) \, dz.$$

Then

$$t_{h,k}^{\text{tr},1,\text{loc}}(\eta)[\omega] = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ k^{(|\alpha|+1)}(0) \neq 0}} \text{Op}^{\text{BW}}(q_\alpha[\omega]) \left[ p_\alpha^{1,\text{loc}} \eta \right]. \quad (5.77)$$

Using the complete homogeneous multiplier  $\dot{m}_{H_\alpha^{(1)}}^{(2)}$  from Definition 4.43, define

$$g_{h,k}^{\text{tr},1}(\eta)[\omega] := \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ k^{(|\alpha|+1)}(0) \neq 0}} \text{Op}^{\text{BW}}(q_\alpha[\omega]) \left[ \dot{m}_{H_\alpha^{(1)}}^{(2)}(D) \eta \right]. \quad (5.78)$$

By Eq. (4.67), for every admissible  $\alpha$ ,

$$p_\alpha^{1,\text{loc}} = \dot{m}_{H_\alpha^{(1)}}^{(2)}(D) + c_{H_\alpha^{(1)}} \text{Id} + \mathfrak{s}_\alpha(D), \quad \mathfrak{s}_\alpha \in \bigcap_{M \in \mathbb{N}} \tilde{\Gamma}^{-M}.$$

Therefore, applying Remark 4.42 term by term to Eq. (5.77) gives

$$t_{h,k}^{\text{tr},1,\text{loc}}(\eta)[\omega] = g_{h,k}^{\text{tr},1}(\eta)[\omega] + \text{Op}^{\text{BW}}(a_{[0,\geq \ell]}[\omega]) \eta + r_{[\ell,\geq \ell]}^{\mathbb{R}}[\omega](\eta) \eta. \quad (5.79)$$

Here the constant corrections  $c_{H_\alpha^{(1)}}$  are included in  $a_{[0,\geq \ell]}[\omega]$ , whereas the multipliers  $\mathfrak{s}_\alpha$  are included in  $r_{[\ell,\geq \ell]}^{\mathbb{R}}[\omega](\eta)$ . Summability follows from Eq. (4.53) and the absolute convergence of Eq. (5.71a) for  $|\nu[\eta]| < r_0$ , after shrinking  $r_0$  if necessary. By Eq. (4.68),

$$\dot{m}_{H_\alpha^{(1)}}^{(2)}(\xi) = \dot{m}_\alpha^{(1)}(\xi) := \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\vartheta \cdot \xi| h(\vartheta) \vartheta^\alpha \, d\sigma(\vartheta).$$

Applying Proposition 4.24 to Eq. (5.78) gives

$$\sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ k^{(|\alpha|+1)}(0) \neq 0}} q_\alpha[\omega] \dot{m}_\alpha^{(1)}(\xi) = \omega(x) \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\vartheta \cdot \xi| h(\vartheta) k'(\nabla \eta(x) \cdot \vartheta) \, d\sigma(\vartheta) = \omega(x) a_1^{h,k;\eta}(\nabla \eta(x), \xi).$$

Since the multiplier has order one, all non-principal terms in the composition have order at most zero. Therefore

$$g_{h,k}^{\text{tr},1}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(\omega(x) a_1^{h,k;\eta}(\nabla \eta(x), \xi) + a_{[0,\geq \ell]}[\omega]\right) \eta + r_{[\ell,\geq \ell]}^{\mathbb{R}}[\omega](\eta) \eta. \quad (5.80)$$

**Subpart 2.2** (Analysis of Eq. (5.75b)). By Eq. (5.16),

$$h(\vartheta) k''(\nabla \eta \cdot \vartheta) (\vartheta^\top \mathbf{H} \eta \vartheta) \quad \text{is odd in } \vartheta,$$

while

$$h(\vartheta) k'(\nabla \eta \cdot \vartheta) (\nabla \omega \cdot \vartheta) \quad \text{is odd in } \vartheta.$$

Hence the angular kernel in Eq. (5.76a) is odd, and therefore

$$\int_{\mathbb{S}^{d-1}} k^{(0)}[\omega](\eta; x, r\vartheta) d\sigma(\vartheta) = 0, \quad r > 0. \quad (5.81)$$

For every  $\alpha \in \mathbb{N}^d$  and every  $1 \leq j, l \leq d$  set

$$\begin{aligned} c_{\alpha,j,l}[\omega](\eta; x) &:= -\frac{k^{(|\alpha|+2)}(0)}{2\alpha!} \omega(x) (\nabla \eta(x))^\alpha \partial_{jl} \eta(x), & h_{\alpha,j,l}^{(0)}(z) &:= \frac{h(\hat{z}) \hat{z}^\alpha \hat{z}_j \hat{z}_l}{|z|^d}, \\ c_{\alpha,j}[\omega](\eta; x) &:= -\frac{k^{(|\alpha|+1)}(0)}{\alpha!} \partial_j \omega(x) (\nabla \eta(x))^\alpha, & h_{\alpha,j}^{(0)}(z) &:= \frac{h(\hat{z}) \hat{z}^\alpha \hat{z}_j}{|z|^d}. \end{aligned} \quad (5.82)$$

By Eqs. (5.71a) and (5.71b), the first and second coefficient families in Eq. (5.82) occur only for indices satisfying

$$|\alpha| + 2 \geq \ell, \quad |\alpha| + 1 \geq \ell,$$

respectively. Since  $c_{\alpha,j,l}[\omega](\eta; x)$  has degree  $|\alpha| + 1$  in  $\eta$ , whereas  $c_{\alpha,j}[\omega](\eta; x)$  has degree  $|\alpha|$  in  $\eta$ , both coefficient families vanish to order at least  $\ell - 1$  at  $\eta = 0$ .

Differentiating Eq. (5.15b) at the origin gives  $k^{(m)}(0) \neq 0 \implies (-1)^m = -\varepsilon$ . Hence  $k^{(|\alpha|+2)}(0) \neq 0 \implies (-1)^{|\alpha|} = -\varepsilon$ . Using also Eq. (5.15a), we obtain

$$h(-\vartheta) (-\vartheta)^\alpha (-\vartheta)_j (-\vartheta)_l = \varepsilon (-1)^{|\alpha|} h(\vartheta) \vartheta^\alpha \vartheta_j \vartheta_l = -h(\vartheta) \vartheta^\alpha \vartheta_j \vartheta_l.$$

Similarly,  $k^{(|\alpha|+1)}(0) \neq 0 \implies (-1)^{|\alpha|} = \varepsilon$ , and therefore

$$h(-\vartheta) (-\vartheta)^\alpha (-\vartheta)_j = -\varepsilon (-1)^{|\alpha|} h(\vartheta) \vartheta^\alpha \vartheta_j = -h(\vartheta) \vartheta^\alpha \vartheta_j.$$

Thus the angular amplitudes of  $h_{\alpha,j,l}^{(0)}$  and  $h_{\alpha,j}^{(0)}$  are odd and therefore satisfy the hypothesis of Item 1.

Let  $t_{h_{\alpha,j,l}^{(0)}}^{\text{loc}}$  and  $t_{h_{\alpha,j}^{(0)}}^{\text{loc}}$  be the localized homogeneous models associated with these amplitudes as per Definition 4.39. Splitting  $\delta_z \eta = \eta - \eta_z$  in Eq. (5.75b), the unshifted contribution vanishes by Eq. (5.81), and Eqs. (5.82) and (5.76a) give

$$\begin{aligned} t_{h,k}^{\text{tr},0,\text{loc}}(\eta)[\omega] &= - \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+2 \geq \ell \\ k^{(|\alpha|+2)}(0) \neq 0}} \sum_{j,l=1}^d \text{Op}^{\text{BW}}(c_{\alpha,j,l}[\omega]) \left[ t_{h_{\alpha,j,l}^{(0)}}^{\text{loc}} \eta \right] \\ &\quad - \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha|+1 \geq \ell \\ k^{(|\alpha|+1)}(0) \neq 0}} \sum_{j=1}^d \text{Op}^{\text{BW}}(c_{\alpha,j}[\omega]) \left[ t_{h_{\alpha,j}^{(0)}}^{\text{loc}} \eta \right]. \end{aligned}$$

Applying Lemma 4.41 and Remark 4.42 term by term removes the cutoff, Item 1 identifies each homogeneous multiplier as  $i$  times a real multiplier homogeneous of degree zero, and Proposition 4.24 recasts the resulting compositions as paradifferential operators of order zero. Summability follows from Eq. (4.53) and the absolute convergence of Eq. (5.71), after shrinking  $r_0$  if necessary. Therefore

$$t_{h,k}^{\text{tr},0,\text{loc}}(\eta)[\omega] = \text{Op}^{\text{BW}}(a_{[0,\geq \ell]}[\omega]) \eta + r_{[\varrho,\geq \ell]}^{\mathbb{R}}[\omega](\eta) \eta. \quad (5.83)$$

**Subpart 2.3** (Analysis of Eq. (5.75c)). By Eq. (5.76b), the kernel  $k^{(\leq -1)}[\cdot]$  is linear in  $\omega$ , has local order  $1 - d > -d$  and background vanishing order at least  $\ell - 1$ . Splitting  $\delta_z \eta = \eta - \eta_z$  in Eq. (5.75c), Remark 4.28 applies to the unshifted contribution and Proposition 4.35 to the translated one, the latter producing a symbol of order  $-1$ , which we absorb in the generic order-zero class. Hence

$$t_{h,k}^{\text{tr},\leq -1,\text{loc}}(\eta)[\omega] = \text{Op}^{\text{BW}}(a_{[0,\geq \ell]}[\omega]) \eta + r_{[\varrho,\geq \ell]}^{\mathbb{R}}[\omega](\eta) \eta. \quad (5.84)$$

**Subpart 2.4** (Analysis of Eq. (5.75d)). Set

$$c_0[\omega](\eta) := \text{Op}^{\text{BW}}(\omega) \circ \text{Op}^{\text{BW}}(k'(\nu[\eta])) - \text{Op}^{\text{BW}}(\omega k'(\nu[\eta])). \quad (5.85)$$

By Eq. (5.16) the map  $z \mapsto h(\hat{z}) c_0[\omega](\eta) [\nu[\eta]]$  is odd, hence

$$\text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} c_0[\omega](\eta) [\nu[\eta]] dz = 0,$$

and Eq. (5.75d) may be rewritten as

$$t_{h,k}^{\text{tr,cp}}(\eta)[\omega] = \text{p.v.} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^d} \{c_z[\omega](\eta) [\Delta_z \eta] - c_0[\omega](\eta) [\nu[\eta]]\} dz. \quad (5.86)$$

Since  $(a, b) \mapsto \text{Op}^{\text{BW}}(a) \circ \text{Op}^{\text{BW}}(b) - \text{Op}^{\text{BW}}(ab)$  is bilinear, Eqs. (5.72) and (5.85) give that the integrand in braces equals

$$\begin{aligned} & (\text{Op}^{\text{BW}}(\omega_z - \omega) \circ \text{Op}^{\text{BW}}(k'(\Delta_z \eta)) - \text{Op}^{\text{BW}}((\omega_z - \omega) k'(\Delta_z \eta))) \Delta_z \eta \\ & + (\text{Op}^{\text{BW}}(\omega) \circ \text{Op}^{\text{BW}}(k'(\Delta_z \eta) - k'(\nu[\eta])) - \text{Op}^{\text{BW}}(\omega (k'(\Delta_z \eta) - k'(\nu[\eta])))) \Delta_z \eta \\ & + c_0[\omega](\eta) [\Delta_z \eta - \nu[\eta]]. \end{aligned} \quad (5.87)$$

Moreover,

$$\begin{aligned} \omega_z - \omega &= -|z| \int_0^1 \nabla \omega(x - tz) \cdot \hat{z} dt, \\ \Delta_z \eta - \nu[\eta] &= -|z| \int_0^1 (1-t) \hat{z}^\top \mathbf{H} \eta(x - tz) \hat{z} dt, \\ k'(\Delta_z \eta) - k'(\nu[\eta]) &= (\Delta_z \eta - \nu[\eta]) \int_0^1 k''(\nu[\eta] + t(\Delta_z \eta - \nu[\eta])) dt, \end{aligned} \quad (5.88)$$

so that every summand of Eq. (5.87) carries one factor  $|z|$ . Writing

$$\Delta_z \eta = \int_0^1 \nabla \eta(x - tz) \cdot \hat{z} dt$$

to reserve  $\eta$  as the argument, Lemma 4.33 turns Eq. (5.87) into a kernel-smoothing family of local order  $1 - d > -d$ , linear in  $\omega$  and of background vanishing order at least  $\ell - 1$ . Applying Lemma 4.34 to Eq. (5.86) we conclude that

$$t_{h,k}^{\text{tr,cp}}(\eta)[\omega] = r_{[\varrho, \geq \ell]}^{\mathbb{R}}[\omega](\eta) \eta. \quad (5.89)$$

Combining Eqs. (5.74), (5.83), (5.84) and (5.89) gives

$$t_{h,k}^{\text{tr,I}}(\eta)[\omega] = t_{h,k}^{\text{tr,I,loc}}(\eta)[\omega] + \text{Op}^{\text{BW}}(a_{[0, \geq \ell]}[\omega]) \eta + r_{[\varrho, \geq \ell]}^{\mathbb{R}}[\omega](\eta) \eta. \quad (5.90)$$

**Part 3** (Analysis of Eq. (5.70b)). Applying Lemma 4.20 to  $F = \tilde{k}$ , with smoothing gain  $\varrho + 2$ , gives

$$q_k = R_{\tilde{k}} \in \mathcal{R}_{\geq 1}^{\varrho+2}.$$

Moreover,

$$q_k(u)[u] = \tilde{k}(u) - \text{Op}^{\text{BW}}(k'(u)) u. \quad (5.91)$$

By Eq. (5.13) and Remark 4.8, both terms on the right-hand side of Eq. (5.91) vanish to order at least  $\ell$  in  $u$ . Hence  $u \mapsto q_k(u)[u]$  vanishes to order at least  $\ell$ , and its differential vanishes to order at least  $\ell - 1$ .

By Eq. (5.15b),

$$\tilde{k}(-u) = -\varepsilon \tilde{k}(u), \quad k'(-u) = \varepsilon k'(u).$$

Using Eq. (5.91), we therefore get

$$q_k(-u)[-u] = -\varepsilon q_k(u)[u].$$

Hence the integrand obtained from Eq. (5.70b) by replacing  $(\omega_z, \Delta_z \eta)$  with  $(\omega(x), v[\eta])$  is odd. Subtracting it gives

$$\begin{aligned} & \text{Op}^{\text{BW}}(\omega_z) q_k(\Delta_z \eta) [\Delta_z \eta] - \text{Op}^{\text{BW}}(\omega(x)) q_k(v[\eta]) [v[\eta]] \\ &= \text{Op}^{\text{BW}}(\omega_z - \omega(x)) q_k(\Delta_z \eta) [\Delta_z \eta] \\ & \quad + \text{Op}^{\text{BW}}(\omega(x)) q_k(\Delta_z \eta) [\Delta_z \eta - v[\eta]] \\ & \quad + \text{Op}^{\text{BW}}(\omega(x)) (q_k(\Delta_z \eta) - q_k(v[\eta])) [v[\eta]]. \end{aligned} \quad (5.92)$$

By Eq. (5.88),

$$\begin{aligned} \omega_z - \omega(x) &= -|z| \int_0^1 \nabla \omega(x - tz) \cdot \hat{z} dt, \\ \Delta_z \eta - v[\eta] &= -|z| \int_0^1 (1-t) \hat{z}^\top \mathbf{H} \eta(x - tz) \hat{z} dt, \\ q_k(\Delta_z \eta) - q_k(v[\eta]) &= -|z| \int_0^1 \int_0^1 (1-s) dq_k(v[\eta] + t(\Delta_z \eta - v[\eta])) [\hat{z}^\top \mathbf{H} \eta(x - sz) \hat{z}] ds dt. \end{aligned}$$

Here  $dq_k$  is well defined since the remainder map  $q_k = R_{\tilde{k}}$  inherits the analyticity of  $k$ . Thus each term in Eq. (5.92) gains one power of  $|z|$ . We may therefore apply Remark 4.19 and Lemmas 4.33 and 4.34 to Eq. (5.70b) and obtain

$$t_{h,k}^{\text{tr},\text{II}}(\eta) [\omega] = r_{[\ell, \geq \ell]}^{\mathbb{R}} [\omega] (\eta) \eta. \quad (5.93)$$

Combining Eqs. (5.69), (5.90) and (5.93) gives

$$t_{h,k}^{\text{tr}}(\eta) [\omega] = t_{h,k}^{\text{tr},1,\text{loc}}(\eta) [\omega] + \text{Op}^{\text{BW}}(a_{[0, \geq \ell]} [\omega]) \eta + r_{[\ell, \geq \ell]}^{\mathbb{R}} [\omega] (\eta) \eta. \quad (5.94)$$

**Part 4** (Proof of Eq. (5.67)). Combining Eqs. (5.79), (5.80) and (5.94) proves Eq. (5.67), with

$$a_{[0, \geq \ell]} [\cdot] \in \mathcal{L}(H^{s_0}; \Gamma^0), \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} [\cdot] \in \mathcal{L}(H^{s_0}; \mathcal{R}^\ell).$$

If  $\ell \geq 2$ , the corresponding ranges improve to  $\Gamma_{\geq \ell-1}^0$  and  $\mathcal{R}_{\geq \ell-1}^\ell$ , respectively. The contributions to  $a_{[0, \geq \ell]} [\omega]$  satisfy Eq. (5.68) term by term: the lower-order kernel contribution by Eq. (4.22), the homogeneous endpoint contribution by Item 1 and Lemma 4.41, and the compensated principal contribution by Lemma 4.44. The Bony–Weyl composition formula preserves the corresponding real-to-real symmetry. The smoothing remainder is real-to-real by the same results and Lemma 4.34. □

### 5.1.4 Proof of Theorem 5.7

For smooth inputs, combining Eqs. (5.26), (5.34), (5.37) and (5.67) gives

$$\begin{aligned} t_{h,k}^{(\text{d})}(\eta) [\omega] &= \text{Op}^{\text{BW}} \left( \omega(x) a_1^{h,k;\eta}(\nabla \eta, \xi) + a_{[0, \geq \ell]} [\omega] \right) \eta \\ & \quad + \text{Op}^{\text{BW}} \left( i \left( a_0^{h,k;\omega}(\nabla \eta, \xi) + a_{[-2, \geq \ell]}^{\mathbb{R};\omega} \right) \right) \omega \\ & \quad + r_{[\ell, \geq \ell]}^{\mathbb{R}} [\omega] (\eta) \eta + r_{[\ell, \geq \ell]}^{\mathbb{R}} (\eta) \omega. \end{aligned} \quad (5.95)$$

The principal symbols satisfy

$$\begin{aligned} -\partial_p a_0^{h,k;\omega}(p, \xi) [\xi] &= \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} \text{sgn}(\vartheta \cdot \xi) (\vartheta \cdot \xi) h(\vartheta) k'(p \cdot \vartheta) d\sigma(\vartheta) \\ &= \frac{\pi}{2} \int_{\mathbb{S}^{d-1}} |\vartheta \cdot \xi| h(\vartheta) k'(p \cdot \vartheta) d\sigma(\vartheta) \\ &= a_1^{h,k;\eta}(p, \xi), \end{aligned}$$

which proves Eq. (5.23). The symbol and remainder classes, their vanishing orders, and their real-to-real properties follow directly from Lemmas 5.9 to 5.12. In particular, Lemma 5.11 gives the purely imaginary and odd complete quasilinear symbol in Eq. (5.37). Absorbing the tail contribution into the corresponding generic order-zero and smoothing terms, and relabeling the lower-order terms according to Notation 5.1, Eq. (5.95) becomes Eq. (5.20). □

## 5.2 Parilinearization of singular integral operators

In this subsection, we apply Theorem 5.7 to the two singular integral operators  $\mathcal{B}_0$  and  $q$  introduced in Eqs. (2.3a) and (2.3b). For the convenience of the reader, we recall here their definitions:

$$\mathcal{B}_0(\eta)[\omega](x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{z}{\left(|z|^2 + (\delta_z \eta(x))^2\right)^{\frac{3}{2}}} \omega(x-z) dz, \quad (5.96a)$$

$$q(\eta)[\omega](x) := \frac{1}{4\pi} \text{p.v.} \int_{\mathbb{R}^2} \frac{\delta_z \eta(x)}{\left(|z|^2 + (\delta_z \eta(x))^2\right)^{\frac{3}{2}}} \omega(x-z) dz. \quad (5.96b)$$

First of all, let us explain how we can rewrite these operators in a form adapted to the implementation of Theorem 5.7. This is given by the following lemma.

**Lemma 5.13** (Admissible kernels for  $\mathcal{B}_0$  and  $q$ ). *For  $\vartheta \in \mathbb{S}^1$  and  $X \in \mathbb{R}$ , set*

$$\begin{aligned} H^{\mathcal{B}_0}(\vartheta) &:= \frac{\vartheta}{4\pi}, \quad k^{\mathcal{B}_0}(X) := \frac{1}{(1+X^2)^{\frac{3}{2}}}, \\ h^q(\vartheta) &:= \frac{1}{4\pi}, \quad k^q(X) := \frac{X}{(1+X^2)^{\frac{3}{2}}} = X k^{\mathcal{B}_0}(X). \end{aligned} \quad (5.97)$$

Writing  $H^{\mathcal{B}_0} = \begin{bmatrix} h_1^{\mathcal{B}_0} \\ h_2^{\mathcal{B}_0} \end{bmatrix}$ , the pairs  $(h_j^{\mathcal{B}_0}, k^{\mathcal{B}_0})$ ,  $j \in \{1, 2\}$ , are admissible of vanishing order  $\ell = 2$ , whereas  $(h^q, k^q)$  is admissible of vanishing order  $\ell = 1$ . Moreover, recalling Eq. (5.17), one has

$$(\mathcal{B}_0(\eta)[\omega])_j = t_{h_j^{\mathcal{B}_0}, k^{\mathcal{B}_0}}^{(2)}(\eta)[\omega], \quad q(\eta)[\omega] = t_{h^q, k^q}^{(2)}(\eta)[\omega]. \quad (5.98)$$

*Proof.*

**Part 1** (Parity and vanishing order). By construction Eq. (5.97), the vector amplitude  $H^{\mathcal{B}_0}$  is odd,  $k^{\mathcal{B}_0}$  is even, and

$$k^{\mathcal{B}_0}(X) - k^{\mathcal{B}_0}(0) = O(X^2).$$

Thus each component pair is admissible, in the sense of Definition 5.4, with  $\varepsilon = -1$  and  $\ell = 2$ . The amplitude  $h^q$  is even,  $k^q$  is odd, and

$$k^q(0) = 0, \quad k^q(X) = O(X).$$

Hence the second pair is admissible, in the sense of Definition 5.4, with  $\varepsilon = 1$  and  $\ell = 1$ .

**Part 2** (Exact operator identification). Since  $\delta_z \eta = |z| \Delta_z \eta$ , the definitions in Eqs. (5.96a) and (5.96b) become

$$\begin{aligned} \mathcal{B}_0(\eta)[\omega](x) &= \text{p.v.} \int_{\mathbb{R}^2} \frac{H^{\mathcal{B}_0}(\hat{z})}{|z|^2} k^{\mathcal{B}_0}(\Delta_z \eta(x)) \omega(x-z) dz, \\ q(\eta)[\omega](x) &= \text{p.v.} \int_{\mathbb{R}^2} \frac{h^q(\hat{z})}{|z|^2} k^q(\Delta_z \eta(x)) \omega(x-z) dz. \end{aligned}$$

This is precisely Eq. (5.98), with the first identity understood componentwise. □

With this in hand, we can now apply Theorem 5.7 to obtain the parilinearization of the operators  $\mathcal{B}_0$  and  $q$ . The final results are stated in the following two propositions. Their complete proofs are given in the next subsubsections.

**Proposition 5.14** (Parilinearization of  $\mathcal{B}_0(\eta)$ ). *There exists  $\varrho_0 > 0$  such that, for every  $\varrho \geq \varrho_0$ , there exist  $s_0 > 2$ ,  $\mu \geq 0$ , and  $r_0 > 0$  such that, for every  $s \geq s_0$  and every real*

$$\eta \in H^{s+1+\mu}(\mathbb{R}^2), \quad \omega \in H^{s+\mu}(\mathbb{R}^2), \quad \|\eta\|_{H^{s_0+1+\mu}} < r_0,$$

the following identity holds in  $H^s(\mathbb{R}^2; \mathbb{R}^2)$ :

$$\mathcal{B}_0(\eta)[\omega] = \text{Op}^{\text{BW}} \left( A_1^{\mathcal{B}_0}(\eta, \omega; x, \xi) + \left[ A_{[0, \geq 2]}[\omega](\eta; x, \xi) \quad iA_{[-2, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}(\eta; x, \xi) \right] \right) \begin{bmatrix} \eta \\ \omega \end{bmatrix} + \mathbf{R}_{[\varrho, \geq 2]}^{\mathcal{B}_0, \text{lin}, \mathbb{R}}(\eta, \omega) \begin{bmatrix} \eta \\ \omega \end{bmatrix}. \quad (5.99)$$

The objects in Eq. (5.99) are as follows.

- The principal matrix symbol is

$$A_1^{\mathcal{B}_0}(\eta, \omega; x, \xi) := \begin{bmatrix} \omega(x) A_1^{\mathcal{B}_0, \eta}(\eta; x, \xi) & iA_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) \end{bmatrix}. \quad (5.100)$$

- The common potential is the real symbol  $\phi_1^{\mathcal{B}_0} \in \Gamma_{\geq 0}^1$  defined by

$$\phi_1^{\mathcal{B}_0}(\eta; x, \xi) := -\frac{\lambda_1(\eta; x, \xi)}{1 + |\nabla \eta(x)|^2}, \quad (5.101)$$

where  $\lambda_1$  is defined in Eq. (5.4),

- The real vector symbol  $A_0^{\mathcal{B}_0, \omega} \in (\Gamma_{\geq 0}^0)^2$ , defined in Eq. (5.8), satisfies

$$A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) = \nabla_\xi \phi_1^{\mathcal{B}_0}(\eta; x, \xi). \quad (5.102)$$

- The real vector symbol  $A_1^{\mathcal{B}_0, \eta} \in (\Gamma_{\geq 1}^1)^2$  is

$$\begin{aligned} A_1^{\mathcal{B}_0, \eta}(\eta; x, \xi) &:= -\frac{\partial \phi_1^{\mathcal{B}_0}}{\partial(\nabla \eta)}(\eta; x, \xi) \\ &= \frac{A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) \otimes \nabla \eta(x) + \nabla \eta(x) \otimes A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi)}{1 + |\nabla \eta(x)|^2} \xi. \end{aligned} \quad (5.103)$$

- The lower-order symbols satisfy

$$A_{[0, \geq 2]}[\cdot] \in \mathcal{L}(H^{s_0}; (\Gamma_{\geq 1}^0)^2), \quad A_{[-2, \geq 2]}^{\mathcal{B}_0, \mathbb{R}} \in (\Gamma_{\geq 2}^{-2})^2. \quad (5.104)$$

The vector symbol  $A_{[-2, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}$  is real-valued componentwise and odd in  $\xi$ , namely

$$A_{[-2, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}(\eta; x, -\xi) = -A_{[-2, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}(\eta; x, \xi).$$

- The smoothing matrix is

$$\mathbf{R}_{[\varrho, \geq 2]}^{\mathcal{B}_0, \text{lin}, \mathbb{R}}(\eta, \omega) := \begin{bmatrix} R_{[\varrho, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}[\omega](\eta) & R_{[\varrho, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}(\eta) \end{bmatrix},$$

where

$$R_{[\varrho, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}[\cdot] \in \mathcal{L}(H^{s_0}; (\mathcal{R}_{\geq 1}^\varrho)^2), \quad R_{[\varrho, \geq 2]}^{\mathcal{B}_0, \mathbb{R}} \in (\mathcal{R}_{\geq 2}^\varrho)^2. \quad (5.105)$$

All the terms in Eq. (5.99) are real-to-real.

**Proposition 5.15** (Paralinearization of  $q(\eta)$ ). *There exists  $\varrho_0 > 0$  such that, for every  $\varrho \geq \varrho_0$ , there exist  $s_0 > 2$ ,  $\mu \geq 0$ , and  $r_0 > 0$  such that, for every  $s \geq s_0$  and every real*

$$\eta \in H^{s+1+\mu}(\mathbb{R}^2), \quad \omega \in H^{s+\mu}(\mathbb{R}^2), \quad \|\eta\|_{H^{s_0+1+\mu}} < r_0,$$

the following identity holds in  $H^s(\mathbb{R}^2)$ :

$$q(\eta)[\omega] = \text{Op}^{\text{BW}} \left( A_1^q(\eta, \omega; x, \xi) + \left[ a_{[0, \geq 1]}[\omega](\eta; x, \xi) \right] \right) \begin{bmatrix} \eta \\ \omega \end{bmatrix} + R_{[\varrho, \geq 1]}^{q, \text{lin}, \mathbb{R}}(\eta, \omega) \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix}. \quad (5.106)$$

The objects in Eq. (5.106) are as follows.

- The principal vector symbol is

$$A_1^q(\eta, \omega; x, \xi) := \begin{bmatrix} \omega(x) a_1^{q,\eta}(\eta; x, \xi) \\ i a_0^{q,\omega}(\eta; x, \xi) \end{bmatrix}.$$

- The real symbol  $a_0^{q,\omega} \in \Gamma_{\geq 1}^0$  is

$$a_0^{q,\omega}(\eta; x, \xi) := \nabla \eta(x) \cdot A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi). \quad (5.107)$$

- The real symbol  $a_1^{q,\eta} \in \Gamma_{\geq 0}^1$  is

$$a_1^{q,\eta}(\eta; x, \xi) := \nabla \eta(x) \cdot A_1^{\mathcal{B}_0, \eta}(\eta; x, \xi) - A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) \cdot \xi. \quad (5.108)$$

- The lower-order symbols satisfy

$$a_{[0, \geq 1]}[\cdot] \in \mathcal{L}(H^{s_0}; \Gamma^0), \quad a_{[-2, \geq 1]}^{q, \mathbb{R}} \in \Gamma_{\geq 1}^{-2}. \quad (5.109)$$

The symbol  $a_{[-2, \geq 1]}^{q, \mathbb{R}}$  is real-valued and odd in  $\xi$ , namely

$$a_{[-2, \geq 1]}^{q, \mathbb{R}}(\eta; x, -\xi) = -a_{[-2, \geq 1]}^{q, \mathbb{R}}(\eta; x, \xi).$$

- The smoothing vector is

$$R_{[\varrho, \geq 1]}^{q, \text{lin}, \mathbb{R}}(\eta, \omega) := \begin{bmatrix} r_{[\varrho, \geq 1]}^{q, \mathbb{R}}[\omega](\eta) \\ r_{[\varrho, \geq 1]}^{q, \mathbb{R}}(\eta) \end{bmatrix},$$

where

$$r_{[\varrho, \geq 1]}^{q, \mathbb{R}}[\cdot] \in \mathcal{L}(H^{s_0}; \mathcal{R}^\varrho), \quad r_{[\varrho, \geq 1]}^{q, \mathbb{R}} \in \mathcal{R}_{\geq 1}^\varrho. \quad (5.110)$$

All the terms in Eq. (5.106) are real-to-real.

### 5.2.1 Identification of the principal symbols

According to Lemma 5.13, we can apply Eqs. (5.21) and (5.22) with  $d = 2$ . This provides Eq. (5.99), with

$$A_1^{\mathcal{B}_0}(\eta, \omega; x, \xi) := \begin{bmatrix} \omega(x) A_1^{\mathcal{B}_0, \eta}(\nabla \eta(x), \xi) & i A_0^{\mathcal{B}_0, \omega}(\nabla \eta(x), \xi) \end{bmatrix}$$

and Eq. (5.106), with

$$A_1^q(\eta, \omega; x, \xi) := \begin{bmatrix} \omega(x) a_1^{q,\eta}(\nabla \eta(x), \xi) \\ i a_0^{q,\omega}(\nabla \eta(x), \xi) \end{bmatrix}$$

where

$$A_0^{\mathcal{B}_0, \omega}(p, \xi) := -\frac{1}{8} \int_{\mathbb{S}^1} \text{sgn}(\vartheta \cdot \xi) \vartheta k^{\mathcal{B}_0}(p \cdot \vartheta) d\sigma(\vartheta), \quad (5.111a)$$

$$A_1^{\mathcal{B}_0, \eta}(p, \xi) := \frac{1}{8} \int_{\mathbb{S}^1} |\vartheta \cdot \xi| \vartheta (k^{\mathcal{B}_0})'(p \cdot \vartheta) d\sigma(\vartheta), \quad (5.111b)$$

$$a_0^{q,\omega}(p, \xi) := -\frac{1}{8} \int_{\mathbb{S}^1} \text{sgn}(\vartheta \cdot \xi) k^q(p \cdot \vartheta) d\sigma(\vartheta), \quad (5.111c)$$

$$a_1^{q,\eta}(p, \xi) := \frac{1}{8} \int_{\mathbb{S}^1} |\vartheta \cdot \xi| (k^q)'(p \cdot \vartheta) d\sigma(\vartheta). \quad (5.111d)$$

Note that the factor  $1/(4\pi)$  in Eq. (5.97) turns  $\pi/2$  into  $1/8$ , which gives exactly Eq. (5.111). What remains is to give the explicit expressions of these integral formulae. We have the following lemma.

**Lemma 5.16** (Potential representation and Cartesian form of the principal symbols). *Let  $k^{\mathcal{B}_0}$  be defined in Eq. (5.97), and let  $A_0^{\mathcal{B}_0, \omega}$  and  $A_1^{\mathcal{B}_0, \eta}$  be the angular symbols defined in Eqs. (5.111a) and (5.111b). For  $p \in \mathbb{R}^2$  and  $\xi \neq 0$ , define*

$$\phi_{\text{int}}^{\mathcal{B}_0}(p, \xi) := -\frac{1}{8} \int_{\mathbb{S}^1} |\vartheta \cdot \xi| k^{\mathcal{B}_0}(p \cdot \vartheta) d\sigma(\vartheta). \quad (5.112)$$

Then

$$\phi_{\text{int}}^{\mathcal{B}_0}(p, \xi) = -\frac{|\xi|}{2} \frac{\sqrt{1 + (p^\perp \cdot \xi / |\xi|)^2}}{1 + |p|^2}. \quad (5.113)$$

Moreover,

$$\begin{aligned} \nabla_\xi \phi_{\text{int}}^{\mathcal{B}_0}(p, \xi) &= A_0^{\mathcal{B}_0, \omega}(p, \xi), \\ -\partial_p \phi_{\text{int}}^{\mathcal{B}_0}(p, \xi) &= A_1^{\mathcal{B}_0, \eta}(p, \xi). \end{aligned}$$

For  $p = \nabla \eta(x)$ , one has

$$\phi_{\text{int}}^{\mathcal{B}_0}(\nabla \eta(x), \xi) = \phi_1^{\mathcal{B}_0}(\eta; x, \xi), \quad (5.115)$$

where  $\phi_1^{\mathcal{B}_0}$  is defined in Eq. (5.101). Consequently,

$$\begin{aligned} A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) &:= A_0^{\mathcal{B}_0, \omega}(\nabla \eta(x), \xi) = \nabla_\xi \phi_{\text{int}}^{\mathcal{B}_0}(\nabla \eta(x), \xi) \\ &= \nabla_\xi \phi_1^{\mathcal{B}_0}(\eta; x, \xi). \end{aligned} \quad (5.116)$$

Furthermore,

$$A_1^{\mathcal{B}_0, \eta}(\eta; x, \xi) := A_1^{\mathcal{B}_0, \eta}(\nabla \eta(x), \xi) = -\frac{\partial \phi_1^{\mathcal{B}_0}}{\partial(\nabla \eta)}(\eta; x, \xi). \quad (5.117)$$

The latter identity is equivalently the Cartesian formula Eq. (5.103).

*Proof.*

**Part 1** (Evaluation of the angular potential). By rotation covariance, it is enough to take  $\xi = |\xi| \begin{bmatrix} 0 \\ 1 \end{bmatrix}$  and

$p = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$ . Writing  $\vartheta := \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$  and using the symmetry of the integrand, we get

$$\int_{\mathbb{S}^1} \frac{|\vartheta \cdot \xi|}{\left(1 + (p \cdot \vartheta)^2\right)^{\frac{3}{2}}} d\sigma(\vartheta) = 2|\xi| \int_0^\pi \frac{\sin \theta}{\left(1 + (p_1 \cos \theta + p_2 \sin \theta)^2\right)^{\frac{3}{2}}} d\theta.$$

The antiderivative

$$-\frac{(1 + p_1^2) \cos \theta + p_1 p_2 \sin \theta}{(1 + p_1^2 + p_2^2) \sqrt{1 + (p_1 \cos \theta + p_2 \sin \theta)^2}}$$

gives

$$\int_0^\pi \frac{\sin \theta}{\left(1 + (p_1 \cos \theta + p_2 \sin \theta)^2\right)^{\frac{3}{2}}} d\theta = \frac{2\sqrt{1 + p_1^2}}{1 + |p|^2}.$$

Since  $p_1^2 = (p^\perp \cdot \xi / |\xi|)^2$  in the chosen frame, substitution in Eq. (5.112) proves Eq. (5.113).

**Part 2** (Differential identification). With  $p = \nabla \eta(x)$ , Eqs. (5.4) and (5.113) give Eq. (5.115). Differentiating Eq. (5.112) under the integral sign yields

$$\nabla_\xi \phi_{\text{int}}^{\mathcal{B}_0} = A_0^{\mathcal{B}_0, \omega}, \quad -\partial_p \phi_{\text{int}}^{\mathcal{B}_0} = A_1^{\mathcal{B}_0, \eta}.$$

Together with Eqs. (5.8) and (5.115), this proves Eqs. (5.116) and (5.117). Finally, differentiating the Cartesian expression Eq. (5.113) gives Eq. (5.103). □

**Lemma 5.17** (Relation between the principal symbols of  $\mathcal{B}_0$  and  $q$ ). *For  $p = \nabla\eta(x)$ , the angular symbols in Eqs. (5.111c) and (5.111d) satisfy*

$$a_0^{q,\omega}(\eta; x, \xi) := a_0^{q,\omega}(\nabla\eta(x), \xi) = \nabla\eta(x) \cdot A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi), \quad (5.118)$$

and

$$a_1^{q,\eta}(\eta; x, \xi) := a_1^{q,\eta}(\nabla\eta(x), \xi) = \nabla\eta(x) \cdot A_1^{\mathcal{B}_0,\eta}(\eta; x, \xi) - A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) \cdot \xi. \quad (5.119)$$

Thus these are precisely the symbols in Eqs. (5.107) and (5.108).

*Proof.*

**Part 1** (Order-zero symbol). The identities in Eq. (5.97) give

$$k^q(X) = X k^{\mathcal{B}_0}(X). \quad (5.120)$$

Hence Eqs. (5.111a) and (5.111c) imply

$$\begin{aligned} a_0^{q,\omega}(p, \xi) &= -\frac{1}{8} \int_{\mathbb{S}^1} \operatorname{sgn}(\vartheta \cdot \xi) (p \cdot \vartheta) k^{\mathcal{B}_0}(p \cdot \vartheta) d\sigma(\vartheta) \\ &= p \cdot A_0^{\mathcal{B}_0,\omega}(p, \xi). \end{aligned}$$

Taking  $p = \nabla\eta(x)$  and using Eq. (5.116) proves Eq. (5.118).

**Part 2** (Order-one symbol). Differentiating Eq. (5.120),

$$(k^q)'(X) = k^{\mathcal{B}_0}(X) + X (k^{\mathcal{B}_0})'(X).$$

Therefore Eq. (5.111d) becomes

$$\begin{aligned} a_1^{q,\eta}(p, \xi) &= \frac{1}{8} \int_{\mathbb{S}^1} |\vartheta \cdot \xi| k^{\mathcal{B}_0}(p \cdot \vartheta) d\sigma(\vartheta) + p \cdot A_1^{\mathcal{B}_0,\eta}(p, \xi) \\ &= -\phi_{\text{int}}^{\mathcal{B}_0}(p, \xi) + p \cdot A_1^{\mathcal{B}_0,\eta}(p, \xi). \end{aligned}$$

Since  $\phi_{\text{int}}^{\mathcal{B}_0}$  is one-homogeneous in  $\xi$ , Eq. (5.116) and Euler's identity give  $\phi_{\text{int}}^{\mathcal{B}_0} = A_0^{\mathcal{B}_0,\omega} \cdot \xi$ . This proves Eq. (5.119).  $\square$

**Corollary 5.18.** *One has*

$$\phi_1^{\mathcal{B}_0}(\eta; x, \xi) = \xi \cdot A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) = -\frac{\lambda_1(\eta; x, \xi)}{1 + |\nabla\eta(x)|^2}, \quad (5.121)$$

and

$$-(\nabla\eta(x) \cdot \xi) A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) + \left(1 + |\nabla\eta(x)|^2\right) A_1^{\mathcal{B}_0,\eta}(\eta; x, \xi) = -\lambda_1(\eta; x, \xi) \frac{\nabla\eta(x)}{1 + |\nabla\eta(x)|^2}. \quad (5.122)$$

*Proof.* Euler's identity applied to the one-homogeneous symbol  $\phi_1^{\mathcal{B}_0}$ , together with Eqs. (5.101) and (5.102), gives Eq. (5.121). Multiplying Eq. (5.103) by  $1 + |\nabla\eta|^2$  gives

$$(1 + |\nabla\eta|^2) A_1^{\mathcal{B}_0,\eta} = (\nabla\eta \cdot \xi) A_0^{\mathcal{B}_0,\omega} + (A_0^{\mathcal{B}_0,\omega} \cdot \xi) \nabla\eta.$$

Now Eq. (5.121) yields Eq. (5.122).  $\square$

### 5.2.2 Synthesis of the two parilinearizations

*Proof of Propositions 5.14 and 5.15.*

**Part 1** (Application to  $\mathcal{B}_0$ ). Apply Theorem 5.7 componentwise to the two admissible pairs of vanishing order  $\ell = 2$  in Lemma 5.13. The angular symbols produced by the theorem are Eqs. (5.111a) and (5.111b); by Lemma 5.16, they are respectively  $A_0^{\mathcal{B}_0, \omega}$  and  $A_1^{\mathcal{B}_0, \eta}$ . Thus the two principal columns are exactly those of Eq. (5.100). Moreover, Eq. (5.24) gives the second lower symbol componentwise in the form  $iA_{[-2, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}$ , where  $A_{[-2, \geq 2]}^{\mathcal{B}_0, \mathbb{R}}$  is real-valued componentwise and odd in  $\xi$ . Since the bracketed terms are linear in  $\omega$ , their vanishing order is  $\ell = 2$  by Remark 4.31. This gives Eqs. (5.104) and (5.105), proving Eq. (5.99).

**Part 2** (Application to  $q$ ). Apply the same theorem to the admissible pair of vanishing order  $\ell = 1$  in Lemma 5.13. The leading angular symbols are Eqs. (5.111c) and (5.111d); Lemma 5.17 identifies them with Eqs. (5.107) and (5.108). Moreover, Eq. (5.24) gives the second lower symbol in the form  $ia_{[-2, \geq 1]}^{q, \mathbb{R}}$ , where  $a_{[-2, \geq 1]}^{q, \mathbb{R}}$  is real-valued and odd in  $\xi$ . The transposed terms vanish to order  $\ell - 1 = 0$  in  $\eta$  and are linear in  $\omega$ ; hence their total vanishing order is  $\ell = 1$  by Remark 4.31. The quasilinear terms have the same vanishing order. This gives Eqs. (5.109) and (5.110) and proves Eq. (5.106).  $\square$

## 5.3 Parilinearization of evolution operators

Based on the parilinearization of the elementary singular integral operators  $\mathcal{B}_0$  and  $q$  carried out in the previous subsection, we provide here the parilinearization of each component of the evolution equations Eq. (2.9). We analyse each brick separately to finally gather, in the next subsection, the obtained results, thus providing the complete parilinearization of the generalized water-waves system.

### 5.3.1 Parilinearization of $\nu^*$

Let  $\nu^*$  be defined as in Eq. (2.14), which is

$$\nu^*[\Omega](\eta) := \begin{bmatrix} V^*[\Omega](\eta) \\ \nu_3^*[\Omega](\eta) \end{bmatrix} := \begin{bmatrix} (\mathcal{B}_0(\eta) [\nabla \eta \cdot \Omega] - q(\eta) \Omega)^\perp \\ -\mathcal{B}_0(\eta)^\perp \cdot \Omega \end{bmatrix}. \quad (5.123)$$

Then, we have the following

**Proposition 5.19** (Parilinearization of  $\Omega \mapsto \nu^*[\Omega](\eta)$ ). *There exists  $\rho_0 > 0$  such that for any  $\rho \geq \rho_0$ , there exists  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$ ,  $\Omega \in (B_{s_0, \mathbb{R}}(I; r_0))^2$  we have the following:*

1. For  $V^*$  defined as in Eq. (5.123) the following parilinear decomposition holds true

$$V^*[\Omega](\eta; x) = \text{Op}^{\text{BW}} \left( \mathfrak{A}_1^{V^*}(\eta, \Omega; x, \xi) + [A_{[0, \geq 1]}[\Omega](\eta; x, \xi) \quad A_{[-2, \geq 1]}(\eta; x, \xi)] \begin{bmatrix} \eta \\ \Omega \end{bmatrix} + \mathfrak{R}_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix} \right), \quad (5.124)$$

where

$$\mathfrak{A}_1^{V^*}(\eta, \Omega; x, \xi) := \begin{bmatrix} A_1^{V^*, \eta}(\eta; x, \xi) \Omega & (iA_0^{V^*, \Omega} + A_{-1}^{V^*, \Omega})(\eta; x, \xi) \end{bmatrix}, \quad (5.125)$$

with

- $A_1^{V^*, \eta} \in (\Gamma_{\geq 0}^1)^{2 \times 2}$  and is explicitly defined as

$$A_1^{V^*, \eta}(\eta; x, \xi) := -\xi \otimes A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) + \nabla \eta \otimes A_1^{\mathcal{B}_0, \eta; \perp}(\eta; x, \xi), \quad (5.126)$$

where  $A_0^{\mathcal{B}_0, \omega}$  is defined in Eq. (5.8) and  $A_1^{\mathcal{B}_0, \eta}$  is defined in Eq. (5.103);

- $A_0^{V^*, \Omega} \in (\Gamma_{\geq 1}^0)^{2 \times 2}$  is the real-valued symbol explicitly defined as

$$A_0^{V^*, \Omega}(\eta; x, \xi) := \nabla \eta \otimes A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi), \quad (5.127)$$

where  $A_0^{\mathcal{B}_0, \omega}$  is defined in Eq. (5.8);

- $A_{-1}^{V^*, \Omega} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}$  and is explicitly defined as

$$A_{-1}^{V^*, \Omega}(\eta; x, \xi) = \frac{J}{2} J_{\xi} A_0^{\mathcal{B}_0, \omega} \mathbf{H}_x \eta, \quad (5.128)$$

where  $A_0^{\mathcal{B}_0, \omega}$  is defined in Eq. (5.8);

2. For  $v_3^*$  defined as in Eq. (5.123) the following parilinear decomposition holds true

$$v_3^*[\Omega](\eta) = \text{Op}^{\text{BW}} \left( a_1^{v_3^*, \Omega}(\eta, \Omega; x, \xi) + \begin{bmatrix} a_{[0, \geq 1]}[\Omega](\eta; x, \xi) \\ A_{[-2, \geq 1]}(\eta; x, \xi) \end{bmatrix} \right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix} + \mathfrak{r}_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix}, \quad (5.129)$$

where

$$a_1^{v_3^*, \Omega}(\eta, \Omega; x, \xi) := \begin{bmatrix} a_1^{v_3^*, \eta}[\Omega](\eta; x, \xi) \\ iA_0^{v_3^*, \Omega}(\eta; x, \xi) \end{bmatrix} := \begin{bmatrix} -A_1^{\mathcal{B}_0, \eta; \perp}(\eta; x, \xi) \cdot \Omega \\ -iA_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \end{bmatrix}.$$

Here  $a_1^{v_3^*, \eta}[\Omega] \in \Gamma_{\geq 1}^1$  is linear in  $\Omega$ , while  $A_0^{v_3^*, \Omega} = -A_0^{\mathcal{B}_0, \omega; \perp} \in (\Gamma_{\geq 0}^0)^2$ . Equivalently, the  $\Omega$ -slot in Eq. (5.129) is the row symbol  $-i(A_0^{\mathcal{B}_0, \omega; \perp})^\top$ .

*Proof.* We prove separately the horizontal and vertical components of  $v^* = \begin{bmatrix} V^* \\ v_3^* \end{bmatrix}$ . Throughout the proof, all generic symbols and remainders are understood in the sense of Notation 5.1. We also freely enlarge  $s_0$  and decrease  $r_0$ , so that all products below are admissible in the classes stated in Propositions 5.14 and 5.15.

**Part 1** (Parilinearization of  $\Omega \mapsto V^*[\Omega](\eta)$ ). Set

$$\omega_\eta := \nabla \eta \cdot \Omega.$$

Since  $\omega_\eta$  is obtained from  $(\eta, \Omega)$  by a Bony product, Lemma 4.18 and Remark 4.19 and Lemma 4.25 (iii)–(iv) allow us to substitute  $\omega = \omega_\eta$  in the parilinearization of  $\mathcal{B}_0$ . Thus, by Proposition 5.14,

$$\begin{aligned} \mathcal{B}_0(\eta)[\nabla \eta \cdot \Omega] &= \text{Op}^{\text{BW}} \left( iA_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) + A_{[-2, \geq 1]}(\eta; x, \xi) \right) [\nabla \eta \cdot \Omega] \\ &\quad + \text{Op}^{\text{BW}} \left( A_1^{\mathcal{B}_0, \eta}(\eta; x, \xi)(\nabla \eta \cdot \Omega) + A_{[0, \geq 1]}[\Omega](\eta; x, \xi) \right) \eta + \mathfrak{R}_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}. \end{aligned} \quad (5.130)$$

Here the symbol  $A_{[0, \geq 1]}[\Omega]$  is linear in  $\Omega$ , with coefficients depending on  $\eta$ , because the factor  $\nabla \eta$  in  $\omega_\eta$  is absorbed in the coefficient.

We next expand the first term in Eq. (5.130). By Lemma 4.18,

$$\nabla \eta \cdot \Omega = \text{Op}^{\text{BW}}((\nabla \eta)^\top) \Omega + \text{Op}^{\text{BW}}(\Omega^\top) \nabla \eta + \sum_{j=1}^2 R^{\text{BW}}[\partial_j \eta, \Omega_j],$$

and Remark 4.19 together with Lemma 4.25 (i) shows that the contribution of  $\sum_{j=1}^2 R^{\text{BW}}[\partial_j \eta, \Omega_j]$  is absorbed in  $\mathfrak{R}_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}$ . For the first paraproduct we use Proposition 4.24 with asymptotic expansion up to order one. Since  $A_0^{\mathcal{B}_0, \omega} \in (\Gamma_{\geq 0}^0)^2$  and  $(\nabla \eta)^\top \in (\Gamma_{\geq 1}^0)^{1 \times 2}$ ,

$$(iA_0^{\mathcal{B}_0, \omega}) \#_1 (\nabla \eta)^\top = iA_0^{\mathcal{B}_0, \omega} (\nabla \eta)^\top + \frac{1}{2} \{A_0^{\mathcal{B}_0, \omega}, (\nabla \eta)^\top\},$$

where the second term belongs to  $(\Gamma_{\geq 1}^{-1})^{2 \times 2}$ . The composition defect belongs to  $\mathcal{R}_{\geq 1}^\rho$  after increasing  $\rho_0$ , while the contribution of  $A_{[-2, \geq 1]}$  gives a symbol in  $(\Gamma_{\geq 1}^{-2})^{2 \times 2}$ .

For the second paraproduct, we first write  $\nabla \eta = \text{Op}^{\text{BW}}(i\xi) \eta$ . Another application of Proposition 4.24 gives

$$\text{Op}^{\text{BW}}(\Omega^\top) \nabla \eta = \text{Op}^{\text{BW}}(i\Omega \cdot \xi + A_{[0, \geq 1]}[\Omega](\eta; x, \xi)) \eta + R_{[\rho, \geq 1]}^{\mathbb{R}}(\eta, \Omega) \eta.$$

Composing once more with  $\text{Op}^{\text{BW}}\left(\mathbf{i}A_0^{\mathcal{B}_0,\omega} + A_{[-2,\geq 1]}\right)$ , and using Proposition 4.24 and Lemma 4.25, the principal part is  $\mathbf{i}A_0^{\mathcal{B}_0,\omega} \mathbf{i}(\Omega \cdot \xi) = -A_0^{\mathcal{B}_0,\omega}(\Omega \cdot \xi)$ , while all Poisson brackets and the terms generated by  $A_{[-2,\geq 1]}$  are absorbed in  $A_{[0,\geq 1]}[\Omega](\eta; x, \xi)$ . Therefore,

$$\begin{aligned} & \text{Op}^{\text{BW}}\left(\mathbf{i}A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) + A_{[-2,\geq 1]}(\eta; x, \xi)\right) [\nabla\eta \cdot \Omega] \\ &= \text{Op}^{\text{BW}}\left(\mathbf{i}A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) (\nabla\eta)^\top + \frac{1}{2} \left\{A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi), (\nabla\eta)^\top\right\} + A_{[-2,\geq 1]}(\eta; x, \xi)\right) \Omega \\ &+ \text{Op}^{\text{BW}}\left(-A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) (\Omega \cdot \xi) + A_{[0,\geq 1]}[\Omega](\eta; x, \xi)\right) \eta + \mathfrak{R}_{[\varrho,\geq 1]}^{\text{lin},\mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}. \end{aligned} \quad (5.131)$$

Inserting Eq. (5.131) into Eq. (5.130) yields

$$\begin{aligned} \mathcal{B}_0(\eta) [\nabla\eta \cdot \Omega] &= \text{Op}^{\text{BW}}\left(\mathbf{i}A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) \otimes \nabla\eta + \frac{1}{2} \left\{A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi), (\nabla\eta)^\top\right\} + A_{[-2,\geq 1]}(\eta; x, \xi)\right) \Omega \\ &+ \text{Op}^{\text{BW}}\left(\left(-A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) \otimes \xi + A_1^{\mathcal{B}_0,\eta}(\eta; x, \xi) \otimes \nabla\eta\right) \Omega + A_{[0,\geq 1]}[\Omega](\eta; x, \xi)\right) \eta + \mathfrak{R}_{[\varrho,\geq 1]}^{\text{lin},\mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}. \end{aligned} \quad (5.132)$$

We now treat  $q(\eta)\Omega$ . Applying Proposition 5.15 to the two scalar components of  $\Omega$ , and collecting them into a vector identity, gives

$$\begin{aligned} q(\eta)\Omega &= \text{Op}^{\text{BW}}\left(\mathbf{i}a_0^{q,\omega}(\eta; x, \xi) \text{Id}_{\mathbb{R}^2} + A_{[-2,\geq 1]}(\eta; x, \xi)\right) \Omega \\ &+ \text{Op}^{\text{BW}}\left(a_1^{q,\eta}(\eta; x, \xi) \Omega + A_{[0,\geq 1]}[\Omega](\eta; x, \xi)\right) \eta + \mathfrak{R}_{[\varrho,\geq 1]}^{\text{lin},\mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}. \end{aligned} \quad (5.133)$$

Here the lower symbol in the  $\eta$ -column is, more precisely,

$$A_{[0,\geq 1]}[\cdot] \in \mathcal{L}\left((H^{s_0})^2; (\Gamma^0)^2\right),$$

and the corresponding smoothing column belongs to  $\mathcal{L}((H^{s_0})^2; (\mathcal{R}^\varrho)^2)$ , by Eqs. (5.109) and (5.110). Although these coefficients need not vanish at  $\eta = 0$ , they multiply  $\eta$ ; hence the complete terms have total degree at least one in  $(\eta, \Omega)$ . By the definition Eq. (5.123),  $V^*[\Omega](\eta) = J(\mathcal{B}_0(\eta) [\nabla\eta \cdot \Omega] - q(\eta)\Omega)$ . Combining Eq. (5.132) and Eq. (5.133), we obtain Eq. (5.124), with the intermediate symbols

$$A_0^{V^*,\Omega}(\eta; x, \xi) := J\left(A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) \otimes \nabla\eta - a_0^{q,\omega}(\eta; x, \xi) \text{Id}_{\mathbb{R}^2}\right), \quad (5.134a)$$

$$A_{-1}^{V^*,\Omega}(\eta; x, \xi) := J\left(\frac{1}{2} \left\{A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi), (\nabla\eta)^\top\right\}\right), \quad (5.134b)$$

$$A_1^{V^*,\eta}(\eta; x, \xi) := J\left(-A_0^{\mathcal{B}_0,\omega}(\eta; x, \xi) \otimes \xi + A_1^{\mathcal{B}_0,\eta}(\eta; x, \xi) \otimes \nabla\eta - a_1^{q,\eta}(\eta; x, \xi) \text{Id}_{\mathbb{R}^2}\right). \quad (5.134c)$$

The membership of these symbols in the classes stated in the proposition follows from

$$A_0^{\mathcal{B}_0,\omega} \in (\Gamma_{\geq 0}^0)^2, \quad A_1^{\mathcal{B}_0,\eta} \in (\Gamma_{\geq 1}^1)^2, \quad a_0^{q,\omega} \in \Gamma_{\geq 1}^0, \quad a_1^{q,\eta} \in \Gamma_{\geq 0}^1,$$

from Propositions 5.14 and 5.15, and from Proposition 4.24. It remains only to identify the three intermediate expressions with the explicit symbols in the statement.

**Subpart 1.1** (Identification of Eq. (5.127)). For any  $W_1, W_2 \in \mathbb{R}^2$ ,

$$W_1^\perp \otimes W_2^\perp + W_2 \otimes W_1 = (W_1 \cdot W_2) \text{Id}_{\mathbb{R}^2}. \quad (5.135)$$

Using Eq. (5.107), namely  $a_0^{q,\omega} = \nabla\eta \cdot A_0^{\mathcal{B}_0,\omega}$ , and Eq. (5.135) with  $(W_1, W_2) = (\nabla\eta, A_0^{\mathcal{B}_0,\omega})$ , we get

$$A_0^{\mathcal{B}_0,\omega} \otimes \nabla\eta - a_0^{q,\omega} \text{Id}_{\mathbb{R}^2} = -\nabla^\perp \eta \otimes A_0^{\mathcal{B}_0,\omega;\perp}. \quad (5.136)$$

Moreover,

$$-J(W_1^\perp \otimes W_2^\perp) = W_1 \otimes W_2^\perp. \quad (5.137)$$

Inserting Eq. (5.136) into Eq. (5.134a) and using Eq. (5.137), we obtain

$$A_0^{V^*,\Omega}(\eta; x, \xi) = \nabla\eta \otimes A_0^{\mathcal{B}_0,\omega;\perp}(\eta; x, \xi),$$

which is Eq. (5.127).

**Subpart 1.2** (Identification of Eq. (5.128)). Since  $(\nabla\eta)^\top$  is independent of  $\xi$ , the matrix Poisson bracket of Eq. (4.13) gives

$$\{A_0^{\mathcal{B}_0,\omega}, (\nabla\eta)^\top\} = J_\xi A_0^{\mathcal{B}_0,\omega} \mathbf{H}_x \eta.$$

Thus Eq. (5.134b) becomes

$$\mathbf{A}_{-1}^{V^*,\Omega}(\eta; x, \xi) = \frac{J}{2} J_\xi A_0^{\mathcal{B}_0,\omega} \mathbf{H}_x \eta,$$

which is Eq. (5.128). The symbol belongs to  $(\Gamma_{\geq 1}^{-1})^{2 \times 2}$ , because one  $\xi$ -derivative falls on  $A_0^{\mathcal{B}_0,\omega} \in (\Gamma_{\geq 0}^0)^2$ .

**Subpart 1.3** (Identification of Eq. (5.126)). By Eq. (5.108),

$$a_1^{q,\eta} = \nabla\eta \cdot A_1^{\mathcal{B}_0,\eta} - A_0^{\mathcal{B}_0,\omega} \cdot \xi.$$

Substituting this identity in Eq. (5.134c) and applying Eq. (5.135) twice, first with  $(W_1, W_2) = (\xi, A_0^{\mathcal{B}_0,\omega})$  and then with  $(W_1, W_2) = (\nabla\eta, A_1^{\mathcal{B}_0,\eta})$ , gives

$$-A_0^{\mathcal{B}_0,\omega} \otimes \xi + (A_0^{\mathcal{B}_0,\omega} \cdot \xi) \text{Id}_{\mathbb{R}^2} = \xi^\perp \otimes A_0^{\mathcal{B}_0,\omega;\perp},$$

and

$$A_1^{\mathcal{B}_0,\eta} \otimes \nabla\eta - (\nabla\eta \cdot A_1^{\mathcal{B}_0,\eta}) \text{Id}_{\mathbb{R}^2} = -\nabla^\perp \eta \otimes A_1^{\mathcal{B}_0,\eta;\perp}.$$

Therefore

$$\mathbf{A}_1^{V^*,\eta} = J \left( \xi^\perp \otimes A_0^{\mathcal{B}_0,\omega;\perp} - \nabla^\perp \eta \otimes A_1^{\mathcal{B}_0,\eta;\perp} \right).$$

Using Eq. (5.137) on the two tensor products, we conclude

$$\mathbf{A}_1^{V^*,\eta}(\eta; x, \xi) = -\xi \otimes A_0^{\mathcal{B}_0,\omega;\perp}(\eta; x, \xi) + \nabla\eta \otimes A_1^{\mathcal{B}_0,\eta;\perp}(\eta; x, \xi),$$

which is Eq. (5.126).

**Part 2** (Paralinearization of  $\Omega \mapsto \nu_3^*[\Omega](\eta)$ ). Write  $\mathcal{B}_0(\eta) = (\mathcal{B}_{0,1}(\eta), \mathcal{B}_{0,2}(\eta))$ . From Eq. (5.123),

$$\nu_3^*[\Omega](\eta) = -\mathcal{B}_0(\eta)^\perp \cdot \Omega = \mathcal{B}_{0,2}(\eta)[\Omega_1] - \mathcal{B}_{0,1}(\eta)[\Omega_2]. \quad (5.138)$$

We apply Proposition 5.14 to the two scalar inputs  $\Omega_1$  and  $\Omega_2$ , and then take the combination in Eq. (5.138). The  $\Omega$ -principal part is

$$\text{Op}^{\text{BW}}(iA_{0,2}^{\mathcal{B}_0,\omega})\Omega_1 - \text{Op}^{\text{BW}}(iA_{0,1}^{\mathcal{B}_0,\omega})\Omega_2 = \text{Op}^{\text{BW}}\left(-i\left(A_0^{\mathcal{B}_0,\omega;\perp}\right)^\top\right)\Omega,$$

which corresponds to  $A_0^{\nu_3^*,\Omega} = -A_0^{\mathcal{B}_0,\omega;\perp} \in (\Gamma_{\geq 0}^0)^2$ . Likewise, the  $\eta$ -principal part is

$$\text{Op}^{\text{BW}}\left(A_{1,2}^{\mathcal{B}_0,\eta}(\eta; x, \xi)\Omega_1 - A_{1,1}^{\mathcal{B}_0,\eta}(\eta; x, \xi)\Omega_2\right)\eta = \text{Op}^{\text{BW}}\left(-A_1^{\mathcal{B}_0,\eta;\perp}(\eta; x, \xi) \cdot \Omega\right)\eta.$$

The remaining terms in Proposition 5.14 are either  $a_{[0,\geq 1]}[\Omega](\eta; x, \xi)\eta$ , smoothing remainders in  $\mathfrak{r}_{[\ell,\geq 1]}^{\text{lin},\mathbb{R}}(\eta, \Omega) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix}$ , or symbols of order  $-2$  acting on  $\Omega$ ; the latter are absorbed in the generic  $A_{[-2,\geq 1]}(\eta; x, \xi)$ -slot of Eq. (5.129). Hence

$$\nu_3^*[\Omega](\eta) = \text{Op}^{\text{BW}}\left(\begin{bmatrix} -A_1^{\mathcal{B}_0,\eta;\perp} \cdot \Omega \\ -iA_0^{\mathcal{B}_0,\omega;\perp} \end{bmatrix} + \begin{bmatrix} a_{[0,\geq 1]}[\Omega] \\ A_{[-2,\geq 1]} \end{bmatrix}\right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix} + \mathbf{r}_{[\ell,\geq 1]}^{\text{lin},\mathbb{R}}(\eta, \Omega) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix},$$

which is precisely Eq. (5.129). The preceding identities also prove  $a_1^{\nu_3^*,\eta}[\Omega] \in \Gamma_{\geq 1}^1$  and  $A_0^{\nu_3^*,\Omega} \in (\Gamma_{\geq 0}^0)^2$ , as declared.  $\square$

We record the only algebraic identity from the preceding computation that is used below in the paralinearization of the pressure contribution.

**Corollary 5.20** (Quadratic identity for the  $\eta$ -symbol of  $V^\star$ ). *For any  $x \in \mathbb{R}^2$ ,  $\xi \in \mathbb{R}^2 \setminus \{0\}$ ,  $W \in \mathbb{R}^2$ , one has*

$$\left( A_1^{V^\star, \eta}(\eta; x, \xi) W \right) \cdot W^\perp = -\lambda_1(\eta; x, \xi) \left[ \left( 2A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot W \right)^2 - \left( \frac{\nabla^\perp \eta(x) \cdot W}{1 + |\nabla \eta(x)|^2} \right)^2 \right]. \quad (5.139)$$

*Proof.* Both sides of Eq. (5.139) are one-homogeneous in  $\xi$  and invariant under simultaneous rotations of  $\xi$ ,  $\nabla \eta(x)$ , and  $W$ . It is therefore enough to prove the identity for  $|\xi| = 1$  and  $\xi = e_1$ . Write

$$\nabla \eta(x) = \begin{bmatrix} p \\ q \end{bmatrix}, \quad W = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Then

$$\langle \nabla \eta \rangle^2 = 1 + p^2 + q^2, \quad 1 + q^2 = 4\lambda_1^2(\eta; x, e_1),$$

and

$$A_0^{\mathcal{B}_0, \omega}(\eta; x, e_1) = -\frac{1}{4\lambda_1(\eta; x, e_1) \langle \nabla \eta \rangle^2} \begin{bmatrix} 1 + q^2 \\ -pq \end{bmatrix}.$$

Hence

$$A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, e_1) \cdot W = -\frac{pq\alpha + (1 + q^2)\beta}{4\lambda_1(\eta; x, e_1) \langle \nabla \eta \rangle^2}.$$

Moreover,

$$\nabla^\perp \eta(x) \cdot W = -q\alpha + p\beta, \quad e_1 \cdot W^\perp = -\beta, \quad \nabla \eta(x) \cdot W^\perp = q\alpha - p\beta.$$

Using Eqs. (5.122) and (5.126), we get

$$\begin{aligned} \left( A_1^{V^\star, \eta}(\eta; x, e_1) W \right) \cdot W^\perp &= \beta A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, e_1) \cdot W \\ &+ (q\alpha - p\beta) \left[ \frac{p}{\langle \nabla \eta \rangle^2} A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, e_1) \cdot W - \frac{\lambda_1(\eta; x, e_1)}{\langle \nabla \eta(x) \rangle^4} (-q\alpha + p\beta) \right]. \end{aligned}$$

Since

$$\beta + \frac{p(q\alpha - p\beta)}{\langle \nabla \eta \rangle^2} = \frac{pq\alpha + (1 + q^2)\beta}{\langle \nabla \eta \rangle^2},$$

we infer

$$\begin{aligned} \left( A_1^{V^\star, \eta}(\eta; x, e_1) W \right) \cdot W^\perp &= -\frac{(pq\alpha + (1 + q^2)\beta)^2}{4\lambda_1(\eta; x, e_1) \langle \nabla \eta(x) \rangle^4} + \frac{\lambda_1(\eta; x, e_1)}{\langle \nabla \eta(x) \rangle^4} (-q\alpha + p\beta)^2 \\ &= -\lambda_1(\eta; x, e_1) \left[ \left( 2A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, e_1) \cdot W \right)^2 - \left( \frac{\nabla^\perp \eta(x) \cdot W}{\langle \nabla \eta \rangle^2} \right)^2 \right]. \end{aligned}$$

This proves Eq. (5.139).  $\square$

### 5.3.2 Paralinearization of operators $T(\eta)$ and $\tau_0(\eta)$

Consider the operator  $T(\eta)$  of nonlinear convolution type, that has been introduced in Eq. (2.62), and that can be recasted as follows using Eq. (2.14):

$$T(\eta)[\Omega] = -\left( V^\star[\Omega](\eta) + \nu_3^\star[\Omega](\eta) \nabla \eta \right)^\perp. \quad (5.140)$$

The paralinearization of the operator  $T(\eta)$  is given in the following proposition.

**Proposition 5.21** (Paralinearization of  $T(\eta)$ ). *There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exists  $s_0 > 0$  and  $r_0 > 0$  such that: For any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$ ,  $\Omega \in (B_{s_0, \mathbb{R}}(I; r_0))^2$ , the operator  $T(\eta)[\Omega]$  as in Eq. (5.140) admits the following paralinearization:*

$$T(\eta)\Omega = \text{Op}^{\text{BW}}\left(\mathfrak{A}_1^T(\eta, \Omega; x, \xi) + [A_{[0, \geq 1]}[\Omega](\eta; x, \xi) \quad A_{[-2, \geq 1]}(\eta; x, \xi)]\right) \begin{bmatrix} \eta \\ \Omega \end{bmatrix} + \mathfrak{R}_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}, \quad (5.141)$$

where

- $\mathfrak{A}_1^T \in (\Gamma_{\geq 1}^1)^{2 \times 3}$  and is defined as

$$\mathfrak{A}_1^T(\eta, \Omega; x, \xi) := \begin{bmatrix} A_1^{T, \eta}[\Omega](\eta; x, \xi) & A_{-1}^{T, \Omega}(\eta; x, \xi) \end{bmatrix},$$

where

- $A_1^{T, \eta} \in \mathcal{L}\left((H^{s_0})^2; (\Gamma_{\geq 0}^1)^2\right)$  is defined as

$$A_1^{T, \eta}[\Omega](\eta; x, \xi) := \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega - i \nu_3^*[\Omega](\eta; x) \right) \xi^\perp, \quad (5.142)$$

where  $\nu_3^*$  is defined in Eq. (5.123) and  $A_0^{\mathcal{B}_0, \omega}$  is defined in Eq. (5.8);

- $A_{-1}^{T, \Omega} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}$  and has the explicit form

$$A_{-1}^{T, \Omega}(\eta; x, \xi) := \frac{1}{2} \left( \text{tr} \left( \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \mathbf{J} \mathbf{H}_x \eta \right) \mathbf{J} + \Delta_\xi \phi_1^{\mathcal{B}_0} \mathbf{H}_x \eta + \Delta_x \eta \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} - \Delta_\xi \phi_1^{\mathcal{B}_0} \Delta_x \eta \text{Id} \right); \quad (5.143)$$

- $\mathbf{A}_{[m, \geq 1]} \in (\Gamma_{\geq 1}^m)^{2 \times 2}$ ,  $A_{[m, \geq 1]} \in (\Gamma_{\geq 1}^m)^2$ ,  $m \in \mathbb{R}$ , denote generic symbols as per Notation 5.1;
- $\mathfrak{R}_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}} \in (\mathcal{R}_{\geq 1}^\rho)^{2 \times 3}$  denotes generic smoothing remainders as per Notation 5.1.

*Proof.* Since the parilinearization for  $V^*[\Omega](\eta)$  is given in Proposition 5.19 we turn our attention to the term

$$-\nu_3^*[\Omega](\eta) \nabla^\perp \eta = \left( \mathcal{B}_0(\eta)^\perp \cdot \Omega \right) \nabla^\perp \eta$$

in Eq. (5.140). Lemma 4.18 gives

$$\left( -\nu_3^*[\Omega](\eta) \right) \nabla^\perp \eta = \text{Op}^{\text{BW}} \left( -\nu_3^*[\Omega](\eta) \right) \nabla^\perp \eta + \text{Op}^{\text{BW}} \left( \nabla^\perp \eta \right) \left[ -\nu_3^*[\Omega](\eta) \right] + R^{\text{BW}} \left[ \nabla^\perp \eta, -\nu_3^*[\Omega](\eta) \right]. \quad (5.144)$$

Using the parilinearization in Eq. (5.129), combined with Remark 4.17 we obtain that

$$-\nu_3^*[\Omega](\eta) \in \mathcal{F}^{\mathbb{R}} \subset \Gamma_{\geq 1}^0. \quad (5.145)$$

So we can apply Proposition 4.24 and obtain that

$$\text{Op}^{\text{BW}} \left( -\nu_3^*[\Omega](\eta) \right) \nabla^\perp \eta = \text{Op}^{\text{BW}} \left( -i \nu_3^*[\Omega](\eta) \xi^\perp + A_{[0, \geq 1]}[\Omega](\eta; x, \xi) \right) \eta + R_{[\rho, \geq 1]}^{\mathbb{R}}[\Omega](\eta) \eta. \quad (5.146)$$

Next, we invoke Eqs. (4.13) and (5.129), Lemma 4.25, and Proposition 4.24 to get

$$\begin{aligned} \text{Op}^{\text{BW}} \left( \nabla^\perp \eta \right) \left[ -\nu_3^*[\Omega](\eta) \right] &= \text{Op}^{\text{BW}} \left( i \nabla^\perp \eta \otimes A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) + \frac{1}{2} \left\{ \nabla^\perp \eta, \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \right)^\top \right\} + \mathbf{A}_{[-2, \geq 1]}(\eta; x, \xi) \right) \Omega \\ &\quad + \text{Op}^{\text{BW}} \left( \nabla^\perp \eta \left( A_1^{\mathcal{B}_0, \eta; \perp}(\eta; x, \xi) \cdot \Omega \right) + A_{[0]}[\Omega](\eta; x, \xi) \right) \eta + \mathfrak{R}_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}. \end{aligned} \quad (5.147)$$

An application of Remark 4.19, Lemma 4.25, and Eq. (5.145) shows

$$R^{\text{BW}} \left[ \nabla^\perp \eta, -\nu_3^*[\Omega](\eta) \right] = R_{[\rho, \geq 1]}^{\mathbb{R}}(\eta, \Omega) \eta. \quad (5.148)$$

Plugging Eqs. (5.146) to (5.148) into Eq. (5.144), we obtain that

$$\begin{aligned} -\nu_3^*[\Omega](\eta) \nabla^\perp \eta &= \text{Op}^{\text{BW}} \left( -i \nu_3^*[\Omega](\eta) \xi^\perp + \nabla^\perp \eta \left( A_1^{\mathcal{B}_0, \eta; \perp}(\eta; x, \xi) \cdot \Omega \right) + A_{[0, \geq 1]}[\Omega](\eta; x, \xi) \right) \eta \\ &\quad + \text{Op}^{\text{BW}} \left( i \nabla^\perp \eta \otimes A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) + \frac{1}{2} \left\{ \nabla^\perp \eta, \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \right)^\top \right\} + \mathbf{A}_{[-2, \geq 1]}(\eta; x, \xi) \right) \Omega \\ &\quad + \mathfrak{R}_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}. \end{aligned} \quad (5.149)$$

Now, we put Eqs. (5.124) and (5.149) into Eq. (5.140) and get

$$\begin{aligned} T(\eta)[\Omega] &= \text{Op}^{\text{BW}} \left( \mathbf{A}_{-1}^{T, \Omega}(\eta; x, \xi) + \mathbf{A}_{[-2, \geq 1]}(\eta; x, \xi) \right) \Omega \\ &\quad + \text{Op}^{\text{BW}} \left( A_1^{T, \eta}[\Omega](\eta; x, \xi) + A_{[0, \geq 1]}(\eta, \Omega; x, \xi) \right) \eta + \mathbf{R}_{[\rho, \geq 1]}^{\mathbb{R}}(\eta) \Omega + R_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega) \eta, \end{aligned}$$

where

$$\mathbf{A}_{-1}^{T,\Omega}(\eta; x, \xi) := \frac{1}{2} \left\{ \nabla^\perp \eta, \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \right)^\top \right\} - \mathbf{J} \mathbf{A}_{-1}^{V^*, \Omega}(\eta; x, \xi) = \frac{1}{2} \left( \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \mathbf{H}_x \eta - \mathbf{J} \mathbf{H}_x \eta \left( \mathbf{J} \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \right)^\top \right) \quad (5.150)$$

and

$$\begin{aligned} A_1^{T,\eta}[\Omega](\eta; x, \xi) &:= -i \nu_3^*[\Omega](\eta) \xi^\perp + \nabla^\perp \eta \left( A_1^{\mathcal{B}_0, \eta; \perp}(\eta; x, \xi) \cdot \Omega \right) \\ &\quad - \mathbf{J} \left( -\xi \otimes A_0^{\mathcal{B}_0, \omega; \perp}(\eta, x, \xi) + \nabla \eta \otimes A_1^{\mathcal{B}_0, \eta; \perp}(\eta; x, \xi) \right) \Omega. \end{aligned} \quad (5.151)$$

We now focus in simplifying the expressions in Eqs. (5.150) and (5.151).

**Step 1** (Simplification of Eq. (5.150)). We use the identity

$$\mathbf{J} \mathbf{A} \mathbf{J} = \mathbf{A}^\top - \text{tr}(\mathbf{A}) \text{Id},$$

valid for any  $2 \times 2$  matrix  $\mathbf{A}$ , so that if  $\mathbf{S}_1, \mathbf{S}_2$  are symmetric

$$\mathbf{S}_1 \mathbf{S}_2 - \mathbf{J} \mathbf{S}_2 (\mathbf{J} \mathbf{S}_1)^\top = \mathbf{S}_1 \mathbf{S}_2 + \mathbf{J} \mathbf{S}_2 \mathbf{S}_1 \mathbf{J} = 2 \mathbf{S}_1 \mathbf{S}_2 - \text{tr}(\mathbf{S}_1 \mathbf{S}_2) \text{Id}. \quad (5.152)$$

Applying Eq. (5.152) we obtain

$$\mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \mathbf{H}_x \eta - \mathbf{J} \mathbf{H}_x \eta \left( \mathbf{J} \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \right)^\top = 2 \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \mathbf{H}_x \eta - \text{tr} \left( \mathbf{H}_x \eta \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \right) \text{Id}.$$

Next, decomposing  $2\mathbf{S}_1 \mathbf{S}_2 = \llbracket \mathbf{S}_1, \mathbf{S}_2 \rrbracket + (\mathbf{S}_1 \mathbf{S}_2 + \mathbf{S}_2 \mathbf{S}_1)$  and using that for symmetric  $2 \times 2$  matrices

$$\mathbf{S}_1 \mathbf{S}_2 + \mathbf{S}_2 \mathbf{S}_1 = \text{tr}(\mathbf{S}_1) \mathbf{S}_2 + \text{tr}(\mathbf{S}_2) \mathbf{S}_1 + \text{tr}(\mathbf{S}_1 \mathbf{S}_2) \text{Id} - \text{tr}(\mathbf{S}_1) \text{tr}(\mathbf{S}_2) \text{Id},$$

we obtain the skew-symmetric/symmetric decomposition

$$\begin{aligned} \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \mathbf{H}_x \eta - \mathbf{J} \mathbf{H}_x \eta \left( \mathbf{J} \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \right)^\top &= \llbracket \mathbf{H}_\xi \phi_1^{\mathcal{B}_0}, \mathbf{H}_x \eta \rrbracket \\ &\quad + \text{tr} \left( \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \right) \mathbf{H}_x \eta + \text{tr}(\mathbf{H}_x \eta) \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} - \text{tr}(\mathbf{H}_\xi \phi_1^{\mathcal{B}_0}) \text{tr}(\mathbf{H}_x \eta) \text{Id}. \end{aligned}$$

Next, since both matrices are  $2 \times 2$  and symmetric

$$\llbracket \mathbf{H}_\xi \phi_1^{\mathcal{B}_0}, \mathbf{H}_x \eta \rrbracket = \text{tr} \left( \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \mathbf{J} \mathbf{H}_x \eta \right) \mathbf{J}.$$

This yields the desired identity Eq. (5.143).

**Step 2** (Equivalence of Eqs. (5.142) and (5.151)). Explicit computations using in particular Eq. (5.137) show that

$$-\mathbf{J} \left( -\xi \otimes A_0^{\mathcal{B}_0, \omega; \perp} + \nabla \eta \otimes A_1^{\mathcal{B}_0, \eta; \perp} \right) \Omega = \xi^\perp A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega - \nabla^\perp \eta A_1^{\mathcal{B}_0, \eta; \perp} \cdot \Omega. \quad (5.153)$$

Thus, plugging Eq. (5.153) in Eq. (5.151) we obtain Eq. (5.142). □

Next, we give the parilinearization of the operator  $\tau_0(\eta)$  introduced in Eq. (2.12), which takes the form

$$\tau_0(\eta) \psi := \Delta^{-1} \nabla^\perp \cdot (T(\eta) [\nabla^\perp \psi]), \quad (5.154)$$

where  $T(\eta)$  is the trace operator defined in Eq. (5.140). The parilinearization result reads as follows.

**Proposition 5.22** (Parilinearization of  $\tau_0(\eta)$ ). *There exists  $\rho_0 > 0$  such that for any  $\rho \geq \rho_0$  there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta, \psi \in B_{s_0, \mathbb{R}}(I; r_0)$ , the operator  $\tau_0(\eta)$  defined in Eq. (5.154) admits the paradifferential decomposition*

$$\tau_0(\eta) \psi = \text{Op}^{\text{BW}} \left( \left[ \begin{array}{c} a_0^{\tau_0, \eta}[\psi](\eta; x, \xi) \\ a_{-1}^{\tau_0, \psi}(\eta; x, \xi) \end{array} \right] + \left[ \begin{array}{c} a_{[-1, \geq 1]}[\psi](\eta; x, \xi) \\ a_{[-2, \geq 1]}(\eta; x, \xi) \end{array} \right] \cdot \left[ \begin{array}{c} \eta \\ \psi \end{array} \right] + R_{[\rho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \psi) \cdot \left[ \begin{array}{c} \eta \\ \psi \end{array} \right], \right.$$

where

- $a_0^{\tau_0, \eta} \in \mathcal{L}(H^{s_0}; \Gamma_{\geq 0}^0)$  is explicitly defined as

$$a_0^{\tau_0, \eta}[\psi](\eta; x, \xi) := -v_3^*[\nabla^\perp \psi](\eta; x) - i \nabla \psi(x) \cdot A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi), \quad (5.155)$$

where  $v_3^*$  is the vertical velocity component defined in Eq. (5.123) and  $A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi)$  is the paradifferential symbol defined in Eq. (5.8);

- $a_{-1}^{\tau_0, \psi} \in \Gamma_{\geq 1}^{-1}$  is explicitly defined as

$$a_{-1}^{\tau_0, \psi}(\eta; x, \xi) := -\frac{|\xi|^2}{32 \lambda_1(\eta; x, \xi)^3} \hat{\xi}^\perp \cdot \mathbf{H}_x \eta \hat{\xi}^\perp = -\frac{\lambda_0(\eta; x, \xi)}{\lambda_1(\eta; x, \xi)}, \quad (5.156)$$

As a joint symbol of  $(\eta, \psi)$ ,  $a_0^{\tau_0, \eta}[\psi] \in \Gamma_{\geq 1}^0$  and it is linear in  $\psi$ . For fixed  $\psi$ , it belongs to  $\Gamma_{\geq 0}^0$  and need not vanish at  $\eta = 0$ .

- $a_{[m, \geq 1]} \in \Gamma_{\geq 1}^m$  for  $m \in \mathbb{R}$  are lower-order symbol corrections (cf. Notation 5.1);
- $R_{[\ell, \geq 1]}^{\text{lin}, \mathbb{R}} \in (\mathcal{R}_{\geq 1}^\ell)^2$  are remainder terms satisfying the estimates in Notation 5.1.

*Proof.* Remind  $\Delta^{-1} \nabla^\perp \cdot \bullet = \text{Op}^{\text{BW}}\left(-\frac{i \xi^\perp}{|\xi|^2} \cdot\right) \bullet$ . Applying Proposition 5.21 we obtain

$$\begin{aligned} \Delta^{-1} \nabla^\perp \cdot T(\eta)[\nabla^\perp \psi] &= \text{Op}^{\text{BW}}\left(\frac{-i \xi^\perp}{|\xi|^2}\right) \cdot \text{Op}^{\text{BW}}\left(A_{-1}^{T, \Omega}(\eta; x, \xi) + A_{[-2, \geq 1]}(\eta; x, \xi)\right) \text{Op}^{\text{BW}}(i \xi^\perp) \psi \\ &\quad + \text{Op}^{\text{BW}}\left(\frac{-i \xi^\perp}{|\xi|^2}\right) \cdot \text{Op}^{\text{BW}}\left(A_1^{T, \eta}[\nabla^\perp \psi](\eta; x, \xi) + A_{[0, \geq 1]}(\eta, \nabla^\perp \psi; x, \xi)\right) \eta \\ &\quad + \text{Op}^{\text{BW}}\left(\frac{-i \xi^\perp}{|\xi|^2}\right) \cdot \left(R_{[\ell, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta) \text{Op}^{\text{BW}}(i \xi^\perp) \psi + R_{[\ell, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \nabla^\perp \psi) \eta\right). \end{aligned}$$

Thus, an application of Proposition 4.24 and Lemma 4.25 yields

$$\begin{aligned} \Delta^{-1} \nabla^\perp \cdot T(\eta)[\nabla^\perp \psi] &= \text{Op}^{\text{BW}}\left(\hat{\xi}^\perp \cdot A_{-1}^{T, \Omega} \hat{\xi}^\perp + a_{[-2, \geq 1]}(\eta; x, \xi)\right) \psi \\ &\quad + \text{Op}^{\text{BW}}\left(-\frac{i \hat{\xi}^\perp}{|\xi|} \cdot A_1^{T, \eta}[\nabla^\perp \psi](\eta; x, \xi) + a_{[-1, \geq 1]}(\eta; x, \xi)\right) \eta + r_{[\ell, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \end{aligned}$$

Making appeal to Eq. (5.143), we can write

$$\hat{\xi}^\perp \cdot A_{-1}^{T, \Omega} \hat{\xi}^\perp = \frac{1}{2} \hat{\xi}^\perp \cdot \left( \Delta_\xi \phi_1^{\mathcal{B}_0} \mathbf{H}_x \eta - \Delta_\xi \phi_1^{\mathcal{B}_0} \Delta_x \eta \text{Id} + \Delta_x \eta \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \right) \hat{\xi}^\perp = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3,$$

where

$$\mathbf{b}_1 := \frac{\Delta_\xi \phi_1^{\mathcal{B}_0}}{2} \hat{\xi}^\perp \cdot \mathbf{H}_x \eta \hat{\xi}^\perp, \quad \mathbf{b}_2 := -\frac{\Delta_\xi \phi_1^{\mathcal{B}_0} \Delta_x \eta}{2}, \quad \mathbf{b}_3 := \frac{\Delta_x \eta}{2} \hat{\xi}^\perp \cdot \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \hat{\xi}^\perp.$$

Since  $\phi_1^{\mathcal{B}_0}$  is 1-homogeneous in  $\xi$ , Euler's formula implies that

$$\mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \xi = 0.$$

Therefore, in the orthonormal basis  $(\hat{\xi}, \hat{\xi}^\perp)$ ,

$$\Delta_\xi \phi_1^{\mathcal{B}_0} = \text{tr}\left(\mathbf{H}_\xi \phi_1^{\mathcal{B}_0}\right) = \hat{\xi}^\perp{}^\top \mathbf{H}_\xi \phi_1^{\mathcal{B}_0} \hat{\xi}^\perp.$$

In particular, this implies that

$$\mathbf{b}_3 = \frac{\Delta_x \eta}{2} \Delta_\xi \phi_1^{\mathcal{B}_0} = -\mathbf{b}_2.$$

Hence,

$$\hat{\xi}^\perp A_{-1}^{T, \Omega} \hat{\xi}^\perp = \mathbf{b}_1 = \frac{\Delta_\xi \phi_1^{\mathcal{B}_0}}{2} \hat{\xi}^\perp \cdot \mathbf{H}_x \eta \hat{\xi}^\perp.$$

Finally, recalling that

$$\phi_1^{\mathcal{B}_0}(\eta; x, \xi) = -\frac{\lambda_1(\eta; x, \xi)}{1 + |\nabla\eta(x)|^2}, \quad \lambda_1(\eta; x, \xi) = \frac{\sqrt{|\xi|^2 + (\nabla^\perp\eta(x) \cdot \xi)^2}}{2},$$

we can compute explicitly

$$\Delta_\xi \phi_1^{\mathcal{B}_0} = -\frac{|\xi|^2}{16\lambda_1^3(\eta; x, \xi)}.$$

Therefore,

$$\hat{\xi}^\perp A_{-1}^{T, \Omega} \hat{\xi}^\perp = -\frac{|\xi|^2}{32\lambda_1^3(\eta; x, \xi)} \hat{\xi}^\perp \cdot \mathbf{H}_x \eta \hat{\xi}^\perp.$$

Besides, according to Eq. (5.142), one readily has

$$-\frac{i\hat{\xi}^\perp}{|\xi|} \cdot A_1^{T, \eta} [\nabla^\perp \psi](\eta; x, \xi) = -\nu_3^* [\nabla^\perp \psi](\eta; x) - i \nabla \psi(x) \cdot A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi).$$

This gives the desired result.  $\square$

### 5.3.3 Paralinearization of $S(\mathfrak{a}|\eta)$

Let us recall that the operator  $S(\mathfrak{a}|\eta)$  was first defined in Eq. (2.11), which we repeat here for sake of readability:

$$S(\mathfrak{a}|\eta) \psi := \nabla^\perp (1 + \mathfrak{a}\tau_0(\eta))^{-1} \psi, \quad (5.157)$$

where  $\tau_0(\eta)$  is defined in Eq. (2.12). We shall also use the projector onto divergence-free vector fields

$$\mathcal{P} := \nabla^\perp \Delta^{-1} \nabla^\perp, \quad \mathcal{P}(\xi) = \frac{\xi^\perp \otimes \xi^\perp}{|\xi|^2}. \quad (5.158)$$

**Lemma 5.23** (Uniform operator inverse for the canonical trace map). *There exist  $s_0 > 0$  and  $r_0 > 0$  such that, for every  $s \geq s_0$ , every  $\mathfrak{a} \in [-2, 2]$ , and every  $\eta \in B_{s_0, \mathbb{R}}(I; r_0) \cap \mathcal{C}(I; H^s)$ , the operators*

$$\text{Id} + \mathfrak{a}\tau_0(\eta) : H^s \longrightarrow H^s, \quad \text{Id} + \mathfrak{a}\mathcal{P}T(\eta) : (H^{s-1})^2 \longrightarrow (H^{s-1})^2$$

*are invertible, uniformly with respect to  $\mathfrak{a}$ . More precisely, for  $f \in H^s$ ,*

$$\begin{aligned} \sup_{\mathfrak{a} \in [-2, 2]} \left\| (\text{Id} + \mathfrak{a}\tau_0(\eta))^{-1} f \right\|_{H^s} &\lesssim_s \|f\|_{H^s} + \|\eta\|_{H^s} \|f\|_{H^{s_0}}, \\ \sup_{\mathfrak{a} \in [-2, 2]} \left\| \left( (\text{Id} + \mathfrak{a}\tau_0(\eta))^{-1} - \text{Id} \right) f \right\|_{H^s} &\lesssim_s \|\eta\|_{H^{s_0}} \|f\|_{H^s} + \|\eta\|_{H^s} \|f\|_{H^{s_0}}. \end{aligned} \quad (5.159)$$

For  $F \in (H^{s-1})^2$ ,

$$\begin{aligned} \sup_{\mathfrak{a} \in [-2, 2]} \left\| (\text{Id} + \mathfrak{a}\mathcal{P}T(\eta))^{-1} F \right\|_{H^{s-1}} &\lesssim_s \|F\|_{H^{s-1}} + \|\eta\|_{H^s} \|F\|_{H^{s_0-1}}, \\ \sup_{\mathfrak{a} \in [-2, 2]} \left\| \left( (\text{Id} + \mathfrak{a}\mathcal{P}T(\eta))^{-1} - \text{Id} \right) F \right\|_{H^{s-1}} &\lesssim_s \|\eta\|_{H^{s_0}} \|F\|_{H^{s-1}} + \|\eta\|_{H^s} \|F\|_{H^{s_0-1}}. \end{aligned} \quad (5.160)$$

Moreover, for every  $\bar{\Omega} \in \mathbb{R}^2$ ,

$$\|T(\eta) \bar{\Omega}\|_{H^{s-1}} \lesssim_s |\bar{\Omega}| \|\eta\|_{H^s}. \quad (5.161)$$

The same uniform tame bounds hold for differences and derivatives in  $\eta$ , and the inverses depend real-analytically on  $\mathfrak{a} \in [-2, 2]$  and on  $\eta$  in these seminorms. Consequently, the substitution  $f \mapsto (\text{Id} + \mathfrak{a}\tau_0(\eta))^{-1} f$  in the coefficient arguments of the symbol and remainder maps of Proposition 5.22 preserves their  $\Gamma_{\geq q}^m$  and  $\mathcal{R}^q$  classes, with the vanishing index  $q$  counted jointly in all displayed variables.

*Proof.* At low frequency, integration by parts in Eq. (5.154) gives

$$\tau_0(\eta) f(x) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{(x-y) \cdot \nabla \eta(y) - \eta(x) + \eta(y)}{(|x-y|^2 + (\eta(x) - \eta(y))^2)^{3/2}} f(y) dy.$$

The local kernel is bounded by  $C \|\mathbf{H}_x \eta\|_{L^\infty} |x-y|^{-1}$ ; at infinity its gradient term is Hilbert–Schmidt and its remaining terms satisfy Schur’s test. Thus  $\|\tau_0(\eta)\|_{\mathcal{L}(L^2)} \lesssim \|\eta\|_{H^{s_0}}$ . On high frequencies, the tame bounds for  $T(\eta)$  from Eq. (2.62), Propositions 5.14 and 5.15, and Theorem 4.15 and the definition  $\tau_0 = \Delta^{-1} \nabla^\perp \cdot T(\eta) \nabla^\perp$  give

$$\|\tau_0(\eta) f\|_{H^s} \lesssim_s \|\eta\|_{H^{s_0}} \|f\|_{H^s} + \|\eta\|_{H^s} \|f\|_{H^{s_0}}.$$

In particular, after decreasing  $r_0$ , one has  $\sup_{a \in [-2,2]} \|a \tau_0(\eta)\|_{\mathcal{L}(H^{s_0})} < 1/2$ . The Neumann series gives the inverse uniformly on  $H^{s_0}$ , and the tame estimate above, with the order-(-1) input term in Proposition 5.22, lifts it to every  $H^s$ , yielding Eq. (5.159). Differentiating the resolvent identity

$$d_\eta (\text{Id} + a \tau_0)^{-1} = -(\text{Id} + a \tau_0)^{-1} a (d_\eta \tau_0) (\text{Id} + a \tau_0)^{-1}$$

and iterating proves the difference and higher tame estimates. These estimates, together with the locally convergent Neumann expansion, give the stated analytic dependence and stability of the symbol and remainder classes and preserve their joint vanishing order.

The kernel of  $T(\eta)$  in Eq. (2.62) has the same local and tail bounds. Thus Proposition 5.21 and Theorem 4.15 and the boundedness of  $\mathcal{P}$  give

$$\|\mathcal{P} T(\eta) F\|_{H^{s-1}} \lesssim_s \|\eta\|_{H^{s_0}} \|F\|_{H^{s-1}} + \|\eta\|_{H^s} \|F\|_{H^{s_0-1}}.$$

The same Neumann-series and resolvent argument, using the order-(-1) input term in Proposition 5.21, proves Eq. (5.160).

For constant inputs, integration by parts in Eqs. (2.3a) and (2.3b) gives

$$\begin{aligned} \mathcal{B}_0(\eta) [1] &= -q(\eta) \nabla \eta, \\ q(\eta) [1] &= \mathcal{B}_0(\eta) \cdot \nabla \eta, \\ T(\eta) \bar{\Omega} &= \mathcal{B}_0(\eta) [\bar{\Omega} \cdot \nabla \eta] - \bar{\Omega} (\mathcal{B}_0(\eta) \cdot \nabla \eta) - \left( (q(\eta) \nabla \eta)^\perp \cdot \bar{\Omega} \right) \nabla^\perp \eta. \end{aligned} \tag{5.162}$$

The boundary terms vanish by oddness and decay, subtracting the flat kernel. Now Propositions 5.14 and 5.15 and Theorem 4.15 give Eq. (5.161); the same calculation after differentiation gives the asserted analytic dependence.  $\square$

**Proposition 5.24** (Paralinearization of  $S(a|\eta)$ ). *There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exists  $s_0 > 0$  and  $r_0 > 0$  such that for all  $\eta, \psi, \tilde{\psi} \in B_{s_0, \mathbb{R}}(I; r_0)$  and all  $\bar{\Omega} \in \mathbb{R}^2$  the following statements hold.*

1. The operator  $(1 + a \tau_0(\eta))^{-1} \psi$  admits the paralinearization

$$\begin{aligned} (1 + a \tau_0(\eta))^{-1} \psi &= \text{Op}^{\text{BW}} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} + a \left( - \begin{bmatrix} a_0^{\text{inv}, \eta} [\psi] (a|\eta; x, \xi) \\ a_{-1}^{\tau_0, \psi} (\eta; x, \xi) \end{bmatrix} + \begin{bmatrix} a_{[-1, \geq 1]} (a|\eta, \psi; x, \xi) \\ a_{[-2, \geq 1]} (a|\eta; x, \xi) \end{bmatrix} \right) \right) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix} \\ &\quad + a R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (a|\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}, \end{aligned} \tag{5.163}$$

where

- $a$  is the Atwood number defined in Eq. (1.5);
- $a_0^{\text{inv}, \eta} \in \mathcal{L}(H^{s_0}; \Gamma_{\geq 0}^0)$  is defined by

$$a_0^{\text{inv}, \eta} [\psi] (a|\eta; x, \xi) := -v_3 [\psi] (a|\eta; x) - i A_0^{\mathcal{B}_0, \omega, \perp} (\eta; x, \xi) \cdot S(a|\eta) \psi, \tag{5.164}$$

where  $v_3 [\psi] (a|\eta; x)$  is defined in Eq. (2.90),  $A_0^{\mathcal{B}_0, \omega}$  is defined in Eq. (5.8), and  $S(a|\eta)$  is defined in Eq. (5.157);

- $a_{-1}^{\tau_0, \psi} \in \Gamma_{\geq 1}^{-1}$  is defined in Eq. (5.156);
- $a_{[m, \geq 1]} \in \Gamma_{\geq 1}^m$ ,  $m \in \mathbb{R}$ , denotes a generic symbol as per Notation 5.1;
- $R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} \in (\mathcal{R}_{\geq 1}^\varrho)^2$  denotes a generic smoothing remainder as per Notation 5.1.

2. The operator  $S(\mathfrak{a}|\eta)\psi$  admits the parilinearization

$$S(\mathfrak{a}|\eta)\psi = \text{Op}^{\text{BW}} \left( A_1^S(\mathfrak{a}|\eta, \psi; x, \xi) + \mathfrak{a} \begin{bmatrix} A_{[0, \geq 1]}(\mathfrak{a}|\eta, \psi; x, \xi) & A_{[-1, \geq 1]}(\mathfrak{a}|\eta; x, \xi) \end{bmatrix} \begin{bmatrix} \eta \\ \psi \end{bmatrix} + \mathfrak{a} R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\mathfrak{a}|\eta, \psi) \begin{bmatrix} \eta \\ \psi \end{bmatrix} \right), \quad (5.165)$$

where

- $A_1^S \in (\Gamma_{\geq 0}^1)^{2 \times 2}$  is defined by

$$A_1^S(\mathfrak{a}|\eta, \psi; x, \xi) := \begin{bmatrix} A_1^{S, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) & A_1^{S, \psi}(\mathfrak{a}|\eta; x, \xi) \end{bmatrix},$$

with

$$A_1^{S, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) := -i\mathfrak{a}\xi^\perp a_0^{\text{inv}, \eta}[\psi](\mathfrak{a}|\eta; x, \xi), \quad (5.166)$$

and

$$A_1^{S, \psi}(\mathfrak{a}|\eta; x, \xi) := i\xi^\perp \left( 1 - \mathfrak{a} a_{-1}^{\tau_0, \psi}(\eta; x, \xi) \right); \quad (5.167)$$

- $A_{[m, \geq 1]} \in (\Gamma_{\geq 1}^m)^2$ ,  $m \in \mathbb{R}$ , denotes a generic vector symbol as per Notation 5.1;
- $R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} \in (\mathcal{R}_{\geq 1}^\varrho)^{2 \times 2}$  denotes a generic smoothing remainder as per Notation 5.1.

3. Let  $\bar{\psi}$  be the affine background potential defined in Eq. (2.7). The canonical background vorticity  $\tilde{\Omega}[\bar{\Omega}](\eta)$ , defined in Eq. (2.10), satisfies the exact identity

$$\tilde{\Omega}[\bar{\Omega}](\eta) = (\text{Id} + \mathfrak{a}\mathcal{P}T(\eta))^{-1} \bar{\Omega}. \quad (5.168)$$

Moreover,

$$\tilde{\Omega}[\bar{\Omega}](\eta) = \bar{\Omega} + \text{Op}^{\text{BW}} \left( A_{1, \text{bg}}^{S, \eta}[\bar{\Omega}](\mathfrak{a}|\eta; x, \xi) + A_{[0, \geq 1]}(\bar{\Omega})(\mathfrak{a}|\eta; x, \xi) \right) \eta + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}[\bar{\Omega}](\mathfrak{a}|\eta) \eta, \quad (5.169)$$

where

- $A_{1, \text{bg}}^{S, \eta} \in \mathcal{L}(\mathbb{R}^2; (\Gamma_{\geq 0}^1)^2)$  is linear in the separately estimated variable and is defined by

$$\begin{aligned} A_{1, \text{bg}}^{S, \eta}[\bar{\Omega}](\mathfrak{a}|\eta; x, \xi) &:= -\mathfrak{a}\mathcal{P}(\xi) A_1^{T, \eta}[\tilde{\Omega}[\bar{\Omega}](\eta)](\eta; x, \xi) \\ &= -\mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \tilde{\Omega}[\bar{\Omega}](\eta; x) - i\nu_3^*[\tilde{\Omega}[\bar{\Omega}](\eta)](\eta; x) \right) \xi^\perp; \end{aligned} \quad (5.170)$$

- $A_{[0, \geq 1]}[\bar{\Omega}] \in \mathcal{L}(\mathbb{R}^2; (\Gamma_{\geq 1}^0)^2)$  is linear in  $\bar{\Omega}$ ;
- $R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}[\bar{\Omega}] \in \mathcal{L}(\mathbb{R}^2; (\mathcal{R}_{\geq 1}^\varrho)^2)$  is linear in  $\bar{\Omega}$ .

**Lemma 5.25** (Low-frequency trace symbols). *For every  $\varrho \geq 0$ , one has*

$$\text{Op}^{\text{BW}} \left( \left( a_{-1}^{\tau_0, \psi} \right)^{\flat} \right) \in \mathcal{R}_{\geq 1}^\varrho, \quad \text{Op}^{\text{BW}} \left( \left( A_{-1}^{T, \Omega} \right)^{\flat} \right), \text{Op}^{\text{BW}} \left( \left( \mathcal{P}(\xi) A_{-1}^{T, \Omega} \right)^{\flat} \right) \in (\mathcal{R}_{\geq 1}^\varrho)^{2 \times 2}, \quad (5.171)$$

with  $^{\flat}$  as in Notation 4.10 and the symbols defined in Eqs. (5.143), (5.156) and (5.158). These operators are real-to-real.

*Proof.* By Eq. (5.156), for  $\hat{\xi} \in \mathbb{S}^1$ ,

$$a_{-1}^{\tau_0, \psi}(\eta; x, \hat{\xi}) = -\frac{\hat{\xi}^\perp \cdot \mathbf{H}_x \eta \hat{\xi}^\perp}{4 \left(1 + (\nabla^\perp \eta \cdot \hat{\xi})^2\right)^{3/2}}.$$

Differentiating with respect to the angular variable on  $\mathbb{S}^1$  gives, pointwise in  $x$ ,

$$\left\| a_{-1}^{\tau_0, \psi}(\eta; x, \bullet) \right\|_{H^1(\mathbb{S}^1)} \lesssim |\mathbf{H}_x \eta| (1 + |\nabla \eta|).$$

Hence, fixing  $s_0 > 2$  and  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$ , Sobolev embedding in dimension two yields

$$\left\| a_{-1}^{\tau_0, \psi}(\eta; \bullet, \bullet) \right\|_{L_x^2 H_\xi^1(\mathbb{S}^1)} \lesssim \|\eta\|_{H^{s_0}}. \quad (5.172)$$

Since  $a_{-1}^{\tau_0, \psi}$  is homogeneous of degree  $-1$  in its frequency variable, denoting by  $\widehat{\phantom{x}}$  the Fourier transform in  $x$  only, one has

$$\widehat{a_{-1}^{\tau_0, \psi}}(\eta; \theta, \zeta) = |\zeta|^{-1} \widehat{a_{-1}^{\tau_0, \psi}}(\eta; \theta, \hat{\xi}), \quad \zeta \neq 0. \quad (5.173)$$

By Eq. (4.8) and Notation 4.10,

$$\text{Op}^{\text{BW}}\left(\widehat{\left(a_{-1}^{\tau_0, \psi}\right)^b}(\eta)\right) h(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} K_\eta(\xi, \varsigma) \widehat{h}(\varsigma) d\varsigma,$$

where

$$K_\eta(\xi, \varsigma) := \chi_\epsilon\left(\frac{2(\xi - \varsigma)}{|\xi + \varsigma|}\right) \chi\left(\frac{\xi + \varsigma}{2}\right) \widehat{a_{-1}^{\tau_0, \psi}}\left(\eta; \xi - \varsigma, \frac{\xi + \varsigma}{2}\right). \quad (5.174)$$

On the support of this kernel, Eqs. (4.8) and (4.9) gives

$$|\xi - \varsigma| \lesssim |\varsigma|, \quad |\xi + \varsigma| \sim |\varsigma|, \quad (5.175)$$

and the analogous estimates with  $\xi$  and  $\varsigma$  interchanged.

We first verify the first condition in Eq. (4.10). For  $\varsigma \neq 0$ , set  $\theta = \xi - \varsigma$ . Using Eqs. (5.173) and (5.175), Cauchy–Schwarz, and the fact that  $|B(0, C|\varsigma|)|^{1/2} \lesssim |\varsigma|$  in dimension two, we obtain

$$\int_{\mathbb{R}^2} |K_\eta(\xi, \varsigma)| d\xi \lesssim |\varsigma|^{-1} \int_{|\theta| \lesssim |\varsigma|} \sup_{\hat{\xi} \in \mathbb{S}^1} \left| \widehat{a_{-1}^{\tau_0, \psi}}(\eta; \theta, \hat{\xi}) \right| d\theta \lesssim \left( \int_{\mathbb{R}^2} \sup_{\hat{\xi} \in \mathbb{S}^1} \left| \widehat{a_{-1}^{\tau_0, \psi}}(\eta; \theta, \hat{\xi}) \right|^2 d\theta \right)^{1/2}.$$

Using  $H^1(\mathbb{S}^1) \hookrightarrow L^\infty(\mathbb{S}^1)$ , Plancherel's identity, and Eq. (5.172), this gives

$$\sup_{\varsigma \in \mathbb{R}^2} \int_{\mathbb{R}^2} |K_\eta(\xi, \varsigma)| d\xi \lesssim \|\eta\|_{H^{s_0}}. \quad (5.176)$$

For  $\varsigma = 0$  the kernel vanishes by the support of  $\chi_\epsilon$ , so the same estimate holds without restriction. Fixing instead  $\xi \neq 0$  and making the same change of variables  $\theta = \xi - \varsigma$ , the support condition gives

$$|\theta| \lesssim |\xi|, \quad |\xi + \varsigma| \sim |\xi|.$$

The same argument therefore gives the second Schur bound

$$\sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} |K_\eta(\xi, \varsigma)| d\varsigma \lesssim \|\eta\|_{H^{s_0}}. \quad (5.177)$$

Again, the estimate is trivial for  $\xi = 0$ . Applying Lemma 4.12 to Eq. (5.174), with Eqs. (5.176) and (5.177), and then using Plancherel's identity, we obtain

$$\left\| \text{Op}^{\text{BW}}\left(\widehat{\left(a_{-1}^{\tau_0, \psi}\right)^b}(\eta)\right) h \right\|_{L^2} \lesssim \|\eta\|_{H^{s_0}} \|h\|_{L^2}.$$

The cutoff  $\chi\left(\frac{\xi+\varsigma}{2}\right)$  implies  $|\xi + \varsigma| \lesssim 1$  on the support of  $K_\eta$ . Together with Eq. (5.175), this yields  $|\xi| + |\varsigma| \lesssim 1$  on  $\text{supp} K_\eta$ . Consequently, for every  $s \geq s_0$  and every  $\varrho \geq 0$   $\langle \xi \rangle^s \langle \varsigma \rangle^{-(s-\varrho)} \lesssim_{s,\varrho} 1$  on  $\text{supp} K_\eta$ . Hence the weighted kernel  $K_{\eta,s,\varrho}(\xi, \varsigma) := \langle \xi \rangle^s K_\eta(\xi, \varsigma) \langle \varsigma \rangle^{-(s-\varrho)}$  satisfies, by Eqs. (5.176) and (5.177), the same Schur estimates:

$$\sup_{\varsigma \in \mathbb{R}^2} \int_{\mathbb{R}^2} |K_{\eta,s,\varrho}(\xi, \varsigma)| d\xi + \sup_{\xi \in \mathbb{R}^2} \int_{\mathbb{R}^2} |K_{\eta,s,\varrho}(\xi, \varsigma)| d\varsigma \lesssim_{s,\varrho} \|\eta\|_{H^{s_0}}.$$

Applying Lemma 4.12 to  $K_{\eta,s,\varrho}$  and using Plancherel's identity gives

$$\left\| \text{Op}^{\text{BW}} \left( \left( a_{-1}^{T_0, \psi} \right)^b (\eta) \right) h \right\|_{H^s} \lesssim_{s,\varrho} \|\eta\|_{H^{s_0}} \|h\|_{H^{s-\varrho}}. \quad (5.178)$$

We now consider  $\mathbf{A}_{-1}^{T,\Omega}$ . By Eqs. (5.101) and (5.143), its entries are homogeneous of degree  $-1$  in  $\xi$  and, after restriction to  $\mathbb{S}^1$ , are linear in  $\mathbf{H}_x \eta$ , with coefficients depending smoothly on  $(\nabla \eta, \hat{\xi})$ . Consequently,

$$\left\| \mathbf{A}_{-1}^{T,\Omega}(\eta; \bullet, \bullet) \right\|_{L_x^2 H_\xi^1(\mathbb{S}^1)} \lesssim \|\eta\|_{H^{s_0}}.$$

The argument leading from Eq. (5.172) to Eq. (5.178) therefore applies componentwise and gives

$$\text{Op}^{\text{BW}} \left( \left( \mathbf{A}_{-1}^{T,\Omega} \right)^b \right) \in (\mathcal{R}_{\geq 1}^\varrho)^{2 \times 2}.$$

Finally,  $\mathcal{P}(\xi)$  is smooth and homogeneous of degree zero on  $\mathbb{R}^2 \setminus \{0\}$ . Hence  $\mathcal{P}(\xi) \mathbf{A}_{-1}^{T,\Omega}$  satisfies the same homogeneity and angular estimate as  $\mathbf{A}_{-1}^{T,\Omega}$ , and the same argument gives

$$\text{Op}^{\text{BW}} \left( \left( \mathcal{P}(\xi) \mathbf{A}_{-1}^{T,\Omega} \right)^b \right) \in (\mathcal{R}_{\geq 1}^\varrho)^{2 \times 2}.$$

All the estimates above have the fixed baseline required in Definition 4.5 and vanish to order at least one in  $\eta$ . This proves Eq. (5.171). The three symbols are real and even in  $\xi$ ; since the cutoffs in Eqs. (4.7) and (4.8) are real and even, the corresponding Bony–Weyl operators are real-to-real.  $\square$

*Proof of Proposition 5.24.*

**Part 1** (Sobolev input). Applying Proposition 5.22, we have

$$\begin{aligned} & (1 + \mathfrak{a} \tau_0(\eta)) \psi \\ &= \text{Op}^{\text{BW}} \left( 1 + \mathfrak{a} \left( a_{-1}^{T_0, \psi} + a_{[-2, \geq 1]} \right) (\eta; x, \xi) \right) \psi + \mathfrak{a} \text{Op}^{\text{BW}} \left( a_0^{T_0, \eta} [\psi] (\eta; x, \xi) + a_{[-1, \geq 1]} (\eta, \psi; x, \xi) \right) \eta \\ & \quad + \mathfrak{a} R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \end{aligned}$$

By Lemma 5.23, at a fixed Sobolev index and after increasing  $s_0$ , for every  $s \geq s_0$ ,

$$\left\| \chi(D/4) \tau_0(\eta) h \right\|_{H^s} + \left\| \chi(D/4) T(\eta) H \right\|_{H^s} \lesssim_{s,\varrho} \|\eta\|_{H^{s_0}} (\|h\|_{H^{s-\varrho}} + \|H\|_{H^{s-\varrho}}). \quad (5.179)$$

By Lemma 5.25, the contributions obtained by applying  $\chi(D/4)$  to the concrete symbols below are smoothing. We may therefore consider their contribution after applying  $1 - \chi(D/4)$ .

Indeed, by Eq. (4.9), on the Fourier support of

$$(1 - \chi(D/4)) \text{Op}^{\text{BW}}(a)$$

one has

$$\left| \frac{\xi + \varsigma}{2} \right| \gtrsim |\xi| \gtrsim 1.$$

Hence the frequency argument of  $a$  stays away from the origin, and Notation 4.10 gives

$$(1 - \chi(D/4)) \text{Op}^{\text{BW}}(a) = (1 - \chi(D/4)) \text{Op}^{\text{BW}}(a^\sharp).$$

Thus the symbols appearing in the high-frequency contribution may be replaced by their fixed high-frequency extensions, while retaining the notation of Notation 5.1.

By Lemma 5.25, replace  $a_{-1}^{\tau_0, \psi}$  by  $\left(a_{-1}^{\tau_0, \psi}\right)^\sharp$  in the input symbol. By Eq. (5.156), the latter vanishes for  $|\xi| \leq 1.1$  and, after decreasing  $r_0$ ,

$$\sup_{|\alpha| \leq 2, x, \xi} \left| \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp \right| \leq \frac{1}{2}, \quad \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp} - 1 \in \Gamma_{\geq 1}^{-1}.$$

Observe that

$$\left\{ \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp}, 1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp \right\} = \left\{ \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp} - 1, 1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp \right\} = 0.$$

Therefore, composing the previous identity on the left with  $\text{Op}^{\text{BW}} \left( \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp} (\eta; x, \xi) \right)$  gives, by virtue of Proposition 4.24 and Lemma 4.25,

$$\begin{aligned} & \text{Op}^{\text{BW}} \left( \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp} (\eta; x, \xi) \right) (1 + \mathfrak{a} \tau_0 (\eta)) \psi \\ &= \psi + \mathfrak{a} \text{Op}^{\text{BW}} (a_{[-2, \geq 1]} (\mathfrak{a} |\eta; x, \xi)) \psi + \mathfrak{a} \text{Op}^{\text{BW}} (a_0^{\tau_0, \eta} [\psi] (\eta; x, \xi) + a_{[-1, \geq 1]} [\psi] (\mathfrak{a} |\eta; x, \xi)) \eta \\ &+ \mathfrak{a} R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\mathfrak{a} |\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \end{aligned}$$

We deduce that

$$\begin{aligned} \psi &= (1 + \mathfrak{a} \tau_0 (\eta))^{-1} (1 + \mathfrak{a} \tau_0 (\eta)) \psi \\ &= \text{Op}^{\text{BW}} \left( \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp} (\eta; x, \xi) \right) (1 + \mathfrak{a} \tau_0 (\eta)) \psi + \mathfrak{a} \text{Op}^{\text{BW}} (a_{[-2, \geq 1]} (\mathfrak{a} |\eta; x, \xi)) \psi \\ &+ \mathfrak{a} \text{Op}^{\text{BW}} (-a_0^{\tau_0, \eta} [\psi] (\eta; x, \xi) + a_{[-1, \geq 1]} [\psi] (\mathfrak{a} |\eta; x, \xi)) \eta + \mathfrak{a} R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\mathfrak{a} |\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \end{aligned} \tag{5.180}$$

Denoting

$$\psi_1 := (1 + \mathfrak{a} \tau_0 (\eta)) \psi, \tag{5.181}$$

then using Lemma 4.25 and Eq. (5.181) in Eq. (5.180) we infer that

$$\begin{aligned} & (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1 \\ &= \text{Op}^{\text{BW}} \left( \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^\sharp} (\eta; x, \xi) \right) \psi_1 + \mathfrak{a} \text{Op}^{\text{BW}} (a_{[-2, \geq 1]} (\mathfrak{a} |\eta; x, \xi)) (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1 \\ &+ \mathfrak{a} \text{Op}^{\text{BW}} (-a_0^{\tau_0, \eta} (\eta, (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1; x, \xi) + a_{[-1, \geq 1]} (\mathfrak{a} |\eta, (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1; x, \xi)) \eta \\ &+ \mathfrak{a} R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\mathfrak{a} |\eta, (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1) \cdot \begin{bmatrix} \eta \\ (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1 \end{bmatrix}. \end{aligned} \tag{5.182}$$

Notice that Lemma 5.23, the definition of symbols and smoothing remainders (cf. Definitions 4.5 and 4.7) imply that, substituting only in the coefficient arguments, using Notation 2.1

$$\begin{aligned} a_0^{\text{inv}, \eta} [\psi_1] (\mathfrak{a} |\eta; x, \xi) &:= a_0^{\tau_0, \eta} (\eta, (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1; x, \xi) && \in \Gamma_{\geq 0}^0, \\ a_{[m, \geq 1]} (\mathfrak{a} |\eta, \psi_1; x, \xi) &:= a_{[m, \geq 1]} (\mathfrak{a} |\eta, (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1; x, \xi) && \in \Gamma_{\geq 1}^m, \\ R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\mathfrak{a} |\eta, \psi_1) &:= R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\mathfrak{a} |\eta, (1 + \mathfrak{a} \tau_0 (\eta))^{-1} \psi_1) && \in (\mathcal{R}_{\geq 1}^\varrho)^2. \end{aligned} \tag{5.183}$$

Recalling Eqs. (2.11) and (5.155) we obtain Eq. (5.164) from the first equation of Eq. (5.183). Next, plugging Eq. (5.183) into Eq. (5.182) and moving the smoothing term acting on  $(1 + \mathfrak{a}\tau_0(\eta))^{-1}\psi_1$  to the left, we get

$$\begin{aligned} & \left( \text{Id} + \mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}}(\mathfrak{a}|\eta) \right) (1 + \mathfrak{a}\tau_0(\eta))^{-1} \psi_1 \\ &= \text{Op}^{\text{BW}} \left( \frac{1}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^{\sharp}(\eta; x, \xi)} \right) \psi_1 + \mathfrak{a} \text{Op}^{\text{BW}} \left( a_{[-2, \geq 1]}(\mathfrak{a}|\eta; x, \xi) \right) (1 + \mathfrak{a}\tau_0(\eta))^{-1} \psi_1 \\ &+ \mathfrak{a} \text{Op}^{\text{BW}} \left( -a_0^{\text{inv}, \eta}[\psi_1](\mathfrak{a}|\eta; x, \xi) + a_{[-1, \geq 1]}(\mathfrak{a}|\eta, \psi_1; x, \xi) \right) \eta \\ &+ \mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}}[\psi_1](\mathfrak{a}|\eta)\eta. \end{aligned} \quad (5.184)$$

We prove by induction that for any  $n \in \mathbb{N}^*$ , there exists  $b_{[-2^n]} \in \Gamma_{\geq 1}^{-2^n}$ ,  $c_{[-1]}^{[n]}, a_{[-1, \geq 1]}^{[n]} \in \Gamma_{\geq 1}^{-1}$ ,  $r_{[\varrho, \geq 1]}^{[n], \mathbb{R}} \in \mathcal{R}_{\geq 1}^{\varrho}$  and  $R_{[\varrho, \geq 1]}^{\text{lin}, [n], \mathbb{R}} \in (\mathcal{R}_{\geq 1}^{\varrho})^2$  such that

$$\begin{aligned} & \left( \text{Op}^{\text{BW}}(1 + b_{[-2^n]}(\mathfrak{a}|\eta; x, \xi)) + \mathfrak{a}r_{[\varrho, \geq 1]}^{[n], \mathbb{R}}(\mathfrak{a}|\eta) \right) (1 + \mathfrak{a}\tau_0(\eta))^{-1} \psi_1 \\ &= \text{Op}^{\text{BW}} \left( 1 + c_{[-1]}^{[n]}(\mathfrak{a}|\eta; x, \xi) \right) \psi_1 + \mathfrak{a} \text{Op}^{\text{BW}} \left( -a_0^{\text{inv}, \eta}[\psi_1](\mathfrak{a}|\eta; x, \xi) + a_{[-1, \geq 1]}^{[n]}(\mathfrak{a}|\eta, \psi_1; x, \xi) \right) \eta \\ &+ \mathfrak{a}R_{[\varrho, \geq 1]}^{\text{lin}, [n], \mathbb{R}}(\mathfrak{a}|\eta, \psi_1) \cdot \begin{bmatrix} \eta \\ \psi_1 \end{bmatrix}. \end{aligned} \quad (5.185)$$

Here  $b_{[-2^n]} = \mathfrak{a}a_{[-2, \geq 1]}$  vanishes near  $\xi = 0$ . In addition, the symbol  $c_{[-1]}^{[n]}$  admits the expansion

$$c_{[-1]}^{[n]}(\mathfrak{a}|\eta; x, \xi) = -\mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^{\sharp}(\eta; x, \xi) + \mathfrak{a}a_{[-2, \geq 1]}(\mathfrak{a}|\eta; x, \xi).$$

The initialization  $n = 1$  is a consequence of Eq. (5.184), with

$$b_{[-2]} := -\mathfrak{a}a_{[-2, \geq 1]}, \quad a_{[-1, \geq 1]}^{[1]} := a_{[-1, \geq 1]}, \quad r_{[\varrho, \geq 1]}^{[1], \mathbb{R}} := r_{[\varrho, \geq 1]}^{\mathbb{R}}, \quad R_{[\varrho, \geq 1]}^{\text{lin}, [1], \mathbb{R}} := \begin{bmatrix} r_{[\varrho, \geq 1]}^{\mathbb{R}}[\psi_1] \\ 0 \end{bmatrix},$$

and

$$c_{[-1]}^{[1]}(\mathfrak{a}|\eta; x, \xi) := -\frac{\mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^{\sharp}(\eta; x, \xi)}{1 + \mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^{\sharp}(\eta; x, \xi)} = -\mathfrak{a} \left( a_{-1}^{\tau_0, \psi} \right)^{\sharp}(\eta; x, \xi) + \mathfrak{a}a_{[-2, \geq 1]}(\mathfrak{a}|\eta; x, \xi).$$

At each of the finitely many steps, decrease  $r_0$  so that  $|b_{[-2^n]}| \leq 1/2$ . Then Proposition 4.24 implies

$$\text{Op}^{\text{BW}} \left( \frac{1}{1 + b_{[-2^n]}(\mathfrak{a}|\eta; x, \xi)} \right) \text{Op}^{\text{BW}}(1 + b_{[-2^n]}(\mathfrak{a}|\eta; x, \xi)) = \text{Op}^{\text{BW}}(1 + b_{[-2^{n+1}]}(\mathfrak{a}|\eta; x, \xi)) + \mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}},$$

where  $r_{[\varrho, \geq 1]}^{\mathbb{R}} \in \mathcal{R}_{\geq 1}^{\varrho}$  and

$$b_{[-2^{n+1}]} := \left( \frac{1}{1 + b_{[-2^n]}(\mathfrak{a}|\eta; x, \xi)} \right)^{\sharp} \#_{\varrho} (1 + b_{[-2^n]}(\mathfrak{a}|\eta; x, \xi)) - 1 \in \Gamma_{\geq 1}^{-2^{n+1}}.$$

This defect still vanishes near  $\xi = 0$ . Composing Eq. (5.185) on the left with this reciprocal proves the induction argument by Proposition 4.24 and Lemma 4.25, retaining all smoothing operators acting on  $(1 + \mathfrak{a}\tau_0(\eta))^{-1}\psi_1$  on the left. By taking  $n = \underline{n} > \log_2(\varrho)$ , one has  $\text{Op}^{\text{BW}}(b_{[-2^n]}) \in \mathcal{R}_{\geq 1}^{\varrho}$ . Thus the left-hand side of Eq. (5.185) becomes

$$\left( \text{Id} + \mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}}(\mathfrak{a}|\eta) \right) (1 + \mathfrak{a}\tau_0(\eta))^{-1} \psi_1.$$

For each  $s$ , decrease the radius in Definition 4.5 so that  $\|\mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}}\|_{\mathcal{L}(H^s)} < 1/2$ . The Neumann series and

$$\begin{aligned} & \left( \text{Id} + \mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}} \right)^{-1} - \text{Id} = -\mathfrak{a} \left( \text{Id} + \mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}} \right)^{-1} r_{[\varrho, \geq 1]}^{\mathbb{R}}, \\ & \left\| \left( \left( \text{Id} + \mathfrak{a}r_{[\varrho, \geq 1]}^{\mathbb{R}} \right)^{-1} - \text{Id} \right) h \right\|_{H^s} \lesssim_{s, \varrho} |\mathfrak{a}| \|\eta\|_{H^{s_0}} \|h\|_{H^{s-\varrho}} \end{aligned} \quad (5.186)$$

give a correction in  $\mathcal{R}_{\geq 1}^\varrho$ , also for finite matrix operators. Inverting this factor in Eq. (5.185) and using Lemmas 4.25 and 5.25 with

$$\text{Op}^{\text{BW}}\left(\left(a_{-1}^{\tau_0, \psi}\right)^\sharp\right) = \text{Op}^{\text{BW}}\left(a_{-1}^{\tau_0, \psi}\right) - \text{Op}^{\text{BW}}\left(\left(a_{-1}^{\tau_0, \psi}\right)^\flat\right)$$

proves Eq. (5.163). To prove Eq. (5.165) with target index  $\varrho$ , we apply Eq. (5.163) first with index  $\varrho + 1$  and then use Eq. (2.11) and Proposition 4.24. Indeed,  $\nabla^\perp = \text{Op}^{\text{BW}}(\mathbf{i}\xi^\perp)$  maps  $\mathcal{R}_{\geq 1}^{\varrho+1}$  into  $(\mathcal{R}_{\geq 1}^\varrho)^2$ ; relabeling the index gives the stated remainder.

**Part 2** (Affine background). Set

$$\begin{aligned} \delta\Omega[\bar{\Omega}](\eta) &:= \tilde{\Omega}[\bar{\Omega}](\eta) - \bar{\Omega}, \\ (\text{Id} + \alpha\mathcal{P}T(\eta))\delta\Omega[\bar{\Omega}](\eta) &= -\alpha\mathcal{P}T(\eta)\bar{\Omega}, \\ \|\delta\Omega[\bar{\Omega}](\eta)\|_{H^{s-1}} &\lesssim_s |\bar{\Omega}| \|\eta\|_{H^s}, \end{aligned} \tag{5.187}$$

by Eqs. (5.160) and (5.161). This gives the unique solution in  $\bar{\Omega} + (H^{s-1})^2$  of

$$\tilde{\Omega}[\bar{\Omega}](\eta) + \alpha\mathcal{P}T(\eta)\tilde{\Omega}[\bar{\Omega}](\eta) = \bar{\Omega}. \tag{5.188}$$

Since  $\mathcal{P}\delta\Omega = \delta\Omega$ , this is the  $\nabla^\perp$  derivative of the canonical trace equation modulo constants, by Eq. (2.12); it proves Eq. (5.168). Equivalently,

$$\tilde{\Omega}[\bar{\Omega}](\eta) = \bar{\Omega} - \alpha\mathcal{P}T(\eta)\tilde{\Omega}[\bar{\Omega}](\eta).$$

Since  $T(\eta)$  is linear in its vorticity argument,  $\tilde{\Omega}[\bar{\Omega}](\eta)$  is linear in  $\bar{\Omega}$ . It suffices to work with  $|\bar{\Omega}| \leq 1$  and extend by linearity. After decreasing  $r_0$ , Eq. (5.187) places  $\delta\Omega[\bar{\Omega}](\eta)$  in the ball required by Proposition 5.21. We use smoothing index  $\varrho + 1$  below.

For the constant part, apply Propositions 5.14 and 5.15 to the Sobolev inputs in Eq. (5.162), with index  $\varrho + 2$ , and use Eqs. (5.142) and (5.151). By Eqs. (2.62) and (3.9),

$$d_\eta T(0)[h]\bar{\Omega} = -\frac{1}{2} \text{Op}^{\text{BW}}\left(\frac{\xi^\perp(\bar{\Omega} \cdot \xi^\perp)}{|\xi|}\right)h = \text{Op}^{\text{BW}}\left(A_1^{T, \eta}[\bar{\Omega}](0; x, \xi)\right)h.$$

Subtracting the flat terms therefore gives lower-order symbols and remainders vanishing at  $\eta = 0$ . In particular,

$$\chi(D/4)\left(T(\eta)\bar{\Omega} - \text{Op}^{\text{BW}}\left(A_1^{T, \eta}[\bar{\Omega}](\eta)\right)\eta\right) = R_{[\varrho+1, \geq 1]}^{\mathbb{R}}[\bar{\Omega}](\eta)\eta:$$

integrate the derivative along  $t\eta$ ,  $0 \leq t \leq 1$ , and use the fixed-index analytic bounds in Lemma 5.23 and compact output support, increasing the baseline.

Apply Eq. (5.141) to  $\delta\Omega[\bar{\Omega}](\eta)$  and add the constant part. By Lemma 5.25 and Eq. (5.179),

$$\begin{aligned} T(\eta)\tilde{\Omega}[\bar{\Omega}](\eta) &= \text{Op}^{\text{BW}}\left(\left(A_{-1}^{T, \Omega}\right)^\sharp(\eta; x, \xi) + A_{[-2, \geq 1]}(\eta; x, \xi)\right)\delta\Omega[\bar{\Omega}](\eta) \\ &\quad + \text{Op}^{\text{BW}}\left(A_1^{T, \eta}[\tilde{\Omega}[\bar{\Omega}](\eta)](\eta; x, \xi) + A_{[0, \geq 1]}[\bar{\Omega}](\eta; x, \xi)\right)\eta \\ &\quad + R_{[\varrho+1, \geq 1]}^{\mathbb{R}}(\alpha|\eta)\delta\Omega[\bar{\Omega}](\eta) + R_{[\varrho+1, \geq 1]}^{\text{lin}, \mathbb{R}}[\bar{\Omega}](\eta)\eta, \end{aligned} \tag{5.189}$$

The matrix symbols and remainders depend only on  $\eta$ ; the vector symbols and remainders are linear in  $\bar{\Omega}$ . Applying  $\mathcal{P}$  to Eq. (5.189), with both factors localized at high frequency in each symbolic composition, yields the following identity. The low-frequency terms are controlled by Lemma 5.25 and the boundedness of  $\mathcal{P}$ ; for the order-one column use the same Schur estimate, retaining its flat multiplier exactly.

$$\begin{aligned} \mathcal{P}T(\eta)\tilde{\Omega}[\bar{\Omega}](\eta) &= \text{Op}^{\text{BW}}\left(\left(\mathcal{P}(\xi)A_{-1}^{T, \Omega}\right)^\sharp(\eta; x, \xi) + A_{[-2, \geq 1]}[\alpha](\eta; x, \xi)\right)\delta\Omega[\bar{\Omega}](\eta) \\ &\quad + \text{Op}^{\text{BW}}\left(\mathcal{P}(\xi)A_1^{T, \eta}[\tilde{\Omega}[\bar{\Omega}](\eta)](\eta; x, \xi) + A_{[0, \geq 1]}[\bar{\Omega}](\alpha|\eta; x, \xi)\right)\eta \\ &\quad + R_{[\varrho+1, \geq 1]}^{\mathbb{R}}(\alpha|\eta)\delta\Omega[\bar{\Omega}](\eta) + R_{[\varrho+1, \geq 1]}^{\text{lin}, \mathbb{R}}[\bar{\Omega}](\alpha|\eta)\eta. \end{aligned} \tag{5.190}$$

Plugging Eq. (5.190) into Eq. (5.188) gives the corrected first reduction

$$\begin{aligned}
& \left( \text{Op}^{\text{BW}} \left( \text{Id} + \mathfrak{a} \left( \mathcal{P}(\xi) A_{-1}^{T, \Omega} \right)^{\sharp} (\eta; x, \xi) \right) + R_{[\varrho+1, \geq 1]}^{\mathbb{R}} (\mathfrak{a}|\eta) \right) \delta \Omega [\bar{\Omega}] (\eta) \\
&= -\mathfrak{a} \text{Op}^{\text{BW}} \left( \mathcal{P}(\xi) A_1^{T, \eta} [\bar{\Omega} [\bar{\Omega}]] (\eta; x, \xi) + A_{[0, \geq 1]} [\bar{\Omega}] (\mathfrak{a}|\eta; x, \xi) \right) \eta \\
&+ \mathfrak{a} \text{Op}^{\text{BW}} \left( A_{[-2, \geq 1]} [\mathfrak{a}] (\eta; x, \xi) \right) \delta \Omega [\bar{\Omega}] (\eta) + R_{[\varrho+1, \geq 1]}^{\text{lin}, \mathbb{R}} [\bar{\Omega}] (\mathfrak{a}|\eta) \eta.
\end{aligned} \tag{5.191}$$

Define

$$P_{[0]} (\mathfrak{a}|\eta; x, \xi) := \left( \text{Id} + \mathfrak{a} \left( \mathcal{P}(\xi) A_{-1}^{T, \Omega} \right)^{\sharp} (\eta; x, \xi) \right)^{-1} \in (\Gamma_{\geq 0}^0)^{2 \times 2}.$$

After decreasing  $r_0$ ,  $\sup_{|\mathfrak{a}| \leq 2, x, \xi} \left\| \mathfrak{a} \left( \mathcal{P}(\xi) A_{-1}^{T, \Omega} \right)^{\sharp} \right\| \leq 1/2$ . Thus  $P_{[0]} - \text{Id} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}$  vanishes near  $\xi = 0$ . By Proposition 4.24 and Lemma 4.25, composing Eq. (5.191) on the left with  $\text{Op}^{\text{BW}} (P_{[0]})$  gives

$$\begin{aligned}
& \left( \text{Id} + R_{[\varrho+1, \geq 1]}^{[1], \mathbb{R}} (\mathfrak{a}|\eta) \right) \delta \Omega [\bar{\Omega}] (\eta) \\
&= \text{Op}^{\text{BW}} \left( A_{1, \text{bg}}^{S, \eta} [\bar{\Omega}] (\mathfrak{a}|\eta; x, \xi) + A_{[0, \geq 1]} [\bar{\Omega}] (\mathfrak{a}|\eta; x, \xi) \right) \eta \\
&+ \mathfrak{a} \text{Op}^{\text{BW}} \left( A_{[-2, \geq 1]}^{[1]} [\mathfrak{a}] (\eta; x, \xi) \right) \delta \Omega [\bar{\Omega}] (\eta) + R_{[\varrho+1, \geq 1]}^{\text{lin}, [1], \mathbb{R}} [\bar{\Omega}] (\mathfrak{a}|\eta) \eta,
\end{aligned} \tag{5.192}$$

where  $R_{[\varrho+1, \geq 1]}^{[1], \mathbb{R}} \in \left( \mathcal{R}_{\geq 1}^{\varrho+1} \right)^{2 \times 2}$  and

$$A_{1, \text{bg}}^{S, \eta} [\bar{\Omega}] (\mathfrak{a}|\eta; x, \xi) := -\mathfrak{a} \mathcal{P}(\xi) A_1^{T, \eta} [\bar{\Omega} [\bar{\Omega}]] (\eta; x, \xi).$$

Indeed, the factor  $P_{[0]} - \text{Id} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}$  only produces order-zero corrections when it is composed with the order-one symbol  $\mathcal{P}(\xi) A_1^{T, \eta} [\bar{\Omega} [\bar{\Omega}]] (\eta)$ .

We now remove the residual term  $\text{Op}^{\text{BW}} \left( A_{[-2, \geq 1]}^{[1]} \right) \delta \Omega [\bar{\Omega}] (\eta)$  by the same parametrix iteration used in the Sobolev-input part. We prove by induction that, for every  $n \in \mathbb{N}^*$ , there exist

$$P_{[-2^n]} \in \left( \Gamma_{\geq 1}^{-2^n} \right)^{2 \times 2}, \quad R_{[\varrho+1, \geq 1]}^{[n], \mathbb{R}} \in \left( \mathcal{R}_{\geq 1}^{\varrho+1} \right)^{2 \times 2},$$

$$A_{[0, \geq 1]}^{[n]} [\bar{\Omega}] \in \mathcal{L} \left( \mathbb{R}^2; (\Gamma_{\geq 1}^0)^2 \right), \quad R_{[\varrho+1, \geq 1]}^{[n], \mathbb{R}} [\bar{\Omega}] \in \mathcal{L} \left( \mathbb{R}^2; \left( \mathcal{R}_{\geq 1}^{\varrho+1} \right)^2 \right),$$

with  $P_{[-2^n]} = 0$  near  $\xi = 0$ , such that

$$\begin{aligned}
& \left( \text{Op}^{\text{BW}} \left( \text{Id} + P_{[-2^n]} (\mathfrak{a}|\eta; x, \xi) \right) + R_{[\varrho+1, \geq 1]}^{[n], \mathbb{R}} (\mathfrak{a}|\eta) \right) \delta \Omega [\bar{\Omega}] (\eta) \\
&= \text{Op}^{\text{BW}} \left( A_{1, \text{bg}}^{S, \eta} [\bar{\Omega}] (\mathfrak{a}|\eta; x, \xi) + A_{[0, \geq 1]}^{[n]} [\bar{\Omega}] (\mathfrak{a}|\eta; x, \xi) \right) \eta + R_{[\varrho+1, \geq 1]}^{\text{lin}, [n], \mathbb{R}} [\bar{\Omega}] (\mathfrak{a}|\eta) \eta.
\end{aligned} \tag{5.193}$$

The initialization  $n = 1$  is obtained from Eq. (5.192) by moving the residual  $\mathfrak{a} \text{Op}^{\text{BW}} \left( A_{[-2, \geq 1]}^{[1]} \right) \delta \Omega [\bar{\Omega}] (\eta)$  to the left. Assume Eq. (5.193) holds at rank  $n$ . Decreasing  $r_0$  at each of the finitely many steps, Proposition 4.24 gives

$$\text{Op}^{\text{BW}} \left( (\text{Id} + P_{[-2^n]})^{-1} \right) \text{Op}^{\text{BW}} (\text{Id} + P_{[-2^n]}) = \text{Op}^{\text{BW}} (\text{Id} + P_{[-2^{n+1}]}) + R_{[\varrho+1, \geq 1]}^{\mathbb{R}},$$

where  $P_{[-2^{n+1}]} \in \left( \Gamma_{\geq 1}^{-2^{n+1}} \right)^{2 \times 2}$ . Composing Eq. (5.193) on the left with  $\text{Op}^{\text{BW}} \left( (\text{Id} + P_{[-2^n]})^{-1} \right)$  proves the induction step: the new defect still vanishes near  $\xi = 0$ , the smoothing terms acting on  $\delta \Omega$  remain on the left, and the extra composition of

$(\text{Id} + P_{[-2^n]})^{-1} - \text{Id} \in (\Gamma_{\geq 1}^{-2^n})^{2 \times 2}$  with  $A_{1, \text{bg}}^{S, \eta} \in (\Gamma_{\geq 0}^1)^2$  has order at most 0 for  $n \geq 1$ , hence is absorbed into  $A_{[0, \geq 1]}^{[n+1]} [\bar{\Omega}]$ .

Taking  $\underline{n} > \log_2(\varrho + 1)$ , one has  $\text{Op}^{\text{BW}}(\mathbf{P}_{[-2\underline{n}]}) \in \left(\mathcal{R}_{\geq 1}^{\varrho+1}\right)^{2 \times 2}$ . The left-hand side of Eq. (5.193) becomes  $\left(\text{Id} + \mathbf{R}_{[\varrho+1, \geq 1]}^{\mathbb{R}}(\mathfrak{a}|\eta)\right) \delta\Omega[\bar{\Omega}](\eta)$ . Invert this factor by Eq. (5.186). Its smoothing correction composed with the order-one column belongs to  $\left(\mathcal{R}_{\geq 1}^{\varrho}\right)^2$  by Lemma 4.25. Adding  $\bar{\Omega}$  gives

$$\bar{\Omega}[\bar{\Omega}](\eta) = \bar{\Omega} + \text{Op}^{\text{BW}}\left(A_{1, \text{bg}}^{S, \eta}[\bar{\Omega}](\mathfrak{a}|\eta; x, \xi) + A_{[0, \geq 1]}[\bar{\Omega}](\mathfrak{a}|\eta; x, \xi)\right)\eta + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}[\bar{\Omega}](\mathfrak{a}|\eta)\eta. \quad (5.194)$$

This proves Eq. (5.169). Finally, using Eq. (5.142) and  $\mathcal{P}(\xi)\xi^\perp = \xi^\perp$ , the principal symbol in Eq. (5.194) is exactly Eq. (5.170).  $\square$

### 5.3.4 Paralinearization of Dirichlet-Neumann-type operators $g^\star(\eta)$ and $g(\mathfrak{a}|\eta)$

Let  $\Omega := \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}$ , where  $\omega_1, \omega_2$  are scalar functions that map from  $\mathbb{R}^2$  to  $\mathbb{R}$ . Consider the operator  $g^\star(\eta)$  of nonlinear convolution type, that has been introduced in Eq. (2.13) and that we recall here for simplicity

$$g^\star(\eta)[\Omega] = -\mathbf{v}^\star[\Omega](\eta) \cdot \begin{bmatrix} -\nabla\eta \\ 1 \end{bmatrix}, \quad (5.195)$$

where  $\mathbf{v}^\star$  is defined in Eq. (2.90).

The paralinearization of the operator  $g^\star(\eta)$  is given in the following proposition.

**Proposition 5.26** (Paralinearization of  $g^\star(\eta)$ ). *There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exists  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$ ,  $\Omega \in \left(B_{s_0, \mathbb{R}}(I; r_0)\right)^2$ , the operator  $g^\star(\eta)[\Omega]$  as in Eq. (5.195) admits the following paralinearization:*

$$\begin{aligned} g^\star(\eta)[\Omega] &= \text{Op}^{\text{BW}}\left(i A_0^{g^\star, \Omega}(\eta; x, \xi) + A_{-1}^{g^\star, \Omega}(\eta; x, \xi) + A_{[-2, \geq 1]}(\eta; x, \xi)\right) \cdot \Omega \\ &\quad + \text{Op}^{\text{BW}}\left(a_1^{g^\star, \eta}[\Omega](\eta; x, \xi) + a_{[0, \geq 1]}(\eta, \Omega; x, \xi)\right)\eta + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta) \cdot \Omega + r_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \Omega)\eta, \end{aligned} \quad (5.196)$$

where

- $A_0^{g^\star, \Omega} \in (\Gamma_{\geq 0}^0)^2$  is explicitly defined as

$$A_0^{g^\star, \Omega}(\eta; x, \xi) := \left(1 + |\nabla\eta|^2\right) A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) = -\nabla_\xi^\perp \lambda_1(\eta; x, \xi), \quad (5.197)$$

where  $A_0^{\mathcal{B}_0, \omega}$  is defined in Eq. (5.8);

- $A_{-1}^{g^\star, \Omega} \in (\Gamma_{\geq 1}^{-1})^2$  is explicitly defined as

$$\left(A_{-1}^{g^\star, \Omega}(\eta; x, \xi)\right)^\top := \frac{1}{2}(\nabla\eta)^\top \nabla_\xi^\perp J_\xi \phi_1^{\mathcal{B}_0} \mathbf{H}_x \eta - \frac{1}{4} J_x |\nabla\eta|^2 \nabla_\xi J_\xi^\perp \phi_1^{\mathcal{B}_0}, \quad (5.198)$$

where  $A_1^{\mathcal{B}_0, \eta}$  is defined in Eq. (5.103);

- $a_1^{g^\star, \eta} \in \mathcal{L}\left((H^{s_0})^2; \Gamma_{\geq 1}^1\right)$  is a symbol 1-positive homogeneous in  $\xi$ , admitting the following expression:

$$a_1^{g^\star, \eta}[\Omega](\eta; x, \xi) := i\xi \cdot V^\star[\Omega](\eta; x) - \frac{\lambda_1(\eta; x, \xi)}{1 + |\nabla\eta|^2} \nabla^\perp \eta \cdot \Omega, \quad (5.199)$$

where  $V^\star[\Omega]$  is defined in Eq. (5.123);

- $a_{[m, \geq 1]} \in \Gamma_{\geq 1}^m$ ,  $A_{[m, \geq 1]} \in (\Gamma_{\geq 1}^m)^2$ ,  $m \in \mathbb{R}$ , denote generic symbols as per Notation 5.1;
- $r_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} \in \mathcal{R}_{\geq 1}^\varrho$  and  $R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} \in \left(\mathcal{R}_{\geq 1}^\varrho\right)^2$  denote generic smoothing remainders as per Notation 5.1.

*Proof.* The part  $-\nu_3 [\Omega] (\eta)$  has already been analyzed in Eq. (5.129), so we only need to parilinearize the term

$$V^\star [\Omega] (\eta) \cdot \nabla \eta$$

in Eq. (5.195). Indeed Lemma 4.18 give us that

$$V^\star [\Omega] (\eta) \cdot \nabla \eta = \text{Op}^{\text{BW}} (V^\star [\Omega] (\eta)) \cdot \nabla \eta + \text{Op}^{\text{BW}} (\nabla \eta) \cdot [V^\star [\Omega] (\eta)] + \sum_{i=1}^2 R^{\text{BW}} [V^\star [\Omega] (\eta)_i, \partial_i \eta]. \quad (5.200)$$

First, applying Proposition 4.24, we have

$$\text{Op}^{\text{BW}} (V^\star [\Omega] (\eta)) \cdot \nabla \eta = \text{Op}^{\text{BW}} (i\xi \cdot V^\star [\Omega] (\eta; x) + a_{[0, \geq 1]} [\Omega] (\eta; x, \xi)) \eta + r_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\Omega) (\eta) \eta. \quad (5.201)$$

We use now Eq. (5.124) and Proposition 4.24 and obtain that

$$\begin{aligned} & \text{Op}^{\text{BW}} (\nabla \eta) \cdot [V^\star [\Omega] (\eta)] \\ &= \text{Op}^{\text{BW}} \left( i \nabla \eta^\top A_0^{V^\star, \Omega} (\eta; x, \xi) + \frac{1}{2} \left\{ \nabla \eta^\top, A_0^{V^\star, \Omega} (\eta; x, \xi) \right\} + \nabla \eta^\top A_{-1}^{V^\star, \Omega} (\eta; x, \xi) + A_{[-2, \geq 1]} (\eta, \Omega; x, \xi) \right) \Omega \\ &+ \text{Op}^{\text{BW}} \left( \nabla \eta^\top A_1^{V^\star, \eta} (\eta; x, \xi) \Omega + a_{[0, \geq 1]} (\eta, \Omega; x, \xi) \right) \eta + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\eta) \cdot \Omega + r_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\eta, \Omega) \eta. \end{aligned} \quad (5.202)$$

Next, using Remark 4.19, Eq. (5.129), and Lemma 4.25 we obtain that

$$\sum_{i=1}^2 R^{\text{BW}} [V^\star [\Omega] (\eta)_i, \partial_i \eta] = r_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\eta, \Omega) \eta. \quad (5.203)$$

We now plug the results of Eqs. (5.129) and (5.201) to (5.203) in Eqs. (5.195) and (5.200) and obtain the paradifferential decomposition stated in Eq. (5.196) where the symbols  $A_0^{g^\star, \Omega}$  and  $A_1^{g^\star, \eta}$  assume the explicit form

$$\left( A_0^{g^\star, \Omega} (\eta; x, \xi) \right)^\top := A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) + \nabla \eta^\top A_0^{V^\star, \Omega} (\eta; x, \xi), \quad (5.204a)$$

$$\left( A_{-1}^{g^\star, \Omega} (\eta; x, \xi) \right)^\top := \nabla \eta^\top A_{-1}^{V^\star, \Omega} (\eta; x, \xi) + \frac{1}{2} \left\{ \nabla \eta^\top, A_0^{V^\star, \Omega} (\eta; x, \xi) \right\}, \quad (5.204b)$$

$$\begin{aligned} a_1^{g^\star, \eta} [\Omega] (\eta; x, \xi) &:= i\xi \cdot V^\star [\Omega] (\eta; x) + \underbrace{\nabla \eta^\top A_1^{V^\star, \eta} (\eta; x, \xi) \Omega + \left( A_1^{\mathcal{B}_0, \eta} (\eta; x, \xi) \right)^\perp \cdot \Omega}_{:= B_1^{g^\star, \eta} (\eta; x, \xi) \cdot \Omega}. \end{aligned}$$

Let us notice that using Eqs. (4.13), (5.127) and (5.128) we obtain that

$$\begin{aligned} \frac{1}{2} \left\{ \nabla \eta^\top, A_0^{V^\star, \Omega} \right\} &= \left[ -\frac{1}{2} \partial_{x_i x_j} \eta \partial_{\xi_j} a_{0; ik}^{V^\star, \Omega} \right]_{k=1,2} = \left[ -\frac{1}{2} \partial_{x_i x_j} \eta \partial_{x_i} \eta \partial_{\xi_j} A_{0; k}^{\mathcal{B}_0, \omega; \perp} \right]_{k=1,2}^\top = -\frac{1}{4} J_x |\nabla_x \eta|^2 \nabla_\xi \left( A_0^{\mathcal{B}_0, \omega; \perp} \right)^\top, \\ \nabla \eta^\top A_{-1}^{V^\star, \Omega} &= \frac{1}{2} (\nabla \eta)^\top J_\xi A_0^{\mathcal{B}_0, \omega; \perp} \mathbf{H}_x \eta. \end{aligned} \quad (5.205)$$

Thus, inserting Eq. (5.205) in Eq. (5.204b) proves Eq. (5.198). It remains hence only to prove that Eqs. (5.204a) and (5.204b) can be expressed as in Eqs. (5.197) and (5.199) respectively. The equivalence of Eq. (5.197) and Eq. (5.204a) follows from Eqs. (5.127) and (5.204a), while using Eqs. (5.103), (5.121) and (5.126) we obtain that

$$B_1^{g^\star, \eta} (\eta; x, \xi) = -\frac{\lambda_1 (\eta; x, \xi)}{1 + |\nabla \eta|^2} \nabla^\perp \eta,$$

and so we conclude.  $\square$

Let us recall that the operator  $\mathcal{G}(\mathfrak{a}|\eta) \psi$  is defined through Eqs. (2.13), (2.15) and (2.90) as:

$$\mathcal{G}(\mathfrak{a}|\eta) \psi := \mathcal{G}^\star (\eta) [S(\mathfrak{a}|\eta) [\psi]] = -\nu [\psi] (\mathfrak{a}|\eta) \cdot \begin{bmatrix} -\nabla \eta \\ 1 \end{bmatrix}, \quad (5.206)$$

where  $\mathcal{G}^\star (\eta)$  is defined in Eq. (5.195) and  $S(\mathfrak{a}|\eta)$  is defined in Eq. (5.157).

**Proposition 5.27** (Parilinearization of the Dirichlet–Neumann operator  $\mathcal{G}(\mathfrak{a}|\eta)\psi$ ). *There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exists  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta, \psi \in B_{s_0, \mathbb{R}}(I; r_0)$ , the operator  $\mathcal{G}(\mathfrak{a}|\eta)\psi$  defined as in Eq. (5.206) admits the following parilinearization:*

$$\mathcal{G}(\mathfrak{a}|\eta)\psi = \text{Op}^{\text{BW}} \left( A_1^{\mathcal{G}}(\mathfrak{a}|\eta, \psi; x, \xi) + \begin{bmatrix} a_{[0, \geq 1]}[\psi](\eta; x, \xi) \\ a_{[-1, \geq 1]}(\eta; x, \xi) \end{bmatrix} \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix} + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}, \right.$$

where

- $\mathfrak{a}$  is the Atwood number defined in Eq. (1.5);
- $A_1^{\mathcal{G}} \in (\Gamma_{\geq 0}^1)^2$  is explicitly defined as

$$A_1^{\mathcal{G}}(\mathfrak{a}|\eta, \psi; x, \xi) := \begin{bmatrix} \alpha_1^{\mathcal{G}, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) \\ \lambda_1(\eta; x, \xi) + \mathfrak{a}\lambda_0(\eta; x, \xi) \end{bmatrix},$$

where

- $\lambda_1 \in \Gamma_{\geq 0}^1$  and  $\lambda_0 \in \Gamma_{\geq 1}^0$  are defined in Eq. (5.4) and Eq. (5.5) respectively;
- $\alpha_1^{\mathcal{G}, \eta} \in \mathcal{L}(H^{s_0}; \Gamma_{\geq 0}^1)$  is a symbol 1-positive homogeneous in  $\xi$ , admitting the following expression:

$$\alpha_1^{\mathcal{G}, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) := \lambda_1(\eta; x, \xi) b[\psi](\mathfrak{a}|\eta; x) + i\mathcal{V}[\psi](\mathfrak{a}|\eta; x) \cdot \xi, \quad (5.207)$$

where  $\lambda_1$  is defined in Eq. (5.4). The coefficient  $\mathcal{V}[\psi](\mathfrak{a}|\eta; x)$  is real-valued,  $\xi$ -independent, and of order zero; it is the product  $i\mathcal{V}[\psi](\mathfrak{a}|\eta; x) \cdot \xi$  that has order one.

- \*  $b[\psi] \in \mathcal{L}(H^{s_0}; \mathcal{F}^{\mathbb{R}})$  is the real-valued coefficient defined in Eq. (5.264);
- \*  $\mathcal{V}[\psi](\mathfrak{a}|\eta) \in \mathcal{L}(H^{s_0}; (\mathcal{F}^{\mathbb{R}})^2)$  is the real-valued coefficient defined in Eq. (5.265).

- $A_{[0, \geq 1]} \in (\Gamma_{\geq 1}^0)^2$  denotes a generic symbol as per Notation 5.1;
- $R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} \in (\mathcal{R}_{\geq 1}^{\varrho})^2$  denotes a generic smoothing remainder as per Notation 5.1.

*Proof.* Choose the baseline regularity one derivative above the threshold in Proposition 5.26, so that  $S\psi$  has the required input regularity. Applying that proposition with smoothing index  $\varrho + 1$  and the composition rules stated in Lemma 4.25, we infer that

$$\begin{aligned} \mathcal{G}^*(\eta) S(\mathfrak{a}|\eta)\psi &= \text{Op}^{\text{BW}} \left( iA_0^{\mathcal{G}^*, \Omega}(\eta; x, \xi) + A_{-1}^{\mathcal{G}^*, \Omega}(\eta; x, \xi) + A_{[-2, \geq 1]}(\eta; x, \xi) \right) \cdot [S(\mathfrak{a}|\eta)\psi] \\ &\quad + \text{Op}^{\text{BW}} \left( \alpha_1^{\mathcal{G}^*, \eta}[S(\mathfrak{a}|\eta)\psi](\eta; x, \xi) + a_{[0, \geq 1]}(\eta, \psi; x, \xi) \right) \eta + R_{[\varrho+1, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \end{aligned} \quad (5.208)$$

Next, we combine Eq. (5.165) and Eq. (5.208) together with the composition rule Proposition 4.24. Since  $S$  has order one, this loses one derivative on the preceding remainder; after relabeling  $\varrho + 1$  as  $\varrho$ , we obtain

$$\begin{aligned} \mathcal{G}^*(\eta) S(\mathfrak{a}|\eta)\psi &= \text{Op}^{\text{BW}} \left( \lambda_1(\eta; x, \xi) + \Lambda_0^{\text{DN}}(\eta; x, \xi) + a_{[-1, \geq 1]}(\eta; x, \xi) \right) \psi \\ &\quad + \text{Op}^{\text{BW}} \left( \alpha_1^{\mathcal{G}, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) + a_{[0, \geq 1]}(\eta, \psi; x, \xi) \right) \eta + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}, \end{aligned}$$

where

$$\lambda_1(\eta; x, \xi) := -\xi^\perp \cdot A_0^{\mathcal{G}^*, \Omega}(\eta; x, \xi), \quad (5.209a)$$

$$\Lambda_0^{\text{DN}}(\eta; x, \xi) := \mathfrak{a}A_0^{\mathcal{G}^*, \Omega}(\eta; x, \xi) \cdot \xi^\perp \cdot a_{-1}^{\tau_0, \psi}(\eta; x, \xi) + i \left( \frac{1}{2} \left\{ \left( A_0^{\mathcal{G}^*, \Omega} \right)^\top, \xi^\perp \right\} + A_{-1}^{\mathcal{G}^*, \Omega}(\eta; x, \xi) \cdot \xi^\perp \right), \quad (5.209b)$$

$$\alpha_1^{\mathcal{G}, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) := \alpha_1^{\mathcal{G}^*, \eta}[S(\mathfrak{a}|\eta)\psi](\eta; x, \xi) + iA_0^{\mathcal{G}^*, \Omega}(\eta; x, \xi) \cdot A_1^{S, \eta}[\psi](\mathfrak{a}|\eta; x, \xi). \quad (5.209c)$$

We now simplify the expressions in Eqs. (5.209a) and (5.209b).

**Part 1** (Equivalence of Eqs. (5.4) and (5.209a)). From Eq. (5.197) and Eq. (5.121), we have

$$\xi^\perp \cdot A_0^{g^*, \Omega}(\eta; x, \xi) = \left(1 + |\nabla \eta|^2\right) \left(A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) \cdot \xi\right) = -\lambda_1(\eta; x, \xi). \quad (5.210)$$

This justifies a posteriori the choice of notation.

**Part 2** (Identification of  $\Lambda_0^{\text{DN}} = \alpha \lambda_0$ ). We decompose this step in two subparts.

**Subpart 2.1** (Proof that  $\text{Im } \Lambda_0^{\text{DN}} \equiv 0$ ). We decompose  $A_{-1}^{g^*, \Omega}(\eta; x, \xi) \cdot \xi^\perp$  using Eq. (5.198) as

$$\begin{aligned} A_{-1}^{g^*, \Omega} \cdot \xi^\perp &= b_0(\eta; x, \xi) + c_0(\eta; x, \xi), \\ b_0(\eta; x, \xi) &:= \frac{1}{2} (\nabla \eta)^\top \left( \nabla_\xi^\perp J_\xi \phi_1^{\mathcal{B}_0} \right) \mathbf{H}_x \eta \xi^\perp, \\ c_0(\eta; x, \xi) &:= -\frac{1}{4} J_x |\nabla \eta|^2 \left( \nabla_\xi J_\xi^\perp \phi_1^{\mathcal{B}_0} \right) \xi^\perp. \end{aligned} \quad (5.211a)$$

We show that  $c_0 = 0$  and that  $b_0$  cancels  $\frac{1}{2} \left\{ \left( A_0^{g^*, \Omega} \right)^\top, \xi^\perp \right\}$ .

**Step 1** ( $c_0 = 0$ ). A componentwise expansion gives for any  $i = 1, 2$

$$\left[ \left( \nabla_\xi J_\xi^\perp \phi_1^{\mathcal{B}_0} \right) \xi^\perp \right]_i = -\partial_{\xi_i} \partial_{\xi_2} \phi_1^{\mathcal{B}_0} (-\xi_2) + \partial_{\xi_i} \partial_{\xi_1} \phi_1^{\mathcal{B}_0} \xi_1 = \left[ J_\xi A_0^{\mathcal{B}_0, \omega} \cdot \xi \right]_i.$$

Since  $A_0^{\mathcal{B}_0, \omega} = \nabla_\xi \phi_1^{\mathcal{B}_0}$  is 0-homogeneous in  $\xi$ , Euler's theorem yields  $J_\xi A_0^{\mathcal{B}_0, \omega} \cdot \xi = 0$ , whence  $c_0 = 0$ .

**Step 2** (Cancellation of  $b_0$  with the Poisson bracket). From Eq. (5.197) and Eq. (4.12), the Poisson bracket contribution reads

$$\frac{1}{2} \left\{ \left( A_0^{g^*, \Omega} \right)^\top, \xi^\perp \right\} = -\frac{1}{2} \nabla^\perp \cdot A_0^{g^*, \Omega} = -\frac{1}{2} \nabla_x \cdot \left( \left( 1 + |\nabla \eta|^2 \right) A_0^{\mathcal{B}_0, \omega} \right) \stackrel{\text{Eq. (5.101)}}{=} \frac{\nabla_x \cdot \nabla_\xi}{2} \lambda_1. \quad (5.212)$$

Moreover, since  $|\nabla \eta|^2$  is independent of  $\xi$ ,

$$\left( 1 + |\nabla \eta|^2 \right) J_\xi A_0^{\mathcal{B}_0, \omega} = -\mathbf{H}_\xi \lambda_1. \quad (5.213)$$

**Substep 2.1** (Reduction of  $b_0$ ). A direct matrix computation shows  $\nabla_\xi^\perp J_\xi \phi_1^{\mathcal{B}_0} = \mathbf{J} J_\xi A_0^{\mathcal{B}_0, \omega}$ , where  $\mathbf{J} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is the standard symplectic matrix. Since  $(\nabla \eta)^\top \mathbf{J} = -(\nabla^\perp \eta)^\top$ , one obtains

$$(\nabla \eta)^\top \left( \nabla_\xi^\perp J_\xi \phi_1^{\mathcal{B}_0} \right) = -(\nabla^\perp \eta)^\top J_\xi A_0^{\mathcal{B}_0, \omega} \stackrel{\text{Eq. (5.213)}}{=} \frac{(\nabla^\perp \eta)^\top \mathbf{H}_\xi \lambda_1}{1 + |\nabla \eta|^2},$$

whence

$$b_0 = \frac{(\nabla^\perp \eta)^\top \mathbf{H}_\xi \lambda_1}{2(1 + |\nabla \eta|^2)} \cdot \mathbf{H}_x \eta \xi^\perp. \quad (5.214)$$

We compute  $(\nabla^\perp \eta)^\top \mathbf{H}_\xi \lambda_1$  explicitly. From Eq. (5.4),  $4\lambda_1^2 = |\xi|^2 + (\nabla^\perp \eta \cdot \xi)^2$ , which gives

$$4\lambda_1 \partial_{\xi_k} \lambda_1 = \xi_k + (\nabla^\perp \eta \cdot \xi) (\nabla^\perp \eta)_k.$$

Differentiating once more in  $\xi_i$  and contracting with  $(\nabla^\perp \eta)_i$ , we find, using  $\nabla^\perp \eta \cdot \xi^\perp = \nabla \eta \cdot \xi$  and  $|\nabla^\perp \eta|^2 = |\nabla \eta|^2$ ,

$$(\nabla^\perp \eta)^\top \mathbf{H}_\xi \lambda_1 = \frac{1 + |\nabla \eta|^2}{16\lambda_1^3} (|\xi|^2 (\nabla^\perp \eta) - (\nabla^\perp \eta \cdot \xi) \xi)^\top. \quad (5.215)$$

Substituting Eq. (5.215) into Eq. (5.214), we obtain

$$b_0 = \frac{1}{32\lambda_1^3} (|\xi|^2 (\nabla^\perp \eta) - (\nabla^\perp \eta \cdot \xi) \xi)^\top \mathbf{H}_x \eta \xi^\perp \stackrel{\mathbf{H}_x \eta = (\mathbf{H}_x \eta)^\top}{=} \frac{(\xi^\perp)^\top \mathbf{H}_x \eta (|\xi|^2 \nabla^\perp \eta - (\nabla^\perp \eta \cdot \xi) \xi)}{32\lambda_1^3}. \quad (5.216)$$

**Substep 2.2** (Reduction of the Poisson bracket). From  $4\lambda_1^2 = |\xi|^2 + (\nabla^\perp \eta \cdot \xi)^2$  and the identities

$$\nabla_{\nabla \eta} (\nabla^\perp \eta \cdot \xi) = -\xi^\perp, \quad \nabla_\xi (\nabla^\perp \eta \cdot \xi) = \nabla^\perp \eta,$$

we compute

$$\partial_{(\nabla \eta)} \partial_{\xi_k} \lambda_1 = \frac{-(\xi^\perp)_j}{16\lambda_1^3} (|\xi|^2 (\nabla^\perp \eta)_k - (\nabla^\perp \eta \cdot \xi) \xi_k) - \frac{\nabla^\perp \eta \cdot \xi}{4\lambda_1} J_{jk},$$

where  $J_{jk}$  are the entries of  $\mathbf{J}$ . Applying the chain rule

$$\frac{\nabla_x \cdot \nabla_\xi}{2} \lambda_1 = \frac{1}{2} \sum_{j,k} (\mathbf{H}_x \eta)_{jk} \partial_{(\nabla \eta)} \partial_{\xi_k} \lambda_1,$$

the antisymmetric term vanishes by the symmetry of  $\mathbf{H}_x \eta$ :

$$\sum_{j,k} (\mathbf{H}_x \eta)_{jk} J_{jk} = \text{tr}(\mathbf{H}_x \eta \mathbf{J}^\top) = (\mathbf{H}_x \eta)_{12} - (\mathbf{H}_x \eta)_{21} = 0,$$

so that

$$\frac{\nabla_x \cdot \nabla_\xi}{2} \lambda_1 = -\frac{(\xi^\perp)^\top \mathbf{H}_x \eta (|\xi|^2 \nabla^\perp \eta - (\nabla^\perp \eta \cdot \xi) \xi)}{32\lambda_1^3}. \quad (5.217)$$

**Step 3** (Synthesis). Combining Eqs. (5.212), (5.216), (5.217), (5.209b) and (5.211a), we conclude

$$\text{Im} \Lambda_0^{\text{DN}} = \frac{\nabla_x \cdot \nabla_\xi}{2} \lambda_1 + b_0 = 0.$$

**Part 3** (Simplification of  $\text{Re} \Lambda_0^{\text{DN}}$ ). Considering the result proved in Subpart 2.1 we have that

$$\Lambda_0^{\text{DN}} = \text{Re} \Lambda_0^{\text{DN}} = \mathfrak{a} A_0^{\mathcal{G}^\star, \Omega} \cdot \xi^\perp \cdot \mathfrak{a}_{-1}^{\tau_0, \psi}.$$

We can now apply Eqs. (5.121), (5.156) and (5.197) and obtain  $\Lambda_0^{\text{DN}} = \mathfrak{a} \lambda_0$ , with  $\lambda_0$  defined in Eq. (5.5).

**Part 4** (Proof of Eq. (5.207)). Using Eqs. (2.90) and (5.199), we obtain

$$\mathfrak{a}_1^{\mathcal{G}^\star, \eta} = \text{i} V[\psi](\eta) \cdot \xi - \lambda_1(\eta; x, \xi) \frac{\nabla^\perp \eta \cdot S(\mathfrak{a}|\eta) \psi}{1 + |\nabla \eta|^2}. \quad (5.218)$$

Moreover, Eqs. (5.164), (5.166) and (5.210) give the additional contribution from the composition of the  $\Omega$ -column of  $\mathcal{G}^\star$  with the  $\eta$ -column of  $S$ :

$$\text{i} A_0^{\mathcal{G}^\star, \Omega} \cdot A_1^{S, \eta}[\psi] = \mathfrak{a} \lambda_1 \nu_3[\psi] + \text{i} \mathfrak{a} \lambda_1 A_0^{\mathcal{B}_0, \omega; \perp} \cdot S(\mathfrak{a}|\eta) \psi. \quad (5.219)$$

We plug Eqs. (5.210), (5.218) and (5.219) into Eq. (5.209c); using Eq. (2.90) yields exactly Eq. (5.207).  $\square$

### 5.3.5 Paralinearization of $\pi^\star$

Let  $\pi^\star$  be defined as in Eq. (2.42a), i.e.

$$\pi^\star(\rho^\pm | \eta, \Omega) := \pi_0(\rho^\pm | \eta) + \pi_2^\star(\mathfrak{a} | \eta)[\Omega, \Omega], \quad (5.220)$$

where  $\pi_0$  and  $\pi_2^\star$  are respectively defined in Eqs. (2.17) and (2.18). The result we prove is the following

**Proposition 5.28** (Paralinearization of  $\pi^\star$ ). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle \rho^\pm \rangle > 0$ , and  $\mathfrak{g} \in \mathbb{R}$ . There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  and  $\Omega \in (B_{s_0, \mathbb{R}}(I; r_0))^2$ , the function  $\pi^\star(\rho^\pm | \eta, \Omega)$  defined in Eq. (5.220) admits the following paralinearization:*

$$\begin{aligned} \pi^\star(\rho^\pm | \eta, \Omega) = & -\mathfrak{g} \mathfrak{a} \eta + \text{Op}^{\text{BW}} \left( \mathfrak{a}_2^{\pi^\star}(\rho^\pm | \eta; x, \xi) + \mathfrak{a}_1^{\pi^\star}(\mathfrak{a} | \eta, \Omega; x, \xi) + \begin{bmatrix} a_{[0, \geq 1]}(\sigma, \rho^\pm, \mathfrak{a} | \eta, \Omega; x, \xi) \\ A_{[-1, \geq 1]}[\Omega](\mathfrak{a} | \eta; x, \xi) \end{bmatrix} \right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix} \\ & + r_{[\varrho, \geq 1]}^{\pi_0, \mathbb{R}}(\sigma, \rho^\pm | \eta) \eta + \mathbf{r}_{[\varrho, \geq 1]}^{\text{bil}, \mathbb{R}}(\mathfrak{a} | \eta, \Omega) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix}, \end{aligned} \quad (5.221)$$

where

- $\mathbf{a}_2^{\pi^*} \in (\Gamma_{\geq 0}^2)^3$  and is explicitly defined as

$$\mathbf{a}_2^{\pi^*}(\rho^\pm|\eta; x, \xi) := \begin{bmatrix} a_2^{\pi^*, \eta}(\rho^\pm|\eta; x, \xi) \\ 0_{\mathbb{R}^2} \end{bmatrix},$$

where  $a_2^{\pi^*, \eta} \in \Gamma_{\geq 0}^2$  is real-valued and is explicitly defined as

$$a_2^{\pi^*, \eta}(\rho^\pm|\eta; x, \xi) := -\frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} \frac{|\xi|^2 + |\nabla^\perp \eta \cdot \xi|^2}{(1 + |\nabla \eta|^2)^{\frac{3}{2}}}. \quad (5.222)$$

- $\mathbf{a}_1^{\pi^*} \in (\Gamma_{\geq 1}^1)^3$  and is explicitly defined as

$$\mathbf{a}_1^{\pi^*}(\mathfrak{a}|\eta, \Omega; x, \xi) := \begin{bmatrix} a_1^{\pi^*, \eta}[\Omega, \Omega](\mathfrak{a}|\eta; x, \xi) \\ A_0^{\pi^*, \Omega}[\Omega](\mathfrak{a}|\eta; x, \xi) \end{bmatrix}, \quad (5.223)$$

where

- $a_1^{\pi^*, \eta} \in \mathcal{L}^2((H^{s_0})^2; \Gamma^1)$  and is explicitly defined as

$$\begin{aligned} & a_1^{\pi^*, \eta}[\Omega, \Omega](\mathfrak{a}|\eta; x, \xi) \\ &:= \lambda_1(\eta; x, \xi) \left[ 4 \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega \right)^2 - \left( \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \right)^2 \right] \\ &+ \mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega \right) V^*[\Omega](\eta; x) \cdot \xi \\ &+ \mathfrak{a} \lambda_1(\eta; x, \xi) v_3^*[\Omega](\eta; x) \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \\ &- i \mathfrak{a} \left[ v_3^*[\Omega](\eta; x) V^*[\Omega](\eta; x) + \frac{1}{4(1 + |\nabla \eta|^2)} \left( (\Omega \cdot \nabla \eta) \Omega - \frac{|\Omega|^2 + |\Omega \cdot \nabla \eta|^2}{1 + |\nabla \eta|^2} \nabla \eta \right) \right] \cdot \xi; \end{aligned} \quad (5.224)$$

where  $\lambda_1 \in \Gamma_{\geq 0}^1$  is defined in Eq. (5.4),  $A_0^{\mathcal{B}_0, \omega} \in (\Gamma_{\geq 0}^0)^2$  and  $A_0^{\mathcal{B}_0, \omega; \perp} := \mathbf{J} A_0^{\mathcal{B}_0, \omega}$  are defined in Eq. (5.8), and  $V^*[\Omega](\eta)$ ,  $v_3^*[\Omega](\eta)$  are the two components of  $\mathbf{v}^*[\Omega](\eta)$  defined in Eq. (2.14);

- $A_0^{\pi^*, \Omega} \in \mathcal{L}((H^{s_0})^2; (\Gamma^0)^2)$  and is explicitly defined as

$$\begin{aligned} A_0^{\pi^*, \Omega}[\Omega](\mathfrak{a}|\eta; x, \xi) &:= \left( V^*[\Omega](\eta; x) + \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} \mathbf{J}(\Omega + (\nabla \eta \cdot \Omega) \nabla \eta) \right)^\perp \\ &- i \left( 1 + |\nabla \eta|^2 \right) \left( \mathfrak{a} v_3^*[\Omega](\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \right) A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi), \end{aligned} \quad (5.225)$$

where  $V^*[\Omega](\eta)$  and  $v_3^*[\Omega](\eta)$  are the two components of  $\mathbf{v}^*[\Omega](\eta)$  defined in Eq. (2.14),  $A_0^{\mathcal{B}_0, \omega; \perp} := \mathbf{J} A_0^{\mathcal{B}_0, \omega}$  with  $A_0^{\mathcal{B}_0, \omega}$  defined in Eq. (5.8), and  $\mathbf{J}$  the  $\pi/2$ -rotation matrix.

- $a_{[0, \geq 1]}(\sigma, \rho^\pm, \mathfrak{a}|\eta, \Omega; x, \xi) \in \Gamma_{\geq 1}^0$  denotes the aggregate generic order-zero  $\eta$ -symbol: it is the sum of the term coming from  $\pi_0$  and a term bilinear in  $\Omega$ . Moreover,  $A_{[-1, \geq 1]}[\Omega] \in \mathcal{L}((H^{s_0})^2; (\Gamma^{-1})^2)$  denotes a generic symbol linear in  $\Omega$ , as per Notation 5.1;
- $r_{[\varrho, \geq 1]}^{\pi_0, \mathbb{R}}(\sigma, \rho^\pm|\eta) \in \mathcal{R}_{\geq 1}^\varrho$  is the smoothing remainder linear in the final  $\eta$  input produced by  $\pi_0$ ;
- $\mathbf{r}_{[\varrho, \geq 1]}^{\text{bil}, \mathbb{R}} \in (\mathcal{R}_{\geq 1}^\varrho)^3$  denotes a bilinear smoothing remainder, whose  $\eta$ -component is bilinear in  $\Omega$  and whose  $\Omega$ -component is linear in  $\Omega$ , in the sense of Remarks 4.30 and 4.31.

**Paralinearization of  $\pi_0$**  In this section, we paralinearize the term  $\pi_0$ , which represents the contributions from surface tension and gravity. For the reader's convenience, we recall from Eq. (2.17) its definition:

$$\pi_0(\rho^\pm|\eta) := \frac{\sigma}{\langle\langle\rho^\pm\rangle\rangle} \nabla \cdot \mathcal{K}(\nabla\eta) - \mathfrak{g}\mathfrak{a}\eta, \quad (5.226)$$

where  $\mathcal{K}(\nabla\eta)$  is defined in Eq. (1.7) as  $\mathcal{K}(\nabla\eta) := \frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}$ .

**Lemma 5.29** (Paralinearization of  $\pi_0$ ). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle\langle\rho^\pm\rangle\rangle > 0$ , and  $\mathfrak{g} \in \mathbb{R}$ . There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$ , the term  $\pi_0(\eta)$  defined in Eq. (5.226) admits the following paralinearization:*

$$\pi_0(\eta) = -\mathfrak{g}\mathfrak{a}\eta + \text{Op}^{\text{BW}}(a_2^{\pi_0, \eta}(\eta; x, \xi) + a_{[0, \geq 1]}(\eta; x, \xi))\eta + r_{[\varrho, \geq 1]}^{\pi_0, \mathbb{R}}(\sigma, \rho^\pm|\eta)\eta, \quad (5.227)$$

where the symbols are defined as follows:

- $a_2^{\pi_0, \eta} \in \Gamma_{\geq 0}^2$  is the real-valued principal symbol, given by

$$a_2^{\pi_0, \eta}(\eta; x, \xi) := -\frac{\sigma}{\langle\langle\rho^\pm\rangle\rangle} \frac{|\xi|^2 + |\nabla^\perp \eta \cdot \xi|^2}{(1 + |\nabla\eta|^2)^{\frac{3}{2}}}; \quad (5.228)$$

- $a_{[0, \geq 1]} \in \Gamma_{\geq 1}^0$ , cf. Notation 5.1;
- $r_{[\varrho, \geq 1]}^{\pi_0, \mathbb{R}}(\sigma, \rho^\pm|\eta) \in \mathcal{R}_{\geq 1}^\varrho$ , cf. Notation 5.1.

*Proof.* Recall from Eq. (2.17) and Eq. (1.7) that

$$\pi_0(\eta) = \frac{\sigma}{\langle\langle\rho^\pm\rangle\rangle} \nabla \cdot \mathcal{K}(\nabla\eta) - \mathfrak{g}\mathfrak{a}\eta, \quad \mathcal{K}(\Omega) = \frac{\Omega}{\sqrt{1+|\Omega|^2}}. \quad (5.229)$$

By vectorial paralinearization formula

$$F(\Omega) - F(0) = \text{Op}^{\text{BW}}(JF(\Omega))\Omega + R_{[\varrho, \geq 1]}^{\mathbb{R}}(\Omega)\Omega, \quad F \in C^\infty(\mathbb{R}^2), \quad (5.230)$$

and composition rule Proposition 4.24, we get

$$\begin{aligned} \mathcal{K}(\nabla\eta) &= \text{Op}^{\text{BW}}(J\mathcal{K}(\nabla\eta))[\nabla\eta] + R_{[\varrho, \geq 1]}^{\mathbb{R}}(\eta)\eta \\ &= \text{Op}^{\text{BW}}\left(iJ\mathcal{K}(\nabla\eta)\xi + \frac{1}{2i}\{J\mathcal{K}(\nabla\eta), i\xi\} + A_{[-1, \geq 1]}(\eta; x, \xi)\right)\eta + R_{[\varrho, \geq 1]}^{\mathbb{R}}(\eta)\eta, \end{aligned}$$

where we used the notation Eq. (4.13). Notice in particular that explicit computations show that

$$J\mathcal{K}(\Omega) = \frac{1}{(1+|\Omega|^2)^{3/2}} (\text{Id} + \Omega^\perp \otimes \Omega^\perp). \quad (5.231)$$

Let us now compute

$$\begin{aligned} \nabla \cdot \mathcal{K}(\nabla\eta) &= \text{Op}^{\text{BW}}(i\xi^\top) \text{Op}^{\text{BW}}\left(iJ\mathcal{K}(\nabla\eta)\xi + \frac{1}{2i}\{J\mathcal{K}(\nabla\eta), i\xi\} + A_{[-1, \geq 1]}(\eta; x, \xi)\right)\eta + r_{[\varrho, \geq 1]}^{\mathbb{R}}(\eta)\eta \\ &= \text{Op}^{\text{BW}}\left(-\xi^\top J\mathcal{K}(\nabla\eta)\xi + \frac{i}{2}(\{\xi^\top, J\mathcal{K}(\nabla\eta)\xi\} + \xi^\top\{J\mathcal{K}(\nabla\eta), \xi\}) + a_{[0, \geq 1]}(\eta; x, \xi)\right)\eta + r_{[\varrho, \geq 1]}^{\mathbb{R}}(\eta)\eta. \end{aligned} \quad (5.232)$$

We use now Eq. (4.13) and Einstein's convention for repeated indexes

$$\{\xi^\top, J\mathcal{K}(\nabla\eta)\xi\} + \xi^\top\{J\mathcal{K}(\nabla\eta), \xi\} = \{\xi_i, J\mathcal{K}(\nabla\eta)_{ij}\xi_j\} + \xi_i\{J\mathcal{K}(\nabla\eta)_{ij}, \xi_j\} \quad (5.233)$$

and Leibnitz's formula  $\{fg, h\} = \{f, h\}g + f\{g, h\}$  to obtain that

$$\{\xi_i, J\mathcal{K}(\nabla\eta)_{ij}\xi_j\} = -\xi_j \{J\mathcal{K}(\nabla\eta)_{ij}, \xi_i\} - \underbrace{J\mathcal{K}(\nabla\eta)_{ij}\{\xi_j, \xi_i\}}_{\equiv 0}. \quad (5.234)$$

We plug Eq. (5.234) into Eq. (5.233), and use the symmetry of  $J\mathcal{K}(\nabla\eta)$ , which evident from Eq. (5.231), to obtain that

$$\{\xi^\top, J\mathcal{K}(\nabla\eta)\xi\} + \xi^\top \{J\mathcal{K}(\nabla\eta), \xi\} \equiv 0. \quad (5.235)$$

We insert now Eq. (5.235) into Eq. (5.232) and use Eq. (5.231) to obtain that

$$\nabla \cdot \mathcal{K}(\nabla\eta) = \text{Op}^{\text{BW}} \left( -\frac{|\xi|^2 + |\nabla^\perp \eta \cdot \xi|^2}{(1 + |\nabla\eta|^2)^{\frac{3}{2}}} + a_{[0, \geq 1]}(\eta; x, \xi) \right) \eta + r_{[\varrho, \geq 1]}^{\mathbb{R}}(\eta) \eta.$$

After multiplication by  $\sigma / \langle\langle \rho^\pm \rangle\rangle$ , we denote the resulting smoothing operator by  $r_{[\varrho, \geq 1]}^{\pi_0, \mathbb{R}}(\sigma, \rho^\pm | \eta)$ . The claim thus follows using Eq. (5.229).  $\square$

**Paralinearization of  $\pi_2^\star$**  Here we paralinearize the term  $\pi_2^\star$  defined in Eq. (2.18) and whose definition we recall here for the reader's convenience

$$\begin{aligned} \pi_2^\star(\mathfrak{a}|\eta)[\Omega_1, \Omega_2] &:= -\pi_{2;I}^\star(\eta)[\Omega_1, \Omega_2] - \pi_{2;II}^\star(\mathfrak{a}|\eta)[\Omega_1, \Omega_2], \\ \pi_{2;I}^\star(\eta)[\Omega_1, \Omega_2] &:= \frac{\Omega_1^\perp \cdot V^\star[\Omega_2](\eta) + \Omega_2^\perp \cdot V^\star[\Omega_1](\eta)}{2}, \\ \pi_{2;II}^\star(\mathfrak{a}|\eta)[\Omega_1, \Omega_2] &:= \frac{\mathfrak{a}}{4} \left[ 2V^\star[\Omega_1](\eta) \cdot V^\star[\Omega_2](\eta) \right. \\ &\quad - \nu_3^\star[\Omega_1](\eta) (\nu_3^\star[\Omega_2](\eta) - 2V^\star[\Omega_2](\eta) \cdot \nabla\eta) \\ &\quad - \nu_3^\star[\Omega_2](\eta) (\nu_3^\star[\Omega_1](\eta) - 2V^\star[\Omega_1](\eta) \cdot \nabla\eta) \\ &\quad \left. + \frac{1}{2} \frac{\Omega_1^\top (\text{Id}_{\mathbb{R}^2} + \nabla\eta \otimes \nabla\eta) \Omega_2}{1 + |\nabla\eta|^2} \right]. \end{aligned} \quad (5.236a)$$

The result we prove is the following

**Lemma 5.30** (Paralinearization of  $\pi_2^\star$ ). *There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  and  $\Omega \in (B_{s_0, \mathbb{R}}(I; r_0))^2$ , the term  $\pi_2^\star(\mathfrak{a}|\eta)[\Omega, \Omega]$  defined in Eq. (5.236a) admits the following paralinearization:*

$$\pi_2^\star(\mathfrak{a}|\eta)[\Omega, \Omega] = \text{Op}^{\text{BW}} \left( \mathfrak{a}_1^{\pi_2^\star}(\mathfrak{a}|\eta, \Omega; x, \xi) + \left[ \frac{a_{[0, \geq 1]}[\Omega, \Omega](\mathfrak{a}|\eta; x, \xi)}{A_{[-1, \geq 1]}[\Omega](\mathfrak{a}|\eta; x, \xi)} \right] \cdot \left[ \frac{\eta}{\Omega} \right] + r_{[\varrho, \geq 1]}^{\text{bil}, \mathbb{R}}(\mathfrak{a}|\eta, \Omega) \cdot \left[ \frac{\eta}{\Omega} \right] \right), \quad (5.237)$$

where

- $\mathfrak{a}_1^{\pi_2^\star} \in (\Gamma_{\geq 1}^1)^3$  and is explicitly defined as

$$\mathfrak{a}_1^{\pi_2^\star}(\mathfrak{a}|\eta, \Omega; x, \xi) := \begin{bmatrix} \mathfrak{a}_1^{\pi_2^\star, \eta}[\Omega, \Omega](\mathfrak{a}|\eta; x, \xi) \\ A_0^{\pi_2^\star, \Omega}[\Omega](\mathfrak{a}|\eta; x, \xi) \end{bmatrix}, \quad (5.238)$$

where

–  $a_1^{\pi_2^*, \eta} \in \mathcal{L}^2((H^{s_0})^2; \Gamma^1)$  and is explicitly defined as

$$\begin{aligned} a_1^{\pi_2^*, \eta}[\Omega, \Omega](\mathfrak{a}|\eta; x, \xi) &:= \lambda_1(\eta; x, \xi) \left[ 4 \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega \right)^2 - \left( \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \right)^2 \right] \\ &\quad + \mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega \right) V^\star[\Omega](\eta; x) \cdot \xi \\ &\quad + \mathfrak{a} \lambda_1(\eta; x, \xi) v_3^\star[\Omega](\eta; x) \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \\ &\quad - i \mathfrak{a} [v_3^\star[\Omega](\eta; x) V^\star[\Omega](\eta; x) \\ &\quad \quad + \frac{1}{4(1 + |\nabla \eta|^2)} \left( (\Omega \cdot \nabla \eta) \Omega - \frac{|\Omega|^2 + |\Omega \cdot \nabla \eta|^2}{1 + |\nabla \eta|^2} \nabla \eta \right)] \cdot \xi, \end{aligned} \quad (5.239)$$

where  $\lambda_1 \in \Gamma_{\geq 0}^1$  is defined in Eq. (5.4),  $A_0^{\mathcal{B}_0, \omega} \in (\Gamma_{\geq 0}^0)^2$  is defined in Eq. (5.8), and  $V^\star[\Omega](\eta)$ ,  $v_3^\star[\Omega](\eta)$  are the two components of  $\mathbf{v}^\star[\Omega](\eta)$  defined in Eq. (2.14);

–  $A_0^{\pi_2^*, \Omega} \in \mathcal{L}((H^{s_0})^2; (\Gamma^0)^2)$  and is explicitly defined as

$$\begin{aligned} A_0^{\pi_2^*, \Omega}[\Omega](\mathfrak{a}|\eta; x, \xi) &:= \left( V^\star[\Omega](\eta; x) + \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} \mathbf{J}(\Omega + (\nabla \eta \cdot \Omega) \nabla \eta) \right)^\perp \\ &\quad - i \left( 1 + |\nabla \eta|^2 \right) \left( \mathfrak{a} v_3^\star[\Omega](\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \right) A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi); \end{aligned} \quad (5.240)$$

where  $V^\star[\Omega](\eta)$  and  $v_3^\star[\Omega](\eta)$  are the two components of  $\mathbf{v}^\star[\Omega](\eta)$  defined in Eq. (2.14),  $A_0^{\mathcal{B}_0, \omega; \perp} := \mathbf{J} A_0^{\mathcal{B}_0, \omega}$  with  $A_0^{\mathcal{B}_0, \omega}$  defined in Eq. (5.8), and  $\mathbf{J}$  the  $\pi/2$ -rotation matrix.

- $a_{[0, \geq 1]}[\Omega, \Omega] \in \mathcal{L}^2((H^{s_0})^2; \Gamma^0)$  denotes a generic symbol bilinear in  $\Omega$ , and  $A_{[-1, \geq 1]}[\Omega] \in \mathcal{L}((H^{s_0})^2; (\Gamma^{-1})^2)$  denotes a generic symbol linear in  $\Omega$ , as per Notation 5.1;
- $\mathbf{r}_{[\varrho, \geq 1]}^{\text{bil}, \mathbb{R}} \in (\mathcal{R}_{\geq 1}^\varrho)^3$  denotes a bilinear smoothing remainder, whose  $\eta$ -component is bilinear in  $\Omega$  and whose  $\Omega$ -component is linear in  $\Omega$ , in the sense of Remarks 4.30 and 4.31.

**Corollary 5.31** (Bilinear parilinearization of  $\pi_2^\star$ ). *There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  and  $\Omega_1, \Omega_2 \in (B_{s_0, \mathbb{R}}(I; r_0))^2$ , the bilinear form  $\pi_2^\star(\mathfrak{a}|\eta)[\Omega_1, \Omega_2]$  defined in Eq. (5.236a) admits the following parilinearization:*

$$\begin{aligned} \pi_2^\star(\mathfrak{a}|\eta)[\Omega_1, \Omega_2] &= \text{Op}^{\text{BW}} \left( a_1^{\pi_2^*, \eta}[\Omega_1, \Omega_2](\mathfrak{a}|\eta; x, \xi) + a_{[0, \geq 1]}[\Omega_1, \Omega_2](\mathfrak{a}|\eta; x, \xi) \right) \eta \\ &\quad + \frac{1}{2} \text{Op}^{\text{BW}} \left( A_0^{\pi_2^*, \Omega}[\Omega_2](\mathfrak{a}|\eta; x, \xi) + A_{[-1, \geq 1]}[\Omega_2](\mathfrak{a}|\eta; x, \xi) \right) \cdot \Omega_1 \\ &\quad + \frac{1}{2} \text{Op}^{\text{BW}} \left( A_0^{\pi_2^*, \Omega}[\Omega_1](\mathfrak{a}|\eta; x, \xi) + A_{[-1, \geq 1]}[\Omega_1](\mathfrak{a}|\eta; x, \xi) \right) \cdot \Omega_2 \\ &\quad + r_{[\varrho, \geq 1]}^{\text{bil}, \mathbb{R}}[\Omega_1, \Omega_2](\mathfrak{a}|\eta) \eta \\ &\quad + \frac{1}{2} R_{[\varrho, \geq 1]}^{\mathbb{R}}[\Omega_2](\mathfrak{a}|\eta) \cdot \Omega_1 + \frac{1}{2} R_{[\varrho, \geq 1]}^{\mathbb{R}}[\Omega_1](\mathfrak{a}|\eta) \cdot \Omega_2, \end{aligned} \quad (5.241)$$

where

- $a_1^{\pi_2^*, \eta} \in \mathcal{L}^2((H^{s_0})^2; \Gamma^1)$  and is explicitly defined as

$$\begin{aligned}
& a_1^{\pi_2^*, \eta} [\Omega_1, \Omega_2] (\mathfrak{a} | \eta; x, \xi) \\
& := \lambda_1 (\eta; x, \xi) \left[ 4 \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega_1 \right) \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega_2 \right) - \frac{(\nabla^\perp \eta \cdot \Omega_1) (\nabla^\perp \eta \cdot \Omega_2)}{(1 + |\nabla \eta|^2)^2} \right] \\
& + \frac{\mathfrak{a}}{2} \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega_1 \right) V^* [\Omega_2] (\eta; x) \cdot \xi + \frac{\mathfrak{a}}{2} \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega_2 \right) V^* [\Omega_1] (\eta; x) \cdot \xi \\
& + \frac{\mathfrak{a} \lambda_1 (\eta; x, \xi)}{2(1 + |\nabla \eta|^2)} \left[ \nu_3^* [\Omega_1] (\eta; x) (\nabla^\perp \eta \cdot \Omega_2) + \nu_3^* [\Omega_2] (\eta; x) (\nabla^\perp \eta \cdot \Omega_1) \right] \\
& - i \mathfrak{a} \left[ \frac{1}{2} \nu_3^* [\Omega_1] (\eta; x) V^* [\Omega_2] (\eta; x) + \frac{1}{2} \nu_3^* [\Omega_2] (\eta; x) V^* [\Omega_1] (\eta; x) \right. \\
& \quad \left. + \frac{1}{4(1 + |\nabla \eta|^2)} \left( \frac{1}{2} (\Omega_1 \cdot \nabla \eta) \Omega_2 + \frac{1}{2} (\Omega_2 \cdot \nabla \eta) \Omega_1 - \frac{\Omega_1 \cdot \Omega_2 + (\Omega_1 \cdot \nabla \eta) (\Omega_2 \cdot \nabla \eta)}{1 + |\nabla \eta|^2} \nabla \eta \right) \right] \cdot \xi;
\end{aligned} \tag{5.242}$$

In particular, setting  $\Omega_1 = \Omega_2 = \Omega$  gives Eq. (5.239).

- $A_0^{\pi_2^*, \Omega} \in \mathcal{L}((H^{s_0})^2; (\Gamma^0)^2)$  and is explicitly defined as

$$\begin{aligned}
A_0^{\pi_2^*, \Omega} [\Omega_j] (\mathfrak{a} | \eta; x, \xi) &:= \left( V^* [\Omega_j] (\eta; x) + \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} J (\Omega_j + (\nabla \eta \cdot \Omega_j) \nabla \eta) \right)^\perp \\
&\quad - i \left( 1 + |\nabla \eta|^2 \right) \left( \mathfrak{a} \nu_3^* [\Omega_j] (\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega_j}{1 + |\nabla \eta|^2} A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \right),
\end{aligned} \tag{5.243}$$

for  $j = 1, 2$ . In particular, setting  $\Omega_1 = \Omega_2 = \Omega$  in Eq. (5.241) gives the  $\Omega$ -component in Eq. (5.240);

- $a_{[m, \geq 1]}$  and  $A_{[m, \geq 1]}$  are generic symbols as per Notation 5.1;
- $r_{[\varrho, \geq 1]}^{\text{bil}, \mathbb{R}}$  and  $R_{[\varrho, \geq 1]}^{\mathbb{R}}$  are smoothing remainders as per Notation 5.1.

*Proof.* The claim follows from the polarization identity

$$\pi_2^* (\mathfrak{a} | \eta) [\Omega_1, \Omega_2] = \frac{1}{2} \left( \pi_2^* (\mathfrak{a} | \eta) [\Omega_1 + \Omega_2, \Omega_1 + \Omega_2] - \pi_2^* (\mathfrak{a} | \eta) [\Omega_1, \Omega_1] - \pi_2^* (\mathfrak{a} | \eta) [\Omega_2, \Omega_2] \right). \tag{5.244}$$

Applying Lemma 5.30 to each term in Eq. (5.244), the  $\eta$ -symbol is the polarization of Eq. (5.239), which gives Eq. (5.242). The  $\Omega$ -symbol  $A_0^{\pi_2^*, \Omega}$  is linear in its argument, hence

$$\frac{1}{2} \left[ A_0^{\pi_2^*, \Omega} [\Omega_1 + \Omega_2] \cdot (\Omega_1 + \Omega_2) - A_0^{\pi_2^*, \Omega} [\Omega_1] \cdot \Omega_1 - A_0^{\pi_2^*, \Omega} [\Omega_2] \cdot \Omega_2 \right]$$

equals

$$\frac{1}{2} A_0^{\pi_2^*, \Omega} [\Omega_2] \cdot \Omega_1 + \frac{1}{2} A_0^{\pi_2^*, \Omega} [\Omega_1] \cdot \Omega_2.$$

The generic symbols and the smoothing remainders polarize in the same way, yielding Eq. (5.241).  $\square$

*Proof of Lemma 5.30.* All along the proof we write

$$\alpha \simeq \beta \quad \Longleftrightarrow \quad \alpha \equiv \beta \quad \text{mod Op}^{\text{BW}} \left( \begin{bmatrix} a_{[0, \geq 1]} [\Omega, \Omega] (\eta; x, \xi) \\ A_{[-1, \geq 1]} [\Omega] (\eta; x, \xi) \end{bmatrix} \right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix} + \begin{bmatrix} r_{[\varrho, \geq 1]}^{\mathbb{R}} [\Omega, \Omega] (\eta) \\ R_{[\varrho, \geq 1]}^{\mathbb{R}} [\Omega] (\eta) \end{bmatrix} \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix}.$$

**Part 1** (Paralinearization of  $\pi_{2;\text{I}}^*$ ). This is a straightforward application of Lemmas 4.18 and 4.25 and Propositions 4.24 and 5.19, which gives that

$$\pi_{2;\text{I}}^*(\eta) [\Omega, \Omega] \simeq \text{Op}^{\text{BW}} \left( \begin{bmatrix} a_1^{\pi_{2;\text{I}}^*, \eta} [\Omega, \Omega] (\eta; x, \xi) \\ A_0^{\pi_{2;\text{I}}^*, \Omega} [\Omega] (\eta; x, \xi) \end{bmatrix} \right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix}, \quad (5.245)$$

where

$$\begin{aligned} a_1^{\pi_{2;\text{I}}^*, \eta} [\Omega, \Omega] (\eta; x, \xi) &:= -\Omega^\top J A_1^{V^*, \eta} (\eta; x, \xi) \Omega, \\ A_0^{\pi_{2;\text{I}}^*, \Omega} [\Omega] (\eta; x, \xi) &:= -V^* [\Omega] (\eta)^\perp + i(\Omega^\perp)^\top A_0^{V^*, \Omega} (\eta; x, \xi). \end{aligned}$$

**Part 2** (Paralinearization of  $\pi_{2;\text{II}}^*$ ). Let us write

$$\pi_{2;\text{II}}^*(\eta) [\Omega, \Omega] = \frac{\mathfrak{a}}{2} \left( |V^* [\Omega] (\eta)|^2 - \nu_3^* [\Omega] (\eta) (\nu_3^* [\Omega] (\eta) - 2V^* [\Omega] (\eta) \cdot \nabla \eta) + \frac{1}{4} \frac{|\Omega|^2 + |\nabla \eta \cdot \Omega|^2}{1 + |\nabla \eta|^2} \right).$$

Using Lemma 4.18 and Proposition 5.19 we obtain that

$$|V^* [\Omega] (\eta)|^2 \simeq \text{Op}^{\text{BW}} \left( 2V^* [\Omega] (\eta)^\top \mathfrak{A}_1^{V^*} (\eta, \Omega; x, \xi) \right) \begin{bmatrix} \eta \\ \Omega \end{bmatrix}, \quad (5.246)$$

where  $\mathfrak{A}_1^{V^*}$  is defined in Eq. (5.125). Again Lemma 4.18 and Proposition 5.19 give us that

$$\begin{aligned} & -\nu_3^* [\Omega] (\eta) (\nu_3^* [\Omega] (\eta) - 2V^* [\Omega] (\eta) \cdot \nabla \eta) \\ & \simeq \text{Op}^{\text{BW}} \left( -\nu_3^* [\Omega] (\eta) \left( \mathfrak{a}_1^{\nu_3^*} (\eta, \Omega) - 2 \left( \left( \mathfrak{A}_1^{V^*} (\eta, \Omega) \right)^\top \nabla \eta + \begin{bmatrix} i\xi \cdot V^* [\Omega] (\eta) \\ 0_{\mathbb{R}^2} \end{bmatrix} \right) \right) - (\nu_3^* [\Omega] (\eta) - 2V^* [\Omega] (\eta) \cdot \nabla \eta) \mathfrak{a}_1^{\nu_3^*} (\eta, \Omega) \right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix}. \end{aligned} \quad (5.247)$$

Let  $h : \mathbb{R}^4 \rightarrow \mathbb{R}_+$  defined as  $h(X, Y) := \frac{1}{4} \frac{X^2 + |X \cdot Y|^2}{1 + |Y|^2}$ , with such notation we have that

$$\frac{1}{4} \frac{|\Omega|^2 + |\nabla \eta \cdot \Omega|^2}{1 + |\nabla \eta|^2} = h(\Omega, \nabla \eta).$$

We apply Eq. (5.230) and Lemma 4.25 we obtain that

$$\frac{1}{4} \frac{|\Omega|^2 + |\nabla \eta \cdot \Omega|^2}{1 + |\nabla \eta|^2} \simeq \text{Op}^{\text{BW}} (J_{X,Y} h(\Omega, \nabla \eta)) \begin{bmatrix} \Omega \\ \nabla \eta \end{bmatrix},$$

where

$$J_{X,Y} h(X, Y) = \frac{1}{2} \left[ \frac{\frac{X + (X \cdot Y)Y}{1 + |Y|^2}}{\frac{(1 + |Y|^2)(X \cdot Y)X - (|X|^2 + (X \cdot Y)^2)Y}{(1 + |Y|^2)^2}} \right]^\top$$

so that, applying Proposition 4.24 we obtain that

$$\frac{1}{4} \frac{|\Omega|^2 + |\nabla \eta \cdot \Omega|^2}{1 + |\nabla \eta|^2} \simeq \text{Op}^{\text{BW}} \left( \mathfrak{a}_1^h (\eta, \Omega; x, \xi) \right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix},$$

where

$$\mathfrak{a}_1^h (\eta, \Omega) := \left[ \frac{i}{2} \frac{1}{1 + |\nabla \eta|^2} \left( \Omega \cdot \nabla \eta \Omega - \frac{|\Omega|^2 + |\Omega \cdot \nabla \eta|^2}{1 + |\nabla \eta|^2} \nabla \eta \right) \cdot \xi \right]. \quad (5.248)$$

**Part 3** (Synthesis). We keep all principal quantities expanded in the notation of the previous proofparts. Let  $\mathfrak{a}_1^{\pi_2^*}$  be as in Eq. (5.238) so that

$$\pi_2^*(\mathfrak{a}|\eta) [\Omega, \Omega] \simeq \text{Op}^{\text{BW}} \left( \mathfrak{a}_1^{\pi_2^*} [\Omega] (\mathfrak{a}|\eta; x, \xi) \right) \cdot \begin{bmatrix} \eta \\ \Omega \end{bmatrix}.$$

We first compute the  $\Omega$ -component. Since  $\pi_2^* = -\pi_{2;\text{I}}^* - \pi_{2;\text{II}}^*$ , from Eqs. (5.245) to (5.248) we obtain

$$\begin{aligned} A_0^{\pi_2^*, \Omega} [\Omega] (\mathfrak{a} | \eta; x, \xi) &= V^* [\Omega] (\eta; x)^\perp - \mathfrak{i} (\Omega^\perp)^\top A_0^{V^*, \Omega} (\eta; x, \xi) \\ &\quad - \frac{\mathfrak{a}}{2} \left[ 2\mathfrak{i} V^* [\Omega] (\eta; x)^\top A_0^{V^*, \Omega} (\eta; x, \xi) \right. \\ &\quad \left. - \nu_3^* [\Omega] (\eta; x) \left( \mathfrak{i} A_0^{\nu_3^*, \Omega} (\eta; x, \xi) - 2\mathfrak{i} \left( A_0^{V^*, \Omega} (\eta; x, \xi) \right)^\top \nabla \eta \right) \right. \\ &\quad \left. - \left( \nu_3^* [\Omega] (\eta; x) - 2V^* [\Omega] (\eta; x) \cdot \nabla \eta \right) \mathfrak{i} A_0^{\nu_3^*, \Omega} (\eta; x, \xi) \right. \\ &\quad \left. + \frac{1}{2} \frac{\Omega + (\Omega \cdot \nabla \eta) \nabla \eta}{1 + |\nabla \eta|^2} \right]. \end{aligned} \quad (5.249)$$

Using

$$A_0^{V^*, \Omega} (\eta; x, \xi) = \nabla \eta \otimes A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi), \quad A_0^{\nu_3^*, \Omega} (\eta; x, \xi) = -A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi),$$

we have

$$(\Omega^\perp)^\top A_0^{V^*, \Omega} (\eta; x, \xi) = -(\nabla^\perp \eta \cdot \Omega) A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi), \quad (5.250)$$

and

$$\begin{aligned} &2\mathfrak{i} V^* [\Omega] (\eta; x)^\top A_0^{V^*, \Omega} (\eta; x, \xi) \\ &- \nu_3^* [\Omega] (\eta; x) \left( \mathfrak{i} A_0^{\nu_3^*, \Omega} (\eta; x, \xi) - 2\mathfrak{i} \left( A_0^{V^*, \Omega} (\eta; x, \xi) \right)^\top \nabla \eta \right) \\ &- \left( \nu_3^* [\Omega] (\eta; x) - 2V^* [\Omega] (\eta; x) \cdot \nabla \eta \right) \mathfrak{i} A_0^{\nu_3^*, \Omega} (\eta; x, \xi) \\ &= 2\mathfrak{i} \left( 1 + |\nabla \eta|^2 \right) \nu_3^* [\Omega] (\eta; x) A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi). \end{aligned} \quad (5.251)$$

Inserting Eq. (5.250) and Eq. (5.251) into Eq. (5.249) gives

$$\begin{aligned} A_0^{\pi_2^*, \Omega} [\Omega] (\mathfrak{a} | \eta; x, \xi) &= V^* [\Omega] (\eta; x)^\perp - \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} (\Omega + (\Omega \cdot \nabla \eta) \nabla \eta) \\ &\quad - \mathfrak{i} \left( 1 + |\nabla \eta|^2 \right) \left( \mathfrak{a} \nu_3^* [\Omega] (\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \right) A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi). \end{aligned} \quad (5.252)$$

Since

$$(J(\Omega + (\nabla \eta \cdot \Omega) \nabla \eta))^\perp = -(\Omega + (\nabla \eta \cdot \Omega) \nabla \eta),$$

the identity Eq. (5.252) is exactly Eq. (5.240).

We now compute the  $\eta$ -component. From Eq. (5.245) and Corollary 5.20, the contribution of  $-\pi_{2;\text{I}}^*$  is

$$\begin{aligned} -a_1^{\pi_{2;\text{I}}^*, \eta} [\Omega, \Omega] (\eta; x, \xi) &= -\left( A_1^{V^*, \eta} (\eta; x, \xi) \Omega \right) \cdot \Omega^\perp \\ &= \lambda_1 (\eta; x, \xi) \left[ \left( 2A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega \right)^2 - \left( \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \right)^2 \right]. \end{aligned} \quad (5.253)$$

For the contribution of  $-\pi_{2;\text{II}}^*$ , we first record the two pointwise identities

$$A_1^{V^*, \eta} (\eta; x, \xi) \Omega + \nabla \eta a_1^{\nu_3^*, \eta} [\Omega] (\eta; x, \xi) = -\xi \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega \right), \quad (5.254)$$

and

$$\left( 1 + |\nabla \eta|^2 \right) a_1^{\nu_3^*, \eta} [\Omega] (\eta; x, \xi) = -(\nabla \eta \cdot \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega \right) + \lambda_1 (\eta; x, \xi) \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2}. \quad (5.255)$$

Indeed, Eq. (5.254) follows from Eq. (5.126) and the definition of  $a_1^{\nu_3^*, \eta}$ , while Eq. (5.255) follows from Eq. (5.122).

Using Eq. (5.254) and Eq. (5.255), the  $\eta$ -symbol coming from Eq. (5.246) and Eq. (5.247) satisfies

$$\begin{aligned}
& -\frac{\mathfrak{a}}{2} \left[ 2V^\star [\Omega] (\eta; x) \cdot \mathbf{A}_1^{V^\star, \eta} (\eta; x, \xi) \Omega - 2\nu_3^\star [\Omega] (\eta; x) a_1^{\nu_3^\star, \eta} [\Omega] (\eta; x, \xi) + 2\nu_3^\star [\Omega] (\eta; x) \nabla \eta \cdot \mathbf{A}_1^{V^\star, \eta} (\eta; x, \xi) \Omega \right. \\
& \left. + 2(V^\star [\Omega] (\eta; x) \cdot \nabla \eta) a_1^{\nu_3^\star, \eta} [\Omega] (\eta; x, \xi) + 2i\nu_3^\star [\Omega] (\eta; x) V^\star [\Omega] (\eta; x) \cdot \xi \right] \\
& = \mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega \right) V^\star [\Omega] (\eta; x) \cdot \xi + \mathfrak{a} \lambda_1 (\eta; x, \xi) \nu_3^\star [\Omega] (\eta; x) \frac{\nabla^\perp \eta \cdot \Omega}{1 + |\nabla \eta|^2} \\
& \quad - i\mathfrak{a} \nu_3^\star [\Omega] (\eta; x) V^\star [\Omega] (\eta; x) \cdot \xi.
\end{aligned} \tag{5.256}$$

Finally, the contribution of Eq. (5.248) to  $-\pi_{2; \Pi}^\star$  is

$$-\frac{\mathfrak{a}}{2} \left[ \frac{i}{2} \frac{1}{1 + |\nabla \eta|^2} \left( (\Omega \cdot \nabla \eta) \Omega - \frac{|\Omega|^2 + |\Omega \cdot \nabla \eta|^2}{1 + |\nabla \eta|^2} \nabla \eta \right) \cdot \xi \right] = -\frac{i\mathfrak{a}}{4(1 + |\nabla \eta|^2)} \left( (\Omega \cdot \nabla \eta) \Omega - \frac{|\Omega|^2 + |\Omega \cdot \nabla \eta|^2}{1 + |\nabla \eta|^2} \nabla \eta \right) \cdot \xi. \tag{5.257}$$

Combining Eq. (5.253), Eq. (5.256), and Eq. (5.257) gives precisely Eq. (5.239). Therefore the parilinearization Eq. (5.237) follows.  $\square$

*Proof of Proposition 5.28.* By Eq. (5.220),

$$\pi^\star (\rho^\pm | \eta, \Omega) = \pi_0 (\rho^\pm | \eta) + \pi_2^\star (\mathfrak{a} | \eta) [\Omega, \Omega].$$

The term  $\pi_0$  is parilinearized in Lemma 5.29. Its principal symbol is Eq. (5.228), which is exactly Eq. (5.222), and it has no  $\Omega$ -component. The quadratic term  $\pi_2^\star$  is parilinearized in Lemma 5.30; its order-one symbol is the one displayed in Eq. (5.223), namely Eq. (5.224) and Eq. (5.225).

Combining these two decompositions gives Eq. (5.221): the order-zero  $\eta$ -symbols and the order  $-1$   $\Omega$ -symbols are absorbed into the generic symbols, the smoothing remainder of  $\pi_0$  gives the separate linear term  $r_{[\varrho, \geq 1]}^{\pi_0, \mathbb{R}} (\sigma, \rho^\pm | \eta) \eta$ , and the smoothing remainders of  $\pi_2^\star$  give the bilinear/linear terms displayed in the last line of Eq. (5.221). The stated symbol classes follow from Lemmas 5.29 and 5.30 and from Lemma 4.25.  $\square$

### 5.3.6 Parilinearization of $\pi$

Let the function  $\pi (\rho^\pm | \eta, \psi)$  be defined as in Eq. (5.258), namely

$$\pi (\rho^\pm | \eta, \psi) := \pi_0 (\rho^\pm | \eta) + \pi_2 (\mathfrak{a} | \eta) [\psi, \psi], \tag{5.258}$$

where  $\pi_0$  is defined in Eq. (2.17) and  $\pi_2$  is defined as

$$\pi_2 (\mathfrak{a} | \eta) [\psi_1, \psi_2] = \pi_2^\star (\mathfrak{a} | \eta) [S(\mathfrak{a} | \eta) \psi_1, S(\mathfrak{a} | \eta) \psi_2]. \tag{5.259}$$

**Proposition 5.32** (Parilinearization of  $\pi$ ). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle \rho^\pm \rangle > 0$ , and  $\mathfrak{g} \in \mathbb{R}$ . There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta, \psi \in B_{s_0, \mathbb{R}} (I; r_0)$ , the function  $\pi (\rho^\pm | \eta, \psi)$  defined in Eq. (5.258) admits the following parilinearization:*

$$\begin{aligned}
\pi (\rho^\pm | \eta, \psi) & = -\mathfrak{g} \mathfrak{a} \eta + \text{Op}^{\text{BW}} \left( A_2^\pi (\rho^\pm | \eta; x, \xi) + A_1^\pi (\mathfrak{a} | \eta, \psi; x, \xi) + A_{[0, \geq 1]} (\sigma, \rho^\pm, \mathfrak{a} | \eta, \psi; x, \xi) \right) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix} \\
& \quad + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} (\sigma, \rho^\pm, \mathfrak{a} | \eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}.
\end{aligned} \tag{5.260}$$

where

- $A_2^\pi \in (\Gamma_{\geq 0}^2)^2$  and is explicitly defined as

$$A_2^\pi (\rho^\pm | \eta; x, \xi) := \begin{bmatrix} a_2^{\pi, \eta} (\rho^\pm | \eta; x, \xi) \\ 0 \end{bmatrix},$$

where

–  $a_2^{\pi,\eta} \in \Gamma_{\geq 0}^2$  is real-valued and is explicitly defined as

$$a_2^{\pi,\eta}(\rho^\pm|\eta; x, \xi) := -\frac{\sigma}{\llbracket \rho^\pm \rrbracket} \frac{|\xi|^2 + |\nabla^\perp \eta \cdot \xi|^2}{(1 + |\nabla \eta|^2)^{\frac{3}{2}}}.$$

In particular,

$$a_2^{\pi,\eta}(\rho^\pm|0; x, \xi) = -\frac{\sigma}{\llbracket \rho^\pm \rrbracket} |\xi|^2.$$

•  $A_1^\pi \in (\Gamma_{\geq 1}^1)^2$  and is explicitly defined as

$$A_1^\pi(\mathfrak{a}|\eta, \psi; x, \xi) := \begin{bmatrix} a_1^{\pi,\eta}[\psi, \psi](\mathfrak{a}|\eta; x, \xi) \\ a_1^{\pi,\psi}[\psi](\mathfrak{a}|\eta; x, \xi) \end{bmatrix},$$

where

–  $a_1^{\pi,\eta} \in \mathcal{L}^2(H^{s_0}; \Gamma^1)$  is real-valued and is explicitly defined as

$$a_1^{\pi,\eta}[\psi, \psi](\mathfrak{a}|\eta; x, \xi) := \lambda_1(\eta; x, \xi) \left[ q_0^\pi[\psi, \psi](\mathfrak{a}|\eta; x, \xi) - (b[\psi](\mathfrak{a}|\eta; x))^2 \right]. \quad (5.261)$$

In particular,  $a_1^{\pi,\eta}[\psi, \psi](\mathfrak{a}|\eta; x, \xi)$  is even in  $\xi$ ;

–  $a_1^{\pi,\psi} \in \mathcal{L}(H^{s_0}; \Gamma^1)$  is explicitly defined as

$$a_1^{\pi,\psi}[\psi](\mathfrak{a}|\eta; x, \xi) := -\lambda_1(\eta; x, \xi) b[\psi](\mathfrak{a}|\eta; x) + i \mathcal{V}[\psi](\mathfrak{a}|\eta; x) \cdot \xi. \quad (5.262)$$

In particular, the first term in Eq. (5.262) is real-valued and even in  $\xi$ , while the second one is the transport symbol;

–  $q_0^\pi \in \mathcal{L}^2(H^{s_0}; \Gamma^0)$  is real-valued and is explicitly defined as

$$q_0^\pi[\psi_1, \psi_2](\mathfrak{a}|\eta; x, \xi) := (4 - \mathfrak{a}^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot S(\mathfrak{a}|\eta) \psi_1 \right) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot S(\mathfrak{a}|\eta) \psi_2 \right). \quad (5.263)$$

In particular,  $q_0^\pi$  is even in  $\xi$ ;

–  $b[\psi] \in \mathcal{L}(H^{s_0}; \mathcal{F}^\mathbb{R})$  is real-valued and is explicitly defined as

$$b[\psi](\mathfrak{a}|\eta; x) := \mathfrak{a} \nu_3[\psi](\mathfrak{a}|\eta; x) - \frac{\nabla^\perp \eta \cdot S(\mathfrak{a}|\eta) \psi}{1 + |\nabla \eta|^2}. \quad (5.264)$$

In particular,  $b[\psi](\mathfrak{a}|\eta; x)$  is independent of  $\xi$ ;

–  $\mathcal{V}[\psi](\mathfrak{a}|\eta) \in \mathcal{L}(H^{s_0}; (\mathcal{F}^\mathbb{R})^2)$  is real-valued and is explicitly defined as

$$\mathcal{V}[\psi](\mathfrak{a}|\eta; x) := V[\psi](\mathfrak{a}|\eta; x) + \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} \mathbf{J}(S(\mathfrak{a}|\eta) \psi + (\nabla \eta \cdot S(\mathfrak{a}|\eta) \psi) \nabla \eta). \quad (5.265)$$

In particular,  $\mathcal{V}[\psi](\mathfrak{a}|\eta; x)$  is independent of  $\xi$ .

- $A_{[0, \geq 1]}$  is a generic vector symbol as per Notation 5.1;
- $R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}$  is a smoothing remainder as per Notation 5.1; its aggregate signature is the one displayed in Eq. (5.260).

*Proof.* Fix the target gain  $\varrho$ . Apply Lemma 5.30 and Proposition 5.24 with smoothing index  $\varrho + 1$ . If  $s_\star$  is the Sobolev threshold in Lemma 5.30, choose the present threshold so that  $s_0 \geq s_\star + 1$ . Since  $S(\mathfrak{a}|\eta)$  has order one,  $\psi \in H^{s_0}$  then gives  $S(\mathfrak{a}|\eta) \psi \in H^{s_\star}$ ; composing a remainder of gain  $\varrho + 1$  with  $S$  leaves gain  $\varrho$ . We relabel all such final remainders as  $R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}$  below.

Set

$$\Omega_\psi := S(\mathfrak{a}|\eta) \psi.$$

Then, by Eq. (5.259),

$$\pi_2(\mathfrak{a}|\eta)[\psi, \psi] = \pi_2^\star(\mathfrak{a}|\eta)[\Omega_\psi, \Omega_\psi].$$

**Part 1** (Composition with  $S(\mathfrak{a}|\eta)$ ). By Lemma 5.30, applied with  $\Omega = \Omega_\psi$ , and by Proposition 5.24, we have

$$\begin{aligned} \pi_2(\mathfrak{a}|\eta)[\psi, \psi] &= \text{Op}^{\text{BW}} \left( \begin{bmatrix} a_1^{\pi_2^*, \eta}[\Omega_\psi, \Omega_\psi](\mathfrak{a}|\eta; x, \xi) & \\ & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix} \\ &\quad + \text{Op}^{\text{BW}} \left( A_0^{\pi_2^*, \Omega}[\Omega_\psi](\mathfrak{a}|\eta; x, \xi) \right) \cdot S(\mathfrak{a}|\eta) \psi \\ &\quad + \text{Op}^{\text{BW}} \left( A_{[0, \geq 1]}(\mathfrak{a}|\eta, \psi; x, \xi) \right) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix} \\ &\quad + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\mathfrak{a}|\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \end{aligned} \quad (5.266)$$

Moreover, Eq. (5.165) gives, at the principal orders needed here,

$$\begin{aligned} S(\mathfrak{a}|\eta) \psi &= \text{Op}^{\text{BW}} \left( \begin{bmatrix} A_1^{S, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) & i\xi^\perp \\ A_{[0, \geq 1]}[\psi](\mathfrak{a}|\eta; x, \xi) & A_{[0, \geq 1]}(\mathfrak{a}|\eta; x, \xi) \end{bmatrix} \right) \begin{bmatrix} \eta \\ \psi \end{bmatrix} \\ &\quad + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\mathfrak{a}|\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}, \end{aligned} \quad (5.267)$$

because the correction  $-i\mathfrak{a}\xi^\perp a_{-1}^{\tau_0, \psi}(\eta; x, \xi)$  in Eq. (5.167) has order 0.

From Eq. (5.166) and Eq. (5.164), the  $\eta$ -column in Eq. (5.267) is

$$A_1^{S, \eta}[\psi](\mathfrak{a}|\eta; x, \xi) = -\mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_\psi - i\nu_3[\psi](\mathfrak{a}|\eta; x) \right) \xi^\perp. \quad (5.268)$$

Composing Eq. (5.266) with Eq. (5.267) and using the symbolic composition formula yields

$$\begin{aligned} \pi_2(\mathfrak{a}|\eta)[\psi, \psi] &= \text{Op}^{\text{BW}} \left( \begin{bmatrix} a_1^{\pi_2^*, \eta}[\Omega_\psi, \Omega_\psi] + A_0^{\pi_2^*, \Omega}[\Omega_\psi] \cdot A_1^{S, \eta}[\psi] & \\ A_0^{\pi_2^*, \Omega}[\Omega_\psi] \cdot i\xi^\perp & \end{bmatrix} + A_{[0, \geq 1]} \right) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix} \\ &\quad + R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\mathfrak{a}|\eta, \psi) \cdot \begin{bmatrix} \eta \\ \psi \end{bmatrix}. \end{aligned} \quad (5.269)$$

All the symbols in Eq. (5.269) are evaluated at  $(\mathfrak{a}|\eta; x, \xi)$ .

**Part 2** ( $\psi$ -component). Using Eq. (5.240), Eq. (5.264), and Eq. (5.265), we compute

$$A_0^{\pi_2^*, \Omega}[\Omega_\psi](\mathfrak{a}|\eta; x, \xi) \cdot i\xi^\perp = i\nu'[\psi](\mathfrak{a}|\eta; x) \cdot \xi + (1 + |\nabla\eta|^2) b[\psi](\mathfrak{a}|\eta; x) A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \xi^\perp.$$

Since

$$(1 + |\nabla\eta|^2) A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \xi^\perp = (1 + |\nabla\eta|^2) A_0^{\mathcal{B}_0, \omega}(\eta; x, \xi) \cdot \xi = -\lambda_1(\eta; x, \xi), \quad (5.270)$$

we obtain

$$A_0^{\pi_2^*, \Omega}[\Omega_\psi](\mathfrak{a}|\eta; x, \xi) \cdot i\xi^\perp = -\lambda_1(\eta; x, \xi) b[\psi](\mathfrak{a}|\eta; x) + i\nu'[\psi](\mathfrak{a}|\eta; x) \cdot \xi. \quad (5.271)$$

This is exactly Eq. (5.262).

**Part 3** ( $\eta$ -component). We now simplify the first component of Eq. (5.269). First, by Eq. (5.240) and Eq. (5.268),

$$A_0^{\pi_2^*, \Omega}[\Omega_\psi] \cdot A_1^{S, \eta}[\psi] = -\mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_\psi - i\nu_3[\psi](\mathfrak{a}|\eta; x) \right) (\nu'[\psi](\mathfrak{a}|\eta; x) \cdot \xi + i\lambda_1(\eta; x, \xi) b[\psi](\mathfrak{a}|\eta; x)), \quad (5.272)$$

where Eq. (5.270) has been used in the second factor.

We shall also use the two pointwise identities

$$\frac{J(\Omega_\psi + (\nabla\eta \cdot \Omega_\psi) \nabla\eta) \cdot \xi}{4(1 + |\nabla\eta|^2)} = \lambda_1(\eta; x, \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_\psi \right), \quad (5.273)$$

and

$$\frac{1}{4(1 + |\nabla\eta|^2)} \left( (\Omega_\psi \cdot \nabla\eta) \Omega_\psi - \frac{|\Omega_\psi|^2 + |\Omega_\psi \cdot \nabla\eta|^2}{1 + |\nabla\eta|^2} \nabla\eta \right) \cdot \xi = \lambda_1(\eta; x, \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_\psi \right) \frac{\nabla^\perp \eta \cdot \Omega_\psi}{1 + |\nabla\eta|^2}. \quad (5.274)$$

Both follow directly from the explicit formula for  $A_0^{\mathcal{B}_0, \omega}$  and from Eq. (5.4).

Combining Eq. (5.239) with Eq. (5.272), and using Eq. (5.273)–Eq. (5.274), the imaginary part cancels exactly. More precisely, the imaginary part is

$$\mathrm{ia} \left[ -\frac{1}{4(1+|\nabla\eta|^2)} \left( (\Omega_\psi \cdot \nabla\eta) \Omega_\psi - \frac{|\Omega_\psi|^2 + |\Omega_\psi \cdot \nabla\eta|^2}{1+|\nabla\eta|^2} \nabla\eta \right) \cdot \xi + \lambda_1(\eta; x, \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_\psi \right) \frac{\nabla^\perp \eta \cdot \Omega_\psi}{1+|\nabla\eta|^2} \right] = 0.$$

The remaining real part is

$$\begin{aligned} & a_1^{\pi_2^*, \eta} [\Omega_\psi, \Omega_\psi] (\mathfrak{a}|\eta; x, \xi) + A_0^{\pi_2^*, \Omega} [\Omega_\psi] (\mathfrak{a}|\eta; x, \xi) \cdot A_1^{S, \eta} [\psi] (\mathfrak{a}|\eta; x, \xi) \\ & = \lambda_1(\eta; x, \xi) \left[ (4 - \mathfrak{a}^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot S(\mathfrak{a}|\eta) \psi \right)^2 - \left( \mathfrak{a} \nu_3[\psi] (\mathfrak{a}|\eta; x) - \frac{\nabla^\perp \eta \cdot S(\mathfrak{a}|\eta) \psi}{1+|\nabla\eta|^2} \right)^2 \right]. \end{aligned} \quad (5.275)$$

By the definitions Eq. (5.263) and Eq. (5.264), Eq. (5.275) is precisely Eq. (5.261).

**Part 4 (Conclusion).** By Lemma 5.29,

$$\begin{aligned} \pi_0(\rho^\pm|\eta) &= -\mathfrak{g}\mathfrak{a}\eta + \mathrm{Op}^{\mathrm{BW}}(a_2^{\pi, \eta}(\rho^\pm|\eta; x, \xi) + a_{[0, \geq 1]}(\rho^\pm|\eta; x, \xi))\eta \\ &\quad + r_{[\varrho, \geq 1]}^{\pi_0, \mathbb{R}}(\sigma, \rho^\pm|\eta)\eta. \end{aligned} \quad (5.276)$$

Combining Eq. (5.276), Eq. (5.269), Eq. (5.271), and Eq. (5.275), we obtain exactly Eq. (5.260). The parity and reality properties stated in the proposition follow from the fact that  $\lambda_1$  is real and even,  $A_0^{\mathcal{B}_0, \omega; \perp}$  is real and odd, while  $b[\psi]$  and  $\mathcal{V}[\psi](\mathfrak{a}|\eta)$  are real-valued and independent of  $\xi$ . □

### 5.3.7 Paralinearization of $\tilde{\pi}$

We now paralinearize the term  $\tilde{\pi}$  which was defined in Eq. (2.16), this is a generalization of the paralinearization of  $\pi$  provided in Proposition 5.32 since  $\pi$  can be recovered from  $\tilde{\pi}$  in the limit  $\bar{\Omega} \rightarrow 0$ . The result we prove is the following

**Proposition 5.33** (Paralinearization of  $\tilde{\pi}$ ). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle\langle \rho^\pm \rangle\rangle > 0$ , and  $\mathfrak{g} \in \mathbb{R}$ . There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta, \tilde{\psi} \in B_{s_0, \mathbb{R}}(I; r_0)$  and any  $\bar{\Omega} \in \mathbb{R}^2$ , the shifted Bernoulli function  $\tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi})$  defined in Eq. (2.16) admits the following paralinearization:*

$$\begin{aligned} \tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}) &= -\mathfrak{g}\mathfrak{a}\eta + \mathrm{Op}^{\mathrm{BW}}(A_2^{\tilde{\pi}}(\rho^\pm|\eta; x, \xi) + A_1^{\tilde{\pi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) + A_{[0, \geq 1]}(\sigma, \rho^\pm, \mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi)) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \\ &\quad + R_{[\varrho, \geq 1]}^{\mathbb{R}}(\sigma, \rho^\pm, \mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}, \end{aligned} \quad (5.277)$$

where

- $A_2^{\tilde{\pi}} \in (\Gamma_{\geq 0}^2)^2$  and is explicitly defined as

$$A_2^{\tilde{\pi}}(\rho^\pm|\eta; x, \xi) := \begin{bmatrix} a_2^{\pi_0, \eta}(\rho^\pm|\eta; x, \xi) \\ 0 \end{bmatrix}, \quad (5.278)$$

where

- $a_2^{\pi_0, \eta} \in \Gamma_{\geq 0}^2$  is real-valued and is explicitly defined in Eq. (5.228). In particular,

$$a_2^{\pi_0, \eta}(\rho^\pm|0; x, \xi) = -\frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2.$$

- $A_1^{\tilde{\pi}} \in (\Gamma_{\geq 0}^1)^2$  and is explicitly defined as

$$A_1^{\tilde{\pi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := \begin{bmatrix} a_1^{\tilde{\pi}, \eta}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) \\ a_1^{\tilde{\pi}, \tilde{\psi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) \end{bmatrix},$$

where

- $a_1^{\tilde{\pi}, \eta} \in \Gamma_{\geq 0}^1$  is real-valued and is explicitly defined as

$$a_1^{\tilde{\pi}, \eta}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := \lambda_1(\eta; x, \xi) \left[ q_0^{\tilde{\pi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) - (b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x))^2 \right]. \quad (5.279)$$

In particular,  $a_1^{\tilde{\pi}, \eta}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi)$  is even in  $\xi$ ;

- $a_1^{\tilde{\pi}, \tilde{\psi}} \in \Gamma_{\geq 0}^1$  is explicitly defined as

$$a_1^{\tilde{\pi}, \tilde{\psi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := -\lambda_1(\eta; x, \xi) b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) + i \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \xi. \quad (5.280)$$

In particular, the first term in Eq. (5.280) is real-valued and even in  $\xi$ , while the second one is the transport symbol;

- $q_0^{\tilde{\pi}} \in \Gamma_{\geq 0}^0$  is real-valued and is explicitly defined as

$$q_0^{\tilde{\pi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := (4 - \mathfrak{a}^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega(\bar{\Omega}|\eta, \tilde{\psi}; x) \right)^2. \quad (5.281)$$

In particular,  $q_0^{\tilde{\pi}}$  is even in  $\xi$ ;

- $\Omega(\bar{\Omega}|\eta, \tilde{\psi})$  is explicitly defined as

$$\Omega(\bar{\Omega}|\eta, \tilde{\psi}; x) := \bar{\Omega}[\bar{\Omega}](\eta; x) + S(\mathfrak{a}|\eta) \tilde{\psi}(x). \quad (5.282)$$

- $b_{\text{can}} \in \mathcal{F}^{\mathbb{R}}$  is real-valued and is explicitly defined as

$$b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) := \mathfrak{a} \nu_3^*[\Omega(\bar{\Omega}|\eta, \tilde{\psi})](\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega(\bar{\Omega}|\eta, \tilde{\psi}; x)}{1 + |\nabla \eta|^2}. \quad (5.283)$$

In particular,

$$b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|0, 0; x) = 0;$$

- $\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) \in (\mathcal{F}_{\geq 0}^{\mathbb{R}})^2$  is real-valued and is explicitly defined as

$$\begin{aligned} \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) &:= V^*[\Omega(\bar{\Omega}|\eta, \tilde{\psi})](\eta; x) \\ &+ \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} \mathbf{J}(\Omega(\bar{\Omega}|\eta, \tilde{\psi}; x) + (\nabla \eta \cdot \Omega(\bar{\Omega}|\eta, \tilde{\psi}; x)) \nabla \eta). \end{aligned} \quad (5.284)$$

In particular, its nonzero flat part is separated as

$$\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) = \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp + \mathcal{V}_{\text{can}}^{\text{nl}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}), \quad \mathcal{V}_{\text{can}}^{\text{nl}} \in (\mathcal{F}^{\mathbb{R}})^2.$$

Thus  $\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x)$  is independent of  $\xi$  and

$$\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|0, 0; x) = \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp;$$

- $A_{[0, \geq 1]}$  is a generic vector symbol as per Notation 5.1;
- $R_{[\rho, \geq 1]}^{\mathbb{R}}$  is a smoothing remainder as per Notation 5.1.

**Paralinearization of  $\tilde{\pi}_0$**  We isolate the background contribution to the zero-th order canonical pressure. Namely, with  $\tilde{\Omega}[\tilde{\Omega}](\eta)$  defined in Eq. (2.10), we set

$$\tilde{\pi}_0[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta) := \pi_2^*(\mathfrak{a}|\eta)[\tilde{\Omega}[\tilde{\Omega}](\eta), \tilde{\Omega}[\tilde{\Omega}](\eta)]. \quad (5.285)$$

Equivalently, the full zero-th order canonical term in Eq. (2.23a) is

$$\pi_{\text{can},0}(\rho^\pm, \tilde{\Omega}|\eta) = \pi_0(\rho^\pm|\eta) + \tilde{\pi}_0[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta) - c_{\tilde{\Omega}}.$$

The local background term has value  $c_{\tilde{\Omega}}$  at  $\eta = 0$ ; this constant must be retained in its paralinearization and cancels the explicit  $-c_{\tilde{\Omega}}$  in the full canonical term.

**Lemma 5.34** (Paralinearization of  $\tilde{\pi}_0$ ). *Let  $\sigma \geq 0$ ,  $\rho^\pm \geq 0$  with  $\langle\langle \rho^\pm \rangle\rangle > 0$ , and  $\mathfrak{g} \in \mathbb{R}$ . There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta \in B_{s_0, \mathbb{R}}(I; r_0)$  and any  $\tilde{\Omega} \in \mathbb{R}^2$ , the term  $\tilde{\pi}_0[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta)$  defined in Eq. (5.285) admits the following paralinearization:*

$$\tilde{\pi}_0[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta) = c_{\tilde{\Omega}} + \text{Op}^{\text{BW}}\left(a_1^{\tilde{\pi}_0, \eta}[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta; x, \xi) + a_{[0, \geq 1]}[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta; x, \xi)\right)\eta + r_{[\varrho, \geq 1]}^{\mathbb{R}}[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta)\eta, \quad (5.286)$$

where

- $a_1^{\tilde{\pi}_0, \eta} \in \mathcal{L}^2(\mathbb{R}^2; \Gamma_{\geq 0}^1)$  is real-valued and is explicitly defined as

$$a_1^{\tilde{\pi}_0, \eta}[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta; x, \xi) := \lambda_1(\eta; x, \xi) \left( a_0^{\tilde{\pi}_0}[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta; x, \xi) - (b_{\text{bg}}[\tilde{\Omega}](\mathfrak{a}|\eta; x))^2 \right), \quad (5.287)$$

where

- $a_0^{\tilde{\pi}_0} \in \mathcal{L}^2(\mathbb{R}^2; \Gamma_{\geq 0}^0)$  is real-valued and is explicitly defined as

$$a_0^{\tilde{\pi}_0}[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta; x, \xi) := (4 - \mathfrak{a}^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \tilde{\Omega}[\tilde{\Omega}](\eta; x) \right)^2;$$

- $b_{\text{bg}}[\tilde{\Omega}] \in \mathcal{L}(\mathbb{R}^2; \mathcal{F}^{\mathbb{R}})$  is real-valued and is explicitly defined as

$$b_{\text{bg}}[\tilde{\Omega}](\mathfrak{a}|\eta; x) := \mathfrak{a} \nu_3^*[\tilde{\Omega}[\tilde{\Omega}](\eta)](\eta; x) - \frac{\nabla^\perp \eta \cdot \tilde{\Omega}[\tilde{\Omega}](\eta; x)}{1 + |\nabla \eta|^2}; \quad (5.288)$$

- $a_{[0, \geq 1]}$  is a generic symbol as per Notation 5.1;
- $r_{[\varrho, \geq 1]}^{\mathbb{R}}$  is a smoothing remainder as per Notation 5.1.

Consequently, the full zero-th order canonical term satisfies

$$\begin{aligned} \pi_{\text{can},0}(\rho^\pm, \tilde{\Omega}|\eta) &= -\mathfrak{g}\mathfrak{a}\eta + \text{Op}^{\text{BW}}\left(a_2^{\pi_0, \eta}(\rho^\pm|\eta; x, \xi) + a_1^{\tilde{\pi}_0, \eta}[\tilde{\Omega}, \tilde{\Omega}](\mathfrak{a}|\eta; x, \xi) + a_{[0, \geq 1]}[\tilde{\Omega}, \tilde{\Omega}](\sigma, \rho^\pm, \mathfrak{a}|\eta; x, \xi)\right)\eta \\ &\quad + r_{[\varrho, \geq 1]}^{\mathbb{R}}[\tilde{\Omega}, \tilde{\Omega}](\sigma, \rho^\pm, \mathfrak{a}|\eta)\eta, \end{aligned} \quad (5.289)$$

where  $a_2^{\pi_0, \eta}$  is defined in Eq. (5.228).

*Proof.* Set

$$\Omega_{\text{bg}} := \tilde{\Omega}[\tilde{\Omega}](\eta), \quad \delta\Omega_{\text{bg}} := \Omega_{\text{bg}} - \tilde{\Omega}.$$

By Eqs. (5.169) and (5.170),  $\delta\Omega_{\text{bg}}$  is the Sobolev correction whose principal  $\eta$ -symbol is  $A_{1, \text{bg}}^{S, \eta}[\tilde{\Omega}]$ . We first use the bilinearity of  $\pi_2^*$  to split

$$\begin{aligned} \pi_2^*(\mathfrak{a}|\eta)[\Omega_{\text{bg}}, \Omega_{\text{bg}}] &= \pi_2^*(\mathfrak{a}|\eta)[\tilde{\Omega}, \tilde{\Omega}] \\ &\quad + 2\pi_2^*(\mathfrak{a}|\eta)[\tilde{\Omega}, \delta\Omega_{\text{bg}}] \\ &\quad + \pi_2^*(\mathfrak{a}|\eta)[\delta\Omega_{\text{bg}}, \delta\Omega_{\text{bg}}]. \end{aligned}$$

The constant–constant block equals  $c_{\bar{\Omega}}$  at  $\eta = 0$  by Eq. (2.21); its difference from  $c_{\bar{\Omega}}$  vanishes at the base point and is parilinearized in  $\eta$  with  $\bar{\Omega}$  as an external parameter. In the two remaining blocks we use the bilinear formula of Corollary 5.31, applying its Sobolev estimates only to  $\delta\Omega_{\text{bg}}$  and using Remark 4.14 for every direct paradifferential action on  $\bar{\Omega}$ . Composing the Sobolev factors with the canonical parilinearization just cited and resumming the three blocks gives

$$\begin{aligned}\tilde{\pi}_0 [\bar{\Omega}, \bar{\Omega}] (\mathfrak{a}|\eta) &= c_{\bar{\Omega}} + \text{Op}^{\text{BW}} \left( a_1^{\pi_2^*, \eta} [\Omega_{\text{bg}}, \Omega_{\text{bg}}] (\mathfrak{a}|\eta; x, \xi) \right) \eta \\ &\quad + \text{Op}^{\text{BW}} \left( A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] (\mathfrak{a}|\eta; x, \xi) \cdot A_{1, \text{bg}}^{S, \eta} [\bar{\Omega}] (\mathfrak{a}|\eta; x, \xi) \right) \eta \\ &\quad + \text{Op}^{\text{BW}} (a_{[0, \geq 1]} [\bar{\Omega}, \bar{\Omega}] (\mathfrak{a}|\eta; x, \xi)) \eta + r_{[\varrho, \geq 1]}^{\mathbb{R}} [\bar{\Omega}, \bar{\Omega}] (\mathfrak{a}|\eta) \eta.\end{aligned}\tag{5.290}$$

We now simplify the order-one symbol in Eq. (5.290). From Eq. (5.240) and Eq. (5.170), one has

$$\begin{aligned}&A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] \cdot A_{1, \text{bg}}^{S, \eta} [\bar{\Omega}] \\ &= -\mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega_{\text{bg}} - i\nu_3^* [\Omega_{\text{bg}}] \right) \left[ V^* [\Omega_{\text{bg}}] \cdot \xi + \frac{\mathfrak{a}}{4(1 + |\nabla\eta|^2)} J(\Omega_{\text{bg}} + (\nabla\eta \cdot \Omega_{\text{bg}}) \nabla\eta) \cdot \xi \right. \\ &\quad \left. + i\lambda_1 \left( \mathfrak{a}\nu_3^* [\Omega_{\text{bg}}] - \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2} \right) \right],\end{aligned}\tag{5.291}$$

where all the symbols in the right-hand side are evaluated at  $(\mathfrak{a}|\eta; x, \xi)$  and all functions at  $x$ .

The two pointwise identities

$$\frac{J(\Omega_{\text{bg}} + (\nabla\eta \cdot \Omega_{\text{bg}}) \nabla\eta) \cdot \xi}{4(1 + |\nabla\eta|^2)} = \lambda_1(\eta; x, \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\text{bg}} \right)\tag{5.292}$$

and

$$\frac{1}{4(1 + |\nabla\eta|^2)} \left( (\Omega_{\text{bg}} \cdot \nabla\eta) \Omega_{\text{bg}} - \frac{|\Omega_{\text{bg}}|^2 + |\Omega_{\text{bg}} \cdot \nabla\eta|^2}{1 + |\nabla\eta|^2} \nabla\eta \right) \cdot \xi = \lambda_1(\eta; x, \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\text{bg}} \right) \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2}\tag{5.293}$$

follow directly from the explicit formula for  $A_0^{\mathcal{B}_0, \omega}$  and Eq. (5.4).

Combining Eq. (5.239) and Eq. (5.291), the imaginary part of the resulting order-one symbol is

$$\begin{aligned}&-\mathfrak{a} \left[ \frac{1}{4(1 + |\nabla\eta|^2)} \left( (\Omega_{\text{bg}} \cdot \nabla\eta) \Omega_{\text{bg}} - \frac{|\Omega_{\text{bg}}|^2 + |\Omega_{\text{bg}} \cdot \nabla\eta|^2}{1 + |\nabla\eta|^2} \nabla\eta \right) \cdot \xi - \lambda_1 \left( A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega_{\text{bg}} \right) \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2} \right] \\ &- \mathfrak{a}^2 \nu_3^* [\Omega_{\text{bg}}] \left[ \lambda_1 \left( A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega_{\text{bg}} \right) - \frac{J(\Omega_{\text{bg}} + (\nabla\eta \cdot \Omega_{\text{bg}}) \nabla\eta) \cdot \xi}{4(1 + |\nabla\eta|^2)} \right],\end{aligned}$$

which vanishes by Eq. (5.292) and Eq. (5.293). Hence the principal symbol is real.

Taking the real part and using again Eq. (5.292), we obtain

$$\text{Re} \left[ a_1^{\pi_2^*, \eta} [\Omega_{\text{bg}}, \Omega_{\text{bg}}] + A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] \cdot A_{1, \text{bg}}^{S, \eta} [\bar{\Omega}] \right] = \lambda_1 \left[ (4 - \mathfrak{a}^2) \left( A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega_{\text{bg}} \right)^2 - \left( \mathfrak{a}\nu_3^* [\Omega_{\text{bg}}] - \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2} \right)^2 \right].$$

This is exactly Eq. (5.287). The generic order-zero symbol and the smoothing remainder in Eq. (5.286) collect the lower symbolic terms from Eq. (5.169), from Lemma 5.30, and from the Bony–Weyl composition formula. Finally, Eq. (5.289) follows by adding Eq. (5.227): the  $c_{\bar{\Omega}}$  in Eq. (5.286) cancels the explicit  $-c_{\bar{\Omega}}$  in the definition of  $\pi_{\text{can}, 0}$  from Eq. (2.23a).  $\square$

**Paralinearization of  $\tilde{\pi}_1$**  We now paralinearize the term which is linear in the canonical perturbation  $\tilde{\psi}$ . Recalling the decomposition Eq. (2.23a), we set

$$\tilde{\pi}_1(\mathfrak{a}, \bar{\Omega}|\eta)[\tilde{\psi}] := 2\pi_2^*(\mathfrak{a}|\eta)[\bar{\Omega}[\bar{\Omega}](\eta), S(\mathfrak{a}|\eta)\tilde{\psi}]. \quad (5.294)$$

**Lemma 5.35** (Paralinearization of  $\tilde{\pi}_1$ ). *There exists  $\varrho_0 > 0$  such that for any  $\varrho \geq \varrho_0$ , there exist  $s_0 > 0$  and  $r_0 > 0$  such that for any  $\eta, \tilde{\psi} \in B_{s_0, \mathbb{R}}(I; r_0)$  and any  $\bar{\Omega} \in \mathbb{R}^2$ , the term  $\tilde{\pi}_1(\mathfrak{a}, \bar{\Omega}|\eta)[\tilde{\psi}]$  defined in Eq. (5.294) admits the following paralinearization:*

$$\begin{aligned} \tilde{\pi}_1(\mathfrak{a}, \bar{\Omega}|\eta)[\tilde{\psi}] &= \text{Op}^{\text{BW}}\left(A_1^{\tilde{\pi}_1}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) + A_{[0, \geq 1]}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi)\right) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \\ &\quad + R_{[\varrho, \geq 1]}^{\mathbb{R}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}, \end{aligned} \quad (5.295)$$

where

- $A_1^{\tilde{\pi}_1} \in (\Gamma_{\geq 0}^1)^2$  and is explicitly defined as

$$A_1^{\tilde{\pi}_1}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) := \begin{bmatrix} a_1^{\tilde{\pi}_1, \eta}[\bar{\Omega}, \tilde{\psi}](\mathfrak{a}|\eta; x, \xi) \\ a_1^{\tilde{\pi}_1, \tilde{\psi}}[\bar{\Omega}](\mathfrak{a}|\eta; x, \xi) \end{bmatrix},$$

where

- $a_1^{\tilde{\pi}_1, \eta} \in \mathcal{L}(\mathbb{R}^2; \mathcal{L}(H^{s_0}; \Gamma_{\geq 0}^1))$  is real-valued and is explicitly defined as

$$\begin{aligned} a_1^{\tilde{\pi}_1, \eta}[\bar{\Omega}, \tilde{\psi}](\mathfrak{a}|\eta; x, \xi) &:= 2\lambda_1(\eta; x, \xi) \left[ (4 - \mathfrak{a}^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \bar{\Omega}[\bar{\Omega}](\eta; x) \right) \right. \\ &\quad \times \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot S(\mathfrak{a}|\eta)\tilde{\psi}(x) \right) \\ &\quad \left. - b_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|\eta; x) b_{\tilde{\psi}}[\tilde{\psi}](\mathfrak{a}|\eta; x) \right]. \end{aligned} \quad (5.296)$$

In particular,  $a_1^{\tilde{\pi}_1, \eta}[\bar{\Omega}, \tilde{\psi}](\mathfrak{a}|\eta; x, \xi)$  is real-valued and even in  $\xi$ ;

- $a_1^{\tilde{\pi}_1, \tilde{\psi}} \in \mathcal{L}(\mathbb{R}^2; \Gamma_{\geq 0}^1)$  is explicitly defined as

$$a_1^{\tilde{\pi}_1, \tilde{\psi}}[\bar{\Omega}](\mathfrak{a}|\eta; x, \xi) := -\lambda_1(\eta; x, \xi) b_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|\eta; x) + i \mathcal{V}_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|\eta; x) \cdot \xi. \quad (5.297)$$

In particular, the first term in Eq. (5.297) is real-valued and even in  $\xi$ , while the second one is the transport symbol generated by the canonical background;

- $b_{\text{bg}}[\bar{\Omega}] \in \mathcal{L}(\mathbb{R}^2; \mathcal{F}^{\mathbb{R}})$  is the real-valued coefficient defined in Eq. (5.288). In particular,

$$b_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|0; x) = 0;$$

- $b_{\tilde{\psi}}[\tilde{\psi}] \in \mathcal{L}(H^{s_0}; \mathcal{F}^{\mathbb{R}})$  is real-valued and is explicitly defined as

$$b_{\tilde{\psi}}[\tilde{\psi}](\mathfrak{a}|\eta; x) := \mathfrak{a} \nu_3[\tilde{\psi}](\mathfrak{a}|\eta; x) - \frac{\nabla^\perp \eta \cdot S(\mathfrak{a}|\eta)\tilde{\psi}(x)}{1 + |\nabla \eta|^2}. \quad (5.298)$$

In particular,  $b_{\tilde{\psi}}[\tilde{\psi}](\mathfrak{a}|\eta; x)$  is independent of  $\xi$ ;

- $\mathcal{V}_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|\eta) \in \mathcal{L}(\mathbb{R}^2; (\mathcal{F}_{\geq 0}^{\mathbb{R}})^2)$  is real-valued and is explicitly defined as

$$\begin{aligned} \mathcal{V}_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|\eta; x) &:= V^*[\bar{\Omega}[\bar{\Omega}](\eta)](\eta; x) \\ &\quad + \frac{\mathfrak{a}}{4(1 + |\nabla \eta|^2)} \mathbf{J}(\bar{\Omega}[\bar{\Omega}](\eta; x) + (\nabla \eta \cdot \bar{\Omega}[\bar{\Omega}](\eta; x)) \nabla \eta). \end{aligned} \quad (5.299)$$

Its flat part and nonlinear correction are

$$\mathcal{V}_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|\eta) = \frac{\mathfrak{a}}{4} \bar{\Omega}^\perp + \mathcal{V}_{\text{bg}}^{\text{nl}}[\bar{\Omega}](\mathfrak{a}|\eta), \quad \mathcal{V}_{\text{bg}}^{\text{nl}} \in \mathcal{L}(\mathbb{R}^2; (\mathcal{F}_{\geq 1}^{\mathbb{R}})^2). \quad (5.300)$$

In particular,  $\mathcal{V}_{\text{bg}}[\bar{\Omega}](\mathfrak{a}|\eta; x)$  is independent of  $\xi$  and equals  $\frac{\mathfrak{a}}{4} \bar{\Omega}^\perp$  at  $\eta = 0$ .

- $A_{[0,\geq 1]}$  is a generic vector symbol as per Notation 5.1;
- $R_{[\varrho,\geq 1]}^{\mathbb{R}}$  is a smoothing remainder as per Notation 5.1.

*Proof.* For the duration of the proof set

$$\Omega_{\text{bg}} := \tilde{\Omega} [\bar{\Omega}] (\eta), \quad \delta\Omega_{\text{bg}} := \Omega_{\text{bg}} - \bar{\Omega}, \quad \Omega_{\tilde{\psi}} := S(\mathfrak{a}|\eta) \tilde{\psi}.$$

Then Eq. (5.294) reads

$$\tilde{\pi}_1(\mathfrak{a}, \bar{\Omega}|\eta) [\tilde{\psi}] = 2\pi_2^*(\mathfrak{a}|\eta) [\Omega_{\text{bg}}, \Omega_{\tilde{\psi}}].$$

**Part 1** (Application of the bilinear parilinearization). For the target gain  $\varrho$ , use Corollary 5.31 and Proposition 5.24 at gain  $\varrho + 1$  and choose the Sobolev threshold one derivative above the one in the corollary. This is legitimate because  $\Omega_{\tilde{\psi}} = S(\mathfrak{a}|\eta) \tilde{\psi}$  loses one derivative, and the subsequent order-one composition turns gain  $\varrho + 1$  into gain  $\varrho$ . We also split

$$\pi_2^*(\mathfrak{a}|\eta) [\Omega_{\text{bg}}, \Omega_{\tilde{\psi}}] = \pi_2^*(\mathfrak{a}|\eta) [\bar{\Omega}, \Omega_{\tilde{\psi}}] + \pi_2^*(\mathfrak{a}|\eta) [\delta\Omega_{\text{bg}}, \Omega_{\tilde{\psi}}].$$

Using the bilinear formula with  $\bar{\Omega}$  as an external parameter, applying its Sobolev estimates to  $\delta\Omega_{\text{bg}}$  and  $\Omega_{\tilde{\psi}}$ , and using Remark 4.14 for the direct actions on  $\bar{\Omega}$ , then resumming, gives

$$\begin{aligned} \tilde{\pi}_1(\mathfrak{a}, \bar{\Omega}|\eta) [\tilde{\psi}] &= 2\text{Op}^{\text{BW}} \left( a_1^{\pi_2^*, \eta} [\Omega_{\text{bg}}, \Omega_{\tilde{\psi}}] (\mathfrak{a}|\eta; x, \xi) \right) \eta \\ &\quad + \text{Op}^{\text{BW}} \left( A_0^{\pi_2^*, \Omega} [\Omega_{\tilde{\psi}}] (\mathfrak{a}|\eta; x, \xi) \right) \cdot \Omega_{\text{bg}} \\ &\quad + \text{Op}^{\text{BW}} \left( A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] (\mathfrak{a}|\eta; x, \xi) \right) \cdot \Omega_{\tilde{\psi}} \\ &\quad + \text{Op}^{\text{BW}} \left( A_{[0,\geq 1]} (\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) \right) \cdot \left[ \frac{\eta}{\tilde{\psi}} \right] + R_{[\varrho,\geq 1]}^{\mathbb{R}} (\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) \cdot \left[ \frac{\eta}{\tilde{\psi}} \right]. \end{aligned} \quad (5.301)$$

For the background factor we use directly Eqs. (5.169) and (5.170); in particular, the principal  $\eta$ -symbol of  $\delta\Omega_{\text{bg}}$  is  $A_{1,\text{bg}}^{S,\eta} [\bar{\Omega}]$ . Likewise, by Proposition 5.24,

$$\Omega_{\tilde{\psi}} = \text{Op}^{\text{BW}} \left( \left[ A_1^{S,\eta} [\tilde{\psi}] (\mathfrak{a}|\eta; x, \xi) \quad i\xi^\perp \right] + A_{[0,\geq 1]} (\mathfrak{a}|\eta, \tilde{\psi}; x, \xi) \right) \left[ \frac{\eta}{\tilde{\psi}} \right] + R_{[\varrho,\geq 1]}^{\mathbb{R}} (\mathfrak{a}|\eta, \tilde{\psi}) \cdot \left[ \frac{\eta}{\tilde{\psi}} \right], \quad (5.302)$$

where

$$A_1^{S,\eta} [\tilde{\psi}] (\mathfrak{a}|\eta; x, \xi) = -\mathfrak{a} \left( A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \Omega_{\tilde{\psi}}(x) - i\nu_3 [\tilde{\psi}] (\mathfrak{a}|\eta; x) \right) \xi^\perp. \quad (5.303)$$

The constant vector  $\bar{\Omega}$  in Eq. (5.169) is annihilated by Bony–Weyl quantization. Therefore, composing Eq. (5.301) with Eq. (5.169) and Eq. (5.302), and using the symbolic composition formula, gives

$$\begin{aligned} \tilde{\pi}_1(\mathfrak{a}, \bar{\Omega}|\eta) [\tilde{\psi}] &= \text{Op}^{\text{BW}} \left( \left[ \begin{aligned} &2a_1^{\pi_2^*, \eta} [\Omega_{\text{bg}}, \Omega_{\tilde{\psi}}] + A_0^{\pi_2^*, \Omega} [\Omega_{\tilde{\psi}}] \cdot A_{1,\text{bg}}^{S,\eta} [\bar{\Omega}] + A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] \cdot A_1^{S,\eta} [\tilde{\psi}] \\ &A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] \cdot i\xi^\perp \end{aligned} \right] \right) \cdot \left[ \frac{\eta}{\tilde{\psi}} \right] \\ &\quad + \text{Op}^{\text{BW}} \left( A_{[0,\geq 1]} (\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) \right) \cdot \left[ \frac{\eta}{\tilde{\psi}} \right] + R_{[\varrho,\geq 1]}^{\mathbb{R}} (\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}) \cdot \left[ \frac{\eta}{\tilde{\psi}} \right]. \end{aligned} \quad (5.304)$$

All symbols in Eq. (5.304) are evaluated at  $(\mathfrak{a}|\eta; x, \xi)$ .

**Part 2** ( $\tilde{\psi}$ -component). Using Eq. (5.243) with  $\Omega_j = \Omega_{\text{bg}}$ , we compute

$$A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] (\mathfrak{a}|\eta; x, \xi) \cdot i\xi^\perp = i\mathcal{V}_{\text{bg}} [\bar{\Omega}] (\mathfrak{a}|\eta; x) \cdot \xi + (1 + |\nabla\eta|^2) b_{\text{bg}} [\bar{\Omega}] (\mathfrak{a}|\eta; x) A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \xi^\perp.$$

Since

$$(1 + |\nabla\eta|^2) A_0^{\mathcal{B}_0, \omega; \perp} (\eta; x, \xi) \cdot \xi^\perp = -\lambda_1 (\eta; x, \xi),$$

we obtain

$$A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] (\mathfrak{a}|\eta; x, \xi) \cdot i\xi^\perp = -\lambda_1 (\eta; x, \xi) b_{\text{bg}} [\bar{\Omega}] (\mathfrak{a}|\eta; x) + i\mathcal{V}_{\text{bg}} [\bar{\Omega}] (\mathfrak{a}|\eta; x) \cdot \xi. \quad (5.305)$$

This is exactly Eq. (5.297).

**Part 3** ( $\eta$ -component). We simplify the first component of Eq. (5.304). The two elementary identities

$$\frac{J(\Omega_{\text{bg}} + (\nabla\eta \cdot \Omega_{\text{bg}}) \nabla\eta) \cdot \xi}{4(1 + |\nabla\eta|^2)} = \lambda_1(\eta; x, \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\text{bg}} \right), \quad (5.306)$$

and

$$\frac{J(\Omega_{\tilde{\psi}} + (\nabla\eta \cdot \Omega_{\tilde{\psi}}) \nabla\eta) \cdot \xi}{4(1 + |\nabla\eta|^2)} = \lambda_1(\eta; x, \xi) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\tilde{\psi}} \right) \quad (5.307)$$

follow from the explicit formula for  $A_0^{\mathcal{B}_0, \omega}$  and from Eq. (5.4). We shall also use

$$\begin{aligned} & \frac{1}{4(1 + |\nabla\eta|^2)} \left( \frac{1}{2} (\Omega_{\text{bg}} \cdot \nabla\eta) \Omega_{\tilde{\psi}} + \frac{1}{2} (\Omega_{\tilde{\psi}} \cdot \nabla\eta) \Omega_{\text{bg}} - \frac{\Omega_{\text{bg}} \cdot \Omega_{\tilde{\psi}} + (\Omega_{\text{bg}} \cdot \nabla\eta) (\Omega_{\tilde{\psi}} \cdot \nabla\eta)}{1 + |\nabla\eta|^2} \nabla\eta \right) \cdot \xi \\ &= \frac{\lambda_1(\eta; x, \xi)}{2} \left[ \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\text{bg}} \right) \frac{\nabla^\perp \eta \cdot \Omega_{\tilde{\psi}}}{1 + |\nabla\eta|^2} + \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\tilde{\psi}} \right) \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2} \right]. \end{aligned} \quad (5.308)$$

This is the polarization of Eq. (5.293).

Combining Eq. (5.242), Eq. (5.170), and Eq. (5.303), the imaginary part of the first component in Eq. (5.304) is a sum of the two brackets

$$\lambda_1 \left( A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega_{\text{bg}} \right) \frac{\nabla^\perp \eta \cdot \Omega_{\tilde{\psi}}}{1 + |\nabla\eta|^2} + \lambda_1 \left( A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega_{\tilde{\psi}} \right) \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2}$$

minus the corresponding expression in Eq. (5.308), together with the two transport brackets cancelled by Eq. (5.306) and Eq. (5.307). Hence the imaginary part vanishes.

The real part is

$$\begin{aligned} & 2a_1^{\pi_2^*, \eta} [\Omega_{\text{bg}}, \Omega_{\tilde{\psi}}] + A_0^{\pi_2^*, \Omega} [\Omega_{\tilde{\psi}}] \cdot A_{1, \text{bg}}^{S, \eta} [\bar{\Omega}] + A_0^{\pi_2^*, \Omega} [\Omega_{\text{bg}}] \cdot A_1^{S, \eta} [\tilde{\psi}] \\ &= 2\lambda_1(\eta; x, \xi) \left[ (4 - \alpha^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\text{bg}} \right) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega_{\tilde{\psi}} \right) \right. \\ & \quad \left. - \left( \alpha \nu_3^* [\Omega_{\text{bg}}] (\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2} \right) \left( \alpha \nu_3 [\tilde{\psi}] (\alpha|\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega_{\tilde{\psi}}}{1 + |\nabla\eta|^2} \right) \right]. \end{aligned} \quad (5.309)$$

By the definitions Eq. (5.288) and Eq. (5.298), this is exactly Eq. (5.296).

**Part 4** (Synthesis). The identities Eq. (5.305) and Eq. (5.309), inserted into Eq. (5.304), give Eq. (5.295). The lower order symbols and smoothing remainders collect the order-zero part of the parilinearizations of  $S(\alpha|\eta)$  and of  $\bar{\Omega}[\bar{\Omega}](\eta)$ , the order  $-1$  column of Corollary 5.31, and the symbolic remainders in the Bony–Weyl composition formula.

□

**Parilinearization of  $\tilde{\pi}_2$**  Recall that  $\tilde{\pi}_2$  is defined in Eq. (2.23c), definition that we recall here for simplicity

$$\tilde{\pi}_2(\alpha|\eta) [\tilde{\psi}, \tilde{\psi}] = \pi_2(\alpha|\eta) [\tilde{\psi}, \tilde{\psi}]$$

so the parilinearization of  $\tilde{\pi}_2$  is provided by Eq. (5.269).

*Proof of Proposition 5.33.* We combine the three homogeneous components in Eq. (2.22), using the shorthand

$$\Omega_{\text{bg}} := \bar{\Omega}[\bar{\Omega}](\eta), \quad \Omega_{\tilde{\psi}} := S(\alpha|\eta) \tilde{\psi},$$

so that, by Eq. (5.282),  $\Omega(\bar{\Omega}|\eta, \tilde{\psi}) = \Omega_{\text{bg}} + \Omega_{\tilde{\psi}}$ . Since the formulas Eq. (5.283) and Eq. (5.284) are linear in  $\Omega$ , this decomposition gives

$$b_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}) = b_{\text{bg}}[\bar{\Omega}](\alpha|\eta) + b_{\tilde{\psi}}[\tilde{\psi}](\alpha|\eta), \quad \mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}) = \mathcal{V}_{\text{bg}}[\bar{\Omega}](\alpha|\eta) + \mathcal{V}[\tilde{\psi}](\alpha|\eta).$$

**Part 1** (Order-two symbol). The only order-two contribution comes from  $\tilde{\pi}_0$ , which is parilinearized in Eq. (5.289); the order-two symbol is therefore exactly Eq. (5.278).

**Part 2** (Order-one  $\eta$ -component). By Lemmas 5.34 and 5.35 and Proposition 5.32, the three contributions to the order-one  $\eta$ -symbol are  $a_1^{\tilde{\pi}_0, \eta}$ ,  $a_1^{\tilde{\pi}_1, \eta}$ , and  $a_1^{\pi, \eta}$  (with  $\psi$  replaced by  $\tilde{\psi}$ ). Summing and using  $\Omega = \Omega_{\text{bg}} + \Omega_{\tilde{\psi}}$ ,

$$a_1^{\tilde{\pi}_0, \eta} + a_1^{\tilde{\pi}_1, \eta} + a_1^{\pi, \eta} = \lambda_1(\eta; x, \xi) \left[ (4 - \alpha^2) \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \Omega(\bar{\Omega}|\eta, \tilde{\psi}; x) \right)^2 - (b_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x))^2 \right],$$

which is precisely  $\lambda_1 [q_0^{\tilde{\pi}} - b_{\text{can}}^2]$ , i.e., Eq. (5.279) by Eq. (5.281).

**Part 3** (Order-one  $\tilde{\psi}$ -component). The two contributions are  $a_1^{\tilde{\pi}_1, \tilde{\psi}}$  from Eq. (5.297) and  $a_1^{\pi, \psi}$  from Eq. (5.262) with  $\psi = \tilde{\psi}$ . Summing and using the additivity of  $b_{\text{can}}$  and  $\mathcal{V}_{\text{can}}$ ,

$$a_1^{\tilde{\pi}_1, \tilde{\psi}} + a_1^{\pi, \psi} = -\lambda_1(\eta; x, \xi) b_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) + i \mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \xi,$$

which is exactly Eq. (5.280).

**Part 4** (Lower-order terms). All remaining contributions have order at most zero: they come from the generic order-zero symbols in Eq. (5.289) and Eq. (5.295), from the lower-order parts of the parilinearizations of  $\bar{\Omega}[\bar{\Omega}](\eta)$  and  $S(\alpha|\eta)\tilde{\psi}$ , and from the Bony–Weyl composition remainders. They are absorbed in  $A_{[0, \geq 1]}$  and  $R_{[\varrho, \geq 1]}^{\mathbb{R}}$ , giving Eq. (5.277) and concluding the proof of Proposition 5.33.  $\square$

## 5.4 Synthesis: proof of Theorem 5.2

We now assemble the parilinearizations obtained in the previous subsections in the canonical variables of Theorem 2.2. The point is that the equation for  $\eta$  contains both the canonical background contribution  $\mathcal{J}^*(\eta)[\bar{\Omega}[\bar{\Omega}](\eta)]$  and the Dirichlet–Neumann contribution  $\mathcal{J}(\alpha|\eta)\tilde{\psi}$ , whereas the equation for  $\tilde{\psi}$  is the shifted Bernoulli equation parilinearized in Proposition 5.33. We keep the expansion of the Dirichlet–Neumann column in  $\tilde{\psi}$  up to order zero, namely  $\lambda_1 + \alpha\lambda_0$ , and absorb only the terms of order at most  $-1$  in the corresponding generic symbol.

*Proof.* Set

$$\Omega_{\text{bg}} := \bar{\Omega}[\bar{\Omega}](\eta), \quad \Omega_{\tilde{\psi}} := S(\alpha|\eta)\tilde{\psi}.$$

Thus, by Eq. (5.11),

$$\Omega(\bar{\Omega}|\eta, \tilde{\psi}) = \Omega_{\text{bg}} + \Omega_{\tilde{\psi}}.$$

**Part 1** (Parilinearization of the  $\eta$ -equation). By Eq. (2.9a), the first equation is

$$\eta_t = \mathcal{J}^*(\eta)[\Omega_{\text{bg}}] + \mathcal{J}(\alpha|\eta)[\tilde{\psi}].$$

We first parilinearize the background term. Write

$$\Omega_{\text{bg}} = \bar{\Omega} + \delta\Omega_{\text{bg}}, \quad \delta\Omega_{\text{bg}} := \Omega_{\text{bg}} - \bar{\Omega}.$$

The canonical parilinearization and its principal symbol are exactly Eqs. (5.169) and (5.170); we do not redefine them here. Since  $\mathcal{J}^*(\eta)$  is linear in its vorticity argument, split

$$\mathcal{J}^*(\eta)[\Omega_{\text{bg}}] = \mathcal{J}^*(\eta)[\bar{\Omega}] + \mathcal{J}^*(\eta)[\delta\Omega_{\text{bg}}].$$

For the first term we use Remark 4.14: every direct Bony–Weyl action on  $\bar{\Omega}$  vanishes, while the dependence on  $\bar{\Omega}$  remains in the symbol acting on  $\eta$ . We apply Proposition 5.26 to the Sobolev correction in the second term and compose it with Eqs. (5.169) and (5.170). Resumming the two pieces gives

$$\begin{aligned} \mathcal{J}^*(\eta)[\Omega_{\text{bg}}] &= \text{Op}^{\text{BW}} \left( a_1^{\mathcal{J}^*, \eta}[\Omega_{\text{bg}}](\eta; x, \xi) + i A_0^{\mathcal{J}^*, \Omega}(\eta; x, \xi) \cdot A_{1, \text{bg}}^{S, \eta}[\bar{\Omega}](\alpha|\eta; x, \xi) \right) \eta \\ &\quad + \text{Op}^{\text{BW}} \left( a_{[0, \geq 1]}[\bar{\Omega}](\alpha|\eta; x, \xi) \right) \eta + r_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}[\bar{\Omega}](\alpha|\eta) \eta. \end{aligned} \tag{5.310}$$

Using Eq. (5.197), Eq. (5.199), and

$$\left(1 + |\nabla\eta|^2\right) A_0^{\mathcal{B}_0, \omega; \perp}(\eta; x, \xi) \cdot \xi^\perp = -\lambda_1(\eta; x, \xi),$$

the principal symbol in Eq. (5.310) becomes

$$\begin{aligned} & \alpha_1^{g^*, \eta}[\Omega_{\text{bg}}](\eta; x, \xi) + i A_0^{g^*, \Omega}(\eta; x, \xi) \cdot A_{1, \text{bg}}^{S, \eta}[\bar{\Omega}](\alpha|\eta; x, \xi) \\ &= \lambda_1(\eta; x, \xi) \left( \alpha v_3^*[\Omega_{\text{bg}}](\eta; x) - \frac{\nabla^\perp \eta \cdot \Omega_{\text{bg}}}{1 + |\nabla\eta|^2} \right) \\ &+ i \left( V^*[\Omega_{\text{bg}}](\eta; x) + \frac{\alpha}{4(1 + |\nabla\eta|^2)} J(\Omega_{\text{bg}} + (\nabla\eta \cdot \Omega_{\text{bg}}) \nabla\eta) \right) \cdot \xi. \end{aligned}$$

Equivalently, using Eq. (5.288) and Eq. (5.299), we have

$$g^*(\eta)[\Omega_{\text{bg}}] = \text{Op}^{\text{BW}} \left( \lambda_1 b_{\text{bg}} + i \mathcal{V}_{\text{bg}}[\bar{\Omega}](\alpha|\eta; x) \cdot \xi + A_{[0, \geq 1]}^\eta \right) \eta + r_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}} \eta. \quad (5.311)$$

By Eq. (5.300), the transport coefficient in this formula is the sum of the flat velocity  $\frac{\alpha}{4} \bar{\Omega}^\perp$  and a nonlinear correction in  $(\mathcal{F}_{\geq 1}^{\mathbb{R}})^2$ .

On the other hand, Proposition 5.27 applied to  $\tilde{\psi}$  gives, with the order-zero DN contribution kept explicit,

$$\begin{aligned} g(\alpha|\eta) \tilde{\psi} &= \text{Op}^{\text{BW}} \left( \left[ \frac{\lambda_1 b_{\tilde{\psi}} + i \mathcal{V}[\tilde{\psi}](\alpha|\eta; x) \cdot \xi}{\lambda_1 + \alpha \lambda_0} \right] + \left[ \frac{A_{[0, \geq 1]}^\eta(\alpha|\eta, \tilde{\psi}; x, \xi)}{A_{[-1, \geq 1]}^\psi(\alpha|\eta, \tilde{\psi}; x, \xi)} \right] \right) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \\ &+ R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\alpha|\eta, \tilde{\psi}) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}. \end{aligned} \quad (5.312)$$

Here the symbol  $\lambda_0$  is the real order-zero symbol defined in Eq. (5.5); the generic symbol in the  $\tilde{\psi}$ -column is now of order  $-1$  because the order-zero part of that column is precisely the term  $\alpha \lambda_0$  displayed in Eq. (5.312).

Adding Eq. (5.311) and Eq. (5.312), and using the linearity of  $\mathbf{v}^*$  in the vorticity argument, gives

$$\begin{aligned} \eta_t &= \text{Op}^{\text{BW}} \left( \left[ \frac{\lambda_1 b_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) + i \mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \xi}{\lambda_1(\eta; x, \xi) + \alpha \lambda_0(\eta; x, \xi)} \right] + \left[ \frac{A_{[0, \geq 1]}^\eta(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi)}{A_{[-1, \geq 1]}^\psi(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi)} \right] \right) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \\ &+ R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}. \end{aligned} \quad (5.313)$$

**Part 2** (Paralinearization of the  $\tilde{\psi}$ -equation). By Eq. (2.9b), the second equation is

$$\tilde{\psi}_t = \tilde{\pi}(\rho^\pm, \bar{\Omega}|\eta, \tilde{\psi}).$$

Therefore Proposition 5.33 gives

$$\begin{aligned} \tilde{\psi}_t &= -g\alpha\eta + \text{Op}^{\text{BW}} \left( \left[ \frac{a_2^{\pi_0, \eta}(\rho^\pm|\eta; x, \xi) + \lambda_1(\eta; x, \xi) (q_0^{\bar{\pi}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi) - b_{\text{can}}^2(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x))}{-\lambda_1(\eta; x, \xi) b_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) + i \mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \xi} \right] \right) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \\ &+ \text{Op}^{\text{BW}} \left( \left[ \frac{A_{[0, \geq 1]}^\eta(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi)}{A_{[-1, \geq 1]}^\psi(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi)} \right] \right) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix} \\ &+ R_{[\varrho, \geq 1]}^{\text{lin}, \mathbb{R}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}) \cdot \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}. \end{aligned} \quad (5.314)$$

All symbols in Eq. (5.314) are evaluated at the displayed arguments whenever the dependence is not written explicitly.

**Part 3** (Matrix synthesis). Combining Eq. (5.313) and Eq. (5.314), the terms that are kept explicit in the quasilinear matrix are the following.

First, the symbols independent of  $b_{\text{can}}$  and of the transport assemble into

$$\mathbf{Q} = \begin{bmatrix} 0 & \lambda_1 + \mathfrak{a}\lambda_0 \\ -\mathfrak{g}\mathfrak{a} + a_2^{\pi_0, \eta} + \lambda_1 q_0^{\tilde{\pi}} & 0 \end{bmatrix}.$$

Second, the real order-one terms involving  $b_{\text{can}}$  assemble into the Alinhac block

$$\mathbf{B} = - \begin{bmatrix} -b_{\text{can}} & 0 \\ b_{\text{can}}^2 & b_{\text{can}} \end{bmatrix} \lambda_1.$$

Indeed, the first row of  $\mathbf{B}$  gives  $\lambda_1 b_{\text{can}}$  in the  $\eta$ -column, while the second row gives  $-\lambda_1 b_{\text{can}}^2$  in the  $\eta$ -column and  $-\lambda_1 b_{\text{can}}$  in the  $\tilde{\psi}$ -column.

Finally, the imaginary first-order transport symbols in the two equations are the same, namely

$$\mathfrak{i} \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x) \cdot \xi,$$

and therefore form the diagonal transport matrix

$$\mathfrak{i}(\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x) \cdot \xi) \text{Id}_{\mathbb{C}^2}.$$

The remaining terms in the  $\eta$ -column are collected in  $A_{[0, \geq 1]}^{\eta}$ , whereas the remaining terms in the  $\tilde{\psi}$ -column are collected in  $A_{[-1, \geq 1]}^{\tilde{\psi}}$ , consistently with the fact that the order-zero DN contribution  $\mathfrak{a}\lambda_0$  has been displayed explicitly in  $\mathbf{Q}$ . The smoothing remainders are collected in  $\mathbf{R}_{[\ell, \geq 1]}^{\text{lin}, \mathbb{R}}$ . This gives exactly Eq. (5.1).  $\square$

## 6 Diagonal reduction and proof of Theorem 1.12

The purpose of this section is to prove Theorem 1.12. The argument proceeds by a cascade of paradifferential conjugations that progressively simplify the canonical CDE Eq. (2.9): the real paradifferential system Eq. (5.1) is brought to a fully diagonal complex form with purely imaginary positive-order symbols, and a modified energy for the resulting scalar system satisfies a closed Riccati-type differential inequality. A prescribed-background regularization followed by a Kato–Bona–Smith argument then yields existence, uniqueness, and continuous dependence. The section is organized as follows.

- Section 6.1 introduces the real Alinhac good unknown  $Y_{\text{Al}} := \begin{bmatrix} \eta_{\text{Al}} \\ \tilde{\psi}_{\text{Al}} \end{bmatrix}$ , defined by the nilpotent triangular conjugation  $\mathbf{G}$  of Definition 6.1. The conjugation is exact by Eq. (6.2), leaves the elevation component unchanged, and removes the off-diagonal Alinhac block  $\mathbf{B}$  from Eq. (5.1); the reduced system is stated in Proposition 6.2.
- Section 6.2 reformulates the Alinhac-reduced system in complex scalar variables. Under the strict Atwood–Weber stability condition Hypothesis 1, the principal symbol of the resulting system is block-diagonal with purely imaginary entries of order  $3/2$ ; the complex paradifferential system is recorded in Proposition 6.4.
- Section 6.3 diagonalizes all positive-order symbols of the complex system via two successive paradifferential conjugations. The quasilinear order- $3/2$  block is treated in Section 6.3.1; the explicit order- $1/2$  off-diagonal block is treated in Section 6.3.2. The outcome is the fully diagonal positive-order system of Proposition 6.7, with diagonalized unknown  $W$ .
- Section 6.4 constructs a modified energy  $E_s$  for the diagonalized system. The multiplier  $\Lambda_s$  is diagonal, with each entry a scalar function of the corresponding diagonal symbol  $d_j$  of  $\mathbf{D}_{>0}^{\mathbb{C}}$ . The resulting Poisson-bracket identity eliminates the potentially derivative-losing subprincipal commutator and yields the closed Riccati estimate of Proposition 6.11; the associated uniform a priori bound is Corollary 6.14.

- Section 6.5 applies the same reductions as linear operators on an increment over a prescribed background. The resulting flow estimate and the tame dependence of  $\mathfrak{P}[Y]$  yields the local Cauchy theory. This local theory is then combined with the inverse-size a priori estimate of Proposition 6.10. The Hamiltonian part remains of size  $O(\varepsilon)$ , whereas the missing undifferentiated low-frequency component of  $\tilde{\psi}$  is controlled by Lemma 6.18; the continuation criterion then gives  $T_\varepsilon \gtrsim \varepsilon^{-1}$ .

## 6.1 A real Alinhac good unknown

The purpose of this subsection is to remove the Alinhac block  $\mathbf{B}$  displayed in Eq. (5.9). We work directly on the real paradifferential system Eq. (5.1), written for

$$Y := \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}.$$

The transformation is triangular and nilpotent. It leaves the elevation component unchanged and replaces the potential component by the usual Alinhac good unknown.

**Definition 6.1** (Real Alinhac good unknown). We define the matrix symbol

$$\mathbf{N}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x) := \begin{bmatrix} 0 & 0 \\ b_{\text{can}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x) & 0 \end{bmatrix}. \quad (6.1)$$

In particular,

$$\mathbf{N}^2 = 0. \quad (6.2)$$

The associated paradifferential operator is

$$\mathbf{G}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}) := \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}(\mathbf{N}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x)). \quad (6.3)$$

Equivalently, the Alinhac variable is

$$Y_{\text{Al}} := \begin{bmatrix} \eta_{\text{Al}} \\ \tilde{\psi}_{\text{Al}} \end{bmatrix} := \mathbf{G}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}) \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}, \quad (6.4)$$

that is,

$$\eta_{\text{Al}} = \eta, \quad \tilde{\psi}_{\text{Al}} = \tilde{\psi} + \text{Op}^{\text{BW}}(b_{\text{can}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x)) \eta. \quad (6.5)$$

By Eq. (6.2), the inverse is exact:

$$\mathbf{G}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi})^{-1} = \text{Id}_{\mathbb{C}^2} - \text{Op}^{\text{BW}}(\mathbf{N}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x)). \quad (6.6)$$

**Proposition 6.2** (Real Alinhac reduction). *Let the assumptions of Theorem 5.2 hold. If  $(\eta, \tilde{\psi})$  solves Eq. (2.9), and if  $Y_{\text{Al}}$  is defined by Eq. (6.4), then*

$$\begin{aligned} \partial_t Y_{\text{Al}} = & \text{Op}^{\text{BW}} \left( \mathbf{Q}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x, \xi) + i(\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x) \cdot \xi) \text{Id}_{\mathbb{C}^2} + A_{\text{lo}}^{\mathbb{R}, \text{Al}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x, \xi) \right) Y_{\text{Al}} \\ & + \mathbf{R}_{[\ell, \geq 1]}^{\mathbb{R}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}) Y_{\text{Al}}. \end{aligned} \quad (6.7)$$

The objects entering Eq. (6.7) are defined as follows.

- $Y_{\text{Al}}$  is the real Alinhac unknown defined in Eq. (6.4); equivalently, its components are given in Eq. (6.5). In particular, the elevation variable is unchanged, while the potential component is corrected by the Alinhac coefficient  $b_{\text{can}}$ , defined in Eq. (5.10).
- $\mathbf{Q}$  is the quasilinear off-diagonal symbol defined in Eq. (5.3).
- $\mathcal{V}_{\text{can}}$  is the canonical transport velocity defined in Eq. (5.12).
- $A_{\text{lo}}^{\mathbb{R}, \text{Al}}$  is a generic lower-order matrix symbol satisfying

$$A_{\text{lo}}^{\mathbb{R}, \text{Al}}(\mathfrak{a}, \bar{\Omega} | \eta, \tilde{\psi}; x, \xi) \in \begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix}. \quad (6.8)$$

- $\mathbf{R}_{[\ell, \geq 1]}^{\mathbb{R}} \in (\mathcal{R}_{\geq 1}^{\ell})^{2 \times 2}$  is a generic smoothing remainder.

*Proof.* By Eqs. (5.1), (5.3) and (5.9), the real paradifferential system can be written as

$$\partial_t Y = \text{Op}^{\text{BW}}(\mathbf{Q} + \mathbf{B} + \mathbf{T} + \mathbf{A}_{\text{lo}}^{\mathbb{R}}) Y + \mathbf{R}_{[\ell, \geq 1]}^{\mathbb{R}} Y, \quad (6.9)$$

where, in this proof only,

$$\mathbf{T} := i(\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \xi) \text{Id}_{\mathbb{C}^2}, \quad (6.10)$$

and

$$\mathbf{A}_{\text{lo}}^{\mathbb{R}} \in \begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix}.$$

The goal is to prove that the triangular good unknown

$$\mathbf{G} = \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}(\mathbf{N})$$

removes the Alinhac block  $\mathbf{B}$  from the positive-order part. We shall use repeatedly the following consequence of Definition 4.22, Proposition 4.24, and Lemma 4.25: after choosing the truncation order  $\ell$  in the Bony–Weyl product sufficiently large,

$$\text{Op}^{\text{BW}}(\mathbf{A}) \text{Op}^{\text{BW}}(\mathbf{C}) = \text{Op}^{\text{BW}}(\mathbf{A} \#_{\ell} \mathbf{C}) + \mathbf{R}_{[\ell, \geq 1]}^{\mathbb{R}}, \quad (6.11)$$

for matrix symbols  $\mathbf{A}, \mathbf{C}$  of the orders occurring below. Moreover, if  $f \in \mathcal{F} \subset \Gamma_{\geq 1}^0$  is independent of  $\xi$  and  $a \in \Gamma_{\geq 0}^m$ , then

$$a \#_{\ell} f = af + a_{[m-1, \geq 1]}, \quad f \#_{\ell} a = fa + a_{[m-1, \geq 1]}, \quad a_{[m-1, \geq 1]} \in \Gamma_{\geq 1}^{m-1}. \quad (6.12)$$

All smoothing errors generated by Eq. (6.11) will be absorbed into the final  $\mathbf{R}_{[\ell, \geq 1]}^{\mathbb{R}}$ .

**Part 1** (Conjugation of the equation). Since

$$Y_{\text{Al}} = \mathbf{G} Y, \quad Y = \mathbf{G}^{-1} Y_{\text{Al}},$$

we have

$$\partial_t Y_{\text{Al}} = (\partial_t \mathbf{G}) \mathbf{G}^{-1} Y_{\text{Al}} + \mathbf{G} \partial_t Y. \quad (6.13)$$

Inserting Eq. (6.13) into Eq. (6.9) and using Eq. (6.6), we obtain

$$\partial_t Y_{\text{Al}} = \left[ (\partial_t \mathbf{G}) \mathbf{G}^{-1} + \mathbf{G} \text{Op}^{\text{BW}}(\mathbf{Q} + \mathbf{B} + \mathbf{T} + \mathbf{A}_{\text{lo}}^{\mathbb{R}}) \mathbf{G}^{-1} \right] Y_{\text{Al}} + \mathbf{G} \mathbf{R}_{[\ell, \geq 1]}^{\mathbb{R}} \mathbf{G}^{-1} Y_{\text{Al}}. \quad (6.14)$$

The first term in the square bracket will be treated separately. The second one is the spatial conjugation to which the paradifferential formula below applies.

**Part 2** (Algebraic cancellation). Let us first highlight the Alinhac cancellation of the  $\mathbf{B}$  term. We write

$$b := b_{\text{can}}, \quad \lambda_1 := \lambda_1(\eta; x, \xi), \quad q_{12} := \lambda_1(\eta; x, \xi) + \mathfrak{a} \lambda_0(\eta; x, \xi)$$

and

$$q_{21} := -\mathfrak{g} \mathfrak{a} + a_2^{\pi_0, \eta}(\rho^{\pm}|\eta; x, \xi) + \lambda_1(\eta; x, \xi) q_0^{\bar{\pi}}(\mathfrak{a}, \bar{\Omega}|\eta, \tilde{\psi}; x, \xi).$$

Then Eqs. (5.3) and (5.9) read

$$\mathbf{Q} = \begin{bmatrix} 0 & q_{12} \\ q_{21} & 0 \end{bmatrix}, \quad \mathbf{B} = \lambda_1 \begin{bmatrix} b & 0 \\ -b^2 & -b \end{bmatrix}. \quad (6.15)$$

Moreover, by Eq. (6.1),

$$\text{Id}_{\mathbb{C}^2} + \mathbf{N} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix}, \quad \text{Id}_{\mathbb{C}^2} - \mathbf{N} = \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix}.$$

At the pointwise symbolic level, isolating the principal upper-right entry  $\lambda$ , one has the exact cancellation

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \left( \begin{bmatrix} 0 & \lambda_1 \\ q_{21} & 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} b & 0 \\ -b^2 & -b \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1 \\ q_{21} & 0 \end{bmatrix}.$$

For the full upper-right entry  $q_{12} = \lambda_1 + \alpha\lambda_0$ , the same pointwise multiplication gives

$$\begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} (\mathbf{Q} + \mathbf{B}) \begin{bmatrix} 1 & 0 \\ -b & 1 \end{bmatrix} = \mathbf{Q} + \alpha\lambda_0 \begin{bmatrix} -b & 0 \\ -b^2 & b \end{bmatrix}. \quad (6.16)$$

Since  $\lambda_0 \in \Gamma_{\geq 1}^0$  and  $b \in \mathcal{F} \subset \Gamma_{\geq 1}^0$ , the residual term in Eq. (6.16) belongs to

$$\begin{bmatrix} \Gamma_{\geq 1}^0 & 0 \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix},$$

and will be absorbed into the lower-order symbol.

**Part 3** (Paradifferential formula for the conjugation). Let  $\mathbf{A}$  be any matrix symbol of the orders occurring below. Since  $\text{Op}^{\text{BW}}(\mathbf{N})$  has only one nonzero lower-left entry, one has

$$(\text{Op}^{\text{BW}}(\mathbf{N}))^2 = 0, \quad \mathbf{G}^{-1} = \text{Id}_{\mathbb{C}^2} - \text{Op}^{\text{BW}}(\mathbf{N}),$$

as recorded in Eq. (6.6). Hence, at the operator level,

$$\mathbf{G} \text{Op}^{\text{BW}}(\mathbf{A}) \mathbf{G}^{-1} = \text{Op}^{\text{BW}}(\mathbf{A}) + \text{Op}^{\text{BW}}(\mathbf{N}) \text{Op}^{\text{BW}}(\mathbf{A}) - \text{Op}^{\text{BW}}(\mathbf{A}) \text{Op}^{\text{BW}}(\mathbf{N}) - \text{Op}^{\text{BW}}(\mathbf{N}) \text{Op}^{\text{BW}}(\mathbf{A}) \text{Op}^{\text{BW}}(\mathbf{N}).$$

Applying Eq. (6.11) to each product, and using Proposition 4.24 and Lemma 4.25, gives

$$\text{Op}^{\text{BW}}(\mathbf{N}) \text{Op}^{\text{BW}}(\mathbf{A}) = \text{Op}^{\text{BW}}(\mathbf{N} \#_{\varrho} \mathbf{A}) + \mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{R}}, \quad (6.17)$$

$$\text{Op}^{\text{BW}}(\mathbf{A}) \text{Op}^{\text{BW}}(\mathbf{N}) = \text{Op}^{\text{BW}}(\mathbf{A} \#_{\varrho} \mathbf{N}) + \mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{R}}, \quad (6.18)$$

$$\text{Op}^{\text{BW}}(\mathbf{N}) \text{Op}^{\text{BW}}(\mathbf{A}) \text{Op}^{\text{BW}}(\mathbf{N}) = \text{Op}^{\text{BW}}((\mathbf{N} \#_{\varrho} \mathbf{A}) \#_{\varrho} \mathbf{N}) + \mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{R}}. \quad (6.19)$$

In the last line, the smoothing remainder produced by the first composition is composed once more with the order-zero operator  $\text{Op}^{\text{BW}}(\mathbf{N})$ , and is still absorbed by  $\mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{R}}$  after decreasing and relabelling  $\varrho$ . Here  $\#_{\varrho}$  denotes the matrix product whose entries are composed with the scalar Bony–Weyl product, and we denote

$$\mathbf{N} \#_{\varrho} \mathbf{A} \#_{\varrho} \mathbf{N} := (\mathbf{N} \#_{\varrho} \mathbf{A}) \#_{\varrho} \mathbf{N}.$$

Therefore

$$\mathbf{G} \text{Op}^{\text{BW}}(\mathbf{A}) \mathbf{G}^{-1} = \text{Op}^{\text{BW}}(\mathbf{A}_{\text{conj}}) + \mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{R}}, \quad (6.20)$$

where

$$\mathbf{A}_{\text{conj}} := \mathbf{A} + \mathbf{N} \#_{\varrho} \mathbf{A} - \mathbf{A} \#_{\varrho} \mathbf{N} - \mathbf{N} \#_{\varrho} \mathbf{A} \#_{\varrho} \mathbf{N}. \quad (6.21)$$

All smoothing remainders generated in Eqs. (6.17) to (6.19) are absorbed in the generic matrix remainder  $\mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{R}}$ , after decreasing and relabelling  $\varrho$ .

**Part 4** (The quasilinear and Alinhac blocks). Set

$$\mathbf{P} := \mathbf{Q} + \mathbf{B}.$$

Applying Eqs. (6.20) and (6.21) with  $\mathbf{A} = \mathbf{P}$ , we have

$$\mathbf{G} \text{Op}^{\text{BW}}(\mathbf{P}) \mathbf{G}^{-1} = \text{Op}^{\text{BW}}(\mathbf{P}_{\text{conj}}) + \mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{R}},$$

where

$$\mathbf{P}_{\text{conj}} := \mathbf{P} + \mathbf{N} \#_{\varrho} \mathbf{P} - \mathbf{P} \#_{\varrho} \mathbf{N} - \mathbf{N} \#_{\varrho} \mathbf{P} \#_{\varrho} \mathbf{N}. \quad (6.22)$$

Using Eq. (6.15), we have

$$\mathbf{P} = \begin{bmatrix} \lambda_1 b & q_{12} \\ q_{21} - \lambda_1 b^2 & -\lambda_1 b \end{bmatrix}. \quad (6.23)$$

Since

$$\mathbf{N} = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix},$$

the matrix structure gives

$$N\#_{\rho}P = \begin{bmatrix} 0 & 0 \\ b\#_{\rho}(\lambda_1 b) & b\#_{\rho}q_{12} \end{bmatrix}, \quad (6.24)$$

$$P\#_{\rho}N = \begin{bmatrix} q_{12}\#_{\rho}b & 0 \\ (-\lambda_1 b)\#_{\rho}b & 0 \end{bmatrix}, \quad (6.25)$$

$$N\#_{\rho}P\#_{\rho}N = \begin{bmatrix} 0 & 0 \\ (b\#_{\rho}q_{12})\#_{\rho}b & 0 \end{bmatrix}. \quad (6.26)$$

The important point is that  $q_{21}$ , although of order 2, is never differentiated by the Bony–Weyl corrections generated by the triangular conjugation. Only  $q_{12}$ ,  $\lambda_1 b$ , and  $b$  enter the composed symbols. Combining Eqs. (6.22) to (6.26), we get

$$P_{\text{conj}} = \begin{bmatrix} \lambda_1 b - q_{12}\#_{\rho}b & q_{12} \\ q_{21} - \lambda_1 b^2 + b\#_{\rho}(\lambda_1 b) - (-\lambda_1 b)\#_{\rho}b - (b\#_{\rho}q_{12})\#_{\rho}b & -\lambda_1 b + b\#_{\rho}q_{12} \end{bmatrix}. \quad (6.27)$$

We now expand each product by Eq. (6.12). Since  $b \in \mathcal{F} \subset \Gamma_{\geq 1}^0$ ,  $\lambda_1 \in \Gamma_{\geq 0}^1$ , and  $q_{12} = \lambda_1 + \alpha\lambda_0 \in \Gamma_{\geq 0}^1$ , one obtains

$$q_{12}\#_{\rho}b = q_{12}b + a_{[0,\geq 1]}, \quad (6.28)$$

$$b\#_{\rho}q_{12} = q_{12}b + a_{[0,\geq 1]}, \quad (6.29)$$

$$b\#_{\rho}(\lambda_1 b) = \lambda_1 b^2 + a_{[0,\geq 1]}, \quad (6.30)$$

$$(-\lambda_1 b)\#_{\rho}b = -\lambda_1 b^2 + a_{[0,\geq 1]}, \quad (6.31)$$

$$(b\#_{\rho}q_{12})\#_{\rho}b = q_{12}b^2 + a_{[0,\geq 1]}. \quad (6.32)$$

Here  $a_{[0,\geq 1]} \in \Gamma_{\geq 1}^0$  is generic: the first neglected Weyl correction contains one  $\xi$ -derivative of an order-one symbol, while all further corrections have lower order. Substituting Eqs. (6.28) to (6.32) into Eq. (6.27) yields

$$P_{\text{conj}} = Q + \alpha\lambda_0 \begin{bmatrix} -b & 0 \\ -b^2 & b \end{bmatrix} + \begin{bmatrix} \Gamma_{\geq 1}^0 & 0 \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix} = Q + \begin{bmatrix} \Gamma_{\geq 1}^0 & 0 \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix}. \quad (6.33)$$

**Part 5** (The transport block). Let

$$\tau := i\mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \xi, \quad T := \tau \text{Id}_{\mathbb{C}^2}.$$

Applying Eqs. (6.20) and (6.21) with  $A = T$ , we have

$$G \text{Op}^{\text{BW}}(T) G^{-1} = \text{Op}^{\text{BW}}(T_{\text{conj}}) + R_{[0,\geq 1]}^{\mathbb{R}},$$

where

$$T_{\text{conj}} := T + N\#_{\rho}T - T\#_{\rho}N - N\#_{\rho}T\#_{\rho}N. \quad (6.34)$$

Since  $T$  is scalar in the matrix indices, it commutes with  $N$  at the pointwise principal-symbol level:

$$NT - TN = 0, \quad NTN = \tau N^2 = 0.$$

At the Bony–Weyl symbolic level,

$$N\#_{\rho}T - T\#_{\rho}N = \begin{bmatrix} 0 & 0 \\ b\#_{\rho}\tau - \tau\#_{\rho}b & 0 \end{bmatrix},$$

and  $N\#_{\rho}T\#_{\rho}N = 0$  by the same lower-left matrix structure. Since we have that  $b$  is independent of  $\xi$  and  $\partial_{\xi}\tau = i\mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x)$ , then using the first Weyl correction in Definition 4.22 and then the composition rules in Proposition 4.24, we get

$$b\#_{\rho}\tau - \tau\#_{\rho}b = -\mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \nabla b + a_{[-1,\geq 1]}.$$

Consequently, by Eq. (6.34),

$$T_{\text{conj}} = T + \begin{bmatrix} 0 & 0 \\ -\mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|\eta, \tilde{\psi}; x) \cdot \nabla b & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \Gamma_{\geq 1}^{-1} & 0 \end{bmatrix} = T + \begin{bmatrix} 0 & 0 \\ \Gamma_{\geq 1}^0 & 0 \end{bmatrix}. \quad (6.35)$$

Thus the transport symbol is unchanged at order 1, and all new transport contributions are lower order.

**Part 6** (The original lower-order symbol). Applying Eqs. (6.20) and (6.21) with  $\mathbf{A} = \mathbf{A}_{\text{lo}}^{\mathbb{R}}$ , we have

$$\mathbf{G} \text{Op}^{\text{BW}}(\mathbf{A}_{\text{lo}}^{\mathbb{R}}) \mathbf{G}^{-1} = \text{Op}^{\text{BW}}(\mathbf{A}_{\text{conj}}^{\mathbb{R}, \text{Al}, \text{lo}}) + \mathbf{R}_{[\rho, \geq 1]}^{\mathbb{R}},$$

where

$$\mathbf{A}_{\text{conj}}^{\mathbb{R}, \text{Al}, \text{lo}} := \mathbf{A}_{\text{lo}}^{\mathbb{R}} + \mathbf{N} \#_{\rho} \mathbf{A}_{\text{lo}}^{\mathbb{R}} - \mathbf{A}_{\text{lo}}^{\mathbb{R}} \#_{\rho} \mathbf{N} - \mathbf{N} \#_{\rho} \mathbf{A}_{\text{lo}}^{\mathbb{R}} \#_{\rho} \mathbf{N}.$$

Write

$$\mathbf{A}_{\text{lo}}^{\mathbb{R}} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad a_{11}, a_{21}, a_{22} \in \Gamma_{\geq 1}^0, \quad a_{12} \in \Gamma_{\geq 1}^{-1}.$$

The pointwise products generated by the triangular matrix are

$$\mathbf{N} \mathbf{A}_{\text{lo}}^{\mathbb{R}} = \begin{bmatrix} 0 & 0 \\ b a_{11} & b a_{12} \end{bmatrix}, \quad \mathbf{A}_{\text{lo}}^{\mathbb{R}} \mathbf{N} = \begin{bmatrix} a_{12} b & 0 \\ a_{22} b & 0 \end{bmatrix}, \quad \mathbf{N} \mathbf{A}_{\text{lo}}^{\mathbb{R}} \mathbf{N} = \begin{bmatrix} 0 & 0 \\ b a_{12} b & 0 \end{bmatrix}.$$

Since  $b \in \mathcal{F} \subset \Gamma_{\geq 1}^0$ , these products have the same matrix order structure as  $\mathbf{A}_{\text{lo}}^{\mathbb{R}}$ . The Bony–Weyl corrections only differentiate symbols in  $x$  and  $\xi$ , and every  $\xi$ -derivative lowers the symbolic order. Therefore, by Proposition 4.24 and Lemma 4.25,

$$\mathbf{A}_{\text{conj}}^{\mathbb{R}, \text{Al}, \text{lo}} \in \begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix}. \quad (6.36)$$

This contribution is included in  $\mathbf{A}_{\text{lo}}^{\mathbb{R}, \text{Al}}$ .

**Part 7** (The time derivative of the good unknown). It remains to treat the first term in Eq. (6.14). Note that

$$\mathbf{N} = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}, \quad \partial_t \mathbf{N} = \begin{bmatrix} 0 & 0 \\ \partial_t b & 0 \end{bmatrix} \in \begin{bmatrix} 0 & 0 \\ \Gamma_{\geq 1}^0 & 0 \end{bmatrix}.$$

In particular, the product of the two corresponding lower-left matrix operators is exactly zero:

$$\text{Op}^{\text{BW}}(\partial_t \mathbf{N}) \text{Op}^{\text{BW}}(\mathbf{N}) = 0.$$

Thus, using the exact inverse  $\mathbf{G}^{-1} = \text{Id}_{\mathbb{C}^2} - \text{Op}^{\text{BW}}(\mathbf{N})$ ,

$$(\partial_t \mathbf{G}) \mathbf{G}^{-1} = \text{Op}^{\text{BW}}(\partial_t \mathbf{N}). \quad (6.37)$$

This term is included in  $\text{Op}^{\text{BW}}(\mathbf{A}_{\text{lo}}^{\mathbb{R}, \text{Al}})$ .

**Part 8** (Smoothing remainders and conclusion). The map

$$\mathbf{A} \mapsto \mathbf{A} + \mathbf{N} \#_{\rho} \mathbf{A} - \mathbf{A} \#_{\rho} \mathbf{N} - \mathbf{N} \#_{\rho} \mathbf{A} \#_{\rho} \mathbf{N}$$

appearing in Eq. (6.21) is linear in  $\mathbf{A}$ . Therefore the spatial conjugation in Eq. (6.14) is obtained by summing the three contributions computed above with  $\mathbf{A} = \mathbf{P}, \mathbf{T}, \mathbf{A}_{\text{lo}}^{\mathbb{R}}$ .

It remains to control the smoothing remainder in Eq. (6.9). The operators  $\mathbf{G}^{\pm 1}$  are the identity plus an order-zero paradifferential operator with coefficient controlled by the same tame norms as  $b_{\text{can}}$ . Therefore, by Lemma 4.25,

$$\mathbf{G} \mathbf{R}_{[\rho, \geq 1]}^{\mathbb{R}} \mathbf{G}^{-1} \in (\mathcal{R}_{\geq 1}^{\rho})^{2 \times 2},$$

after decreasing and relabelling  $\rho$ . The same convention absorbs all smoothing remainders produced by the symbolic compositions in Eq. (6.11).

Collecting Eqs. (6.33) and (6.35) to (6.37) inside the conjugation identity Eq. (6.14), we obtain

$$\partial_t Y_{\text{Al}} = \text{Op}^{\text{BW}}(\mathbf{Q} + \mathbf{T} + \mathbf{A}_{\text{lo}}^{\mathbb{R}, \text{Al}}) Y_{\text{Al}} + \mathbf{R}_{[\rho, \geq 1]}^{\mathbb{R}} Y_{\text{Al}}, \quad (6.38)$$

with

$$\mathbf{A}_{\text{lo}}^{\mathbb{R}, \text{Al}} \in \begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix}, \quad \mathbf{R}_{[\rho, \geq 1]}^{\mathbb{R}} \in (\mathcal{R}_{\geq 1}^{\rho})^{2 \times 2}.$$

Recalling the definition of  $\mathbf{T}$  in Eq. (6.10), this is exactly Eq. (6.7).

□

## 6.2 Complex formulation

Throughout this subsection we assume Hypothesis 1. Hence, by Proposition 3.4,  $\mathfrak{d}(\mathfrak{a}, \bar{\Omega}|\xi) < 0$  for every  $\xi \in \mathbb{R}^2 \setminus \{0\}$ , and the two branches of the flat linearized CDE are purely imaginary and simple. We use the positive frequency introduced below to complexify the real Alinhac unknown and diagonalize the positive-order flat dynamics.

**Definition 6.3** (Complex Fourier multipliers and non-homogeneous coordinate). Assume Hypothesis 1, fix the even low-frequency cut-off  $\chi$  of Eq. (4.7), and let  $\xi \in \mathbb{R}^2 \setminus \{0\}$ . We introduce the following objects.

1. *Homogeneous dispersive multipliers.* The order-3/2 dispersive frequency is

$$\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) := \sqrt{-\mathfrak{d}_{\mathfrak{a}, \bar{\Omega}}(\xi)} \in \tilde{\Gamma}^{\frac{3}{2}}. \quad (6.39)$$

In particular, by Eq. (3.4),

$$\omega_{\frac{3}{2}}^2(\mathfrak{a}, \bar{\Omega}|\xi) = -\frac{|\xi|}{2} \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi). \quad (6.40)$$

Thus the two branches in Eq. (3.5) are obtained by replacing  $\sqrt{\mathfrak{d}}$  with  $i\omega_{\frac{3}{2}}$ .

The order-(-1/4) symmetrizing multiplier is

$$\mathfrak{m}_{-\frac{1}{4}}(\mathfrak{a}, \bar{\Omega}|\xi) := \sqrt{\frac{|\xi|}{2\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi)}} \in \tilde{\Gamma}^{-\frac{1}{4}}. \quad (6.41)$$

Equivalently,

$$\mathfrak{m}_{-\frac{1}{4}}^2 = \frac{\frac{1}{2}|\xi|}{\omega_{\frac{3}{2}}} = -\frac{\omega_{\frac{3}{2}}}{\mathfrak{q}}, \quad \mathfrak{m}_{-\frac{1}{4}}^{-2} = \frac{2\omega_{\frac{3}{2}}}{|\xi|}, \quad (6.42)$$

where all quantities in Eq. (6.42) are evaluated at  $(\mathfrak{a}, \bar{\Omega}|\xi)$ .

The homogeneous leading capillary frequency and symmetrizer are

$$\dot{\omega}_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) := \sqrt{\frac{\sigma}{2\langle\langle \rho^\pm \rangle\rangle}} |\xi|^{\frac{3}{2}} \in \tilde{\Gamma}^{\frac{3}{2}} \quad (6.43)$$

and

$$\dot{\mathfrak{m}}_{-\frac{1}{4}}(\mathfrak{a}, \bar{\Omega}|\xi) := \sqrt{\frac{|\xi|}{2\dot{\omega}_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi)}} \in \tilde{\Gamma}^{-\frac{1}{4}}. \quad (6.44)$$

In particular,

$$\omega_{\frac{3}{2}} - \dot{\omega}_{\frac{3}{2}} \in \tilde{\Gamma}^{\frac{1}{2}}, \quad \mathfrak{m}_{-\frac{1}{4}}^2 - \dot{\mathfrak{m}}_{-\frac{1}{4}}^2 \in \tilde{\Gamma}^{-\frac{3}{2}}, \quad \mathfrak{m}_{-\frac{1}{4}}^{-2} - \dot{\mathfrak{m}}_{-\frac{1}{4}}^{-2} \in \tilde{\Gamma}^{-\frac{1}{2}}. \quad (6.45)$$

2. *Non-homogeneous complex coordinate and flat defect.* Define, initially for  $\xi \neq 0$ ,

$$\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}(\mathfrak{a}, \bar{\Omega}|\xi) := \chi(\xi) + (1 - \chi(\xi)) \mathfrak{m}_{-\frac{1}{4}}(\mathfrak{a}, \bar{\Omega}|\xi), \quad (6.46)$$

and extend it smoothly through  $\xi = 0$ , where it equals 1. In the following two matrix displays, every occurrence of  $\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}$  is evaluated at  $(\mathfrak{a}, \bar{\Omega}|\xi)$ . The non-homogeneous complex diagonalizing matrix is

$$\mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi) := \frac{1}{\sqrt{2}} \begin{bmatrix} \left(\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}\right)^{-1} & i\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} \\ \left(\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}\right)^{-1} & -i\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} \end{bmatrix}. \quad (6.47)$$

Its inverse is

$$\mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi)^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} & \mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} \\ -i\left(\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}\right)^{-1} & i\left(\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}\right)^{-1} \end{bmatrix}. \quad (6.48)$$

The complex coordinate is

$$U := \begin{bmatrix} u \\ \bar{u} \end{bmatrix} := \mathbf{M}(\mathfrak{a}, \bar{\Omega}|D) Y_{\text{Al}}, \quad Y_{\text{Al}} := \begin{bmatrix} \eta_{\text{Al}} \\ \tilde{\psi}_{\text{Al}} \end{bmatrix}, \quad (6.49)$$

where  $Y_{\text{Al}}$  is defined in Eq. (6.4).

Finally, define the diagonal sign matrix

$$\mathbf{J}_{\text{d}} := \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad (6.50)$$

and the flat low-frequency defect

$$\begin{aligned} A_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi) &:= \mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi) \begin{bmatrix} 0 & \frac{|\xi|}{2} \\ \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) & 0 \end{bmatrix} \mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi)^{-1} \\ &\quad - (1 - \chi(\xi)) i\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \mathbf{J}_{\text{d}}. \end{aligned} \quad (6.51)$$

For later use, we recover the real Alinhac variable from  $U$  by

$$\begin{bmatrix} \eta_{\text{Al}}(U) \\ \tilde{\psi}_{\text{Al}}(U) \end{bmatrix} := \mathbf{M}(\mathfrak{a}, \bar{\Omega}|D)^{-1} U. \quad (6.52)$$

The coefficients of the real Alinhac-reduced system are evaluated on the physical variables

$$\begin{bmatrix} \eta(U) \\ \tilde{\psi}(U) \end{bmatrix} := \mathbf{G}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U))^{-1} \begin{bmatrix} \eta_{\text{Al}}(U) \\ \tilde{\psi}_{\text{Al}}(U) \end{bmatrix}, \quad (6.53)$$

where  $\mathbf{G}$  is defined in Eq. (6.3). More explicitly, by Eqs. (6.5) and (6.6),

$$\eta(U) = \eta_{\text{Al}}(U), \quad \tilde{\psi}(U) = \tilde{\psi}_{\text{Al}}(U) - \text{Op}^{\text{BW}}(b_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x)) \eta_{\text{Al}}(U). \quad (6.54)$$

**Proposition 6.4** (Complex form of the Alinhac-reduced paradifferential system). *Let the assumptions of Theorem 5.2 and Hypothesis 1 hold. If  $(\eta, \tilde{\psi})$  solves the canonical CDE Eq. (2.9), if  $Y_{\text{Al}}$  is the real Alinhac unknown defined in Eq. (6.4), and if  $U$  is defined by Eq. (6.49), then*

$$\begin{aligned} \partial_t U &= \text{Op}^{\text{BW}} \left( A_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi) + (1 - \chi(\xi)) i\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \mathbf{J}_{\text{d}} (\text{Id}_{\mathbb{C}^2} + \dot{\mathbf{E}}_{\text{ql},0}^{\mathbb{C}}(U; x, \hat{\xi})) \right) U \\ &\quad + \text{Op}^{\text{BW}} \left( i(\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi) \text{Id}_{\mathbb{C}^2} \right) U \\ &\quad + \text{Op}^{\text{BW}} \left( (1 - \chi(\xi)) \dot{A}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) |\xi|^{\frac{1}{2}} \right) U \\ &\quad + \text{Op}^{\text{BW}} \left( A_{[0, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \right) U + \mathbf{R}_{[\ell, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U) U. \end{aligned} \quad (6.55)$$

where:

- $U = \begin{bmatrix} u \\ \bar{u} \end{bmatrix}$  is the complex Alinhac-reduced unknown defined in Eq. (6.49); the physical variables  $\eta(U)$  and  $\tilde{\psi}(U)$  are recovered through the inverse real Alinhac map in Eqs. (6.53) and (6.54);
- $A_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi) \in (\tilde{\Gamma}^0)^{2 \times 2}$  is the Fourier-localized symbol introduced in Eq. (6.51).
- $\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \in \tilde{\Gamma}^{3/2}$  is the scalar Fourier multiplier defined in Eq. (6.39);
- $\mathbf{J}_{\text{d}}$  is the diagonal sign matrix defined in Eq. (6.50);
- $\dot{\mathbf{E}}_{\text{ql},0}^{\mathbb{C}}(U; x, \hat{\xi}) \in (\Gamma_{\geq 1}^0)^{2 \times 2}$  is the homogeneous degree-zero quasilinear matrix coefficient

$$\dot{\mathbf{E}}_{\text{ql},0}^{\mathbb{C}}(U; x, \hat{\xi}) := \begin{bmatrix} \dot{e}_0 & \dot{f}_0 \\ \dot{f}_0 & \dot{e}_0 \end{bmatrix} (U; x, \hat{\xi}), \quad (6.56)$$

where the scalar homogeneous coefficients are

–  $\dot{e}_0 \in \Gamma_{\geq 1}^0$  and is explicitly defined as

$$\dot{e}_0(U; x, \hat{\xi}) := \frac{1}{2} \left( \sqrt{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2} + \frac{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2}{(1 + |\nabla \eta(U)|^2)^{\frac{3}{2}}} \right) - 1; \quad (6.57)$$

–  $\dot{f}_0 \in \Gamma_{\geq 1}^0$  and is explicitly defined as

$$\dot{f}_0(U; x, \hat{\xi}) := \frac{1}{2} \left( \frac{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2}{(1 + |\nabla \eta(U)|^2)^{\frac{3}{2}}} - \sqrt{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2} \right); \quad (6.58)$$

- the transport is kept in the explicit scalar form

$$\mathbf{i}(\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi) \text{Id}_{\mathbb{C}^2},$$

where  $\mathcal{V}_{\text{can}}$  is defined in Eq. (5.12);

- $\dot{A}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) \in (\Gamma_{\geq 1}^0)^{2 \times 2}$  is the homogeneous degree-zero coefficient of the explicit order-1/2 block. It is defined by

$$\dot{A}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) := \mathbf{i} \begin{bmatrix} \dot{\alpha}_0^{\mathbb{C}} & \dot{\beta}_0^{\mathbb{C}} \\ -\dot{\beta}_0^{\mathbb{C}} & -\dot{\alpha}_0^{\mathbb{C}} \end{bmatrix} (\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}), \quad (6.59)$$

where

$$\dot{\alpha}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) := \frac{\dot{b}_0^{\mathbb{C}} - \dot{a}_0^{\mathbb{C}}}{2} (\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}), \quad (6.60a)$$

$$\dot{\beta}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) := \frac{\dot{a}_0^{\mathbb{C}} + \dot{b}_0^{\mathbb{C}}}{2} (\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}), \quad (6.60b)$$

$$\dot{a}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) := \mathfrak{a} \sqrt{\frac{2\sigma}{\langle\langle \rho^\pm \rangle\rangle}} \lambda_0(\eta(U); x, \xi), \quad (6.60c)$$

$$\begin{aligned} \dot{b}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) &:= \frac{4 - \mathfrak{a}^2}{16} \sqrt{\frac{2\langle\langle \rho^\pm \rangle\rangle}{\sigma}} \sqrt{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2} \\ &\times \left[ 4 \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta(U); x, \xi) \cdot \Omega(\bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \right)^2 - \frac{(\bar{\Omega}^\perp \cdot \hat{\xi})^2 \sqrt{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2}}{(1 + |\nabla \eta(U)|^2)^{\frac{3}{2}}} \right]. \end{aligned} \quad (6.60d)$$

Here  $\lambda_0$  is defined in Eq. (5.5),  $A_0^{\mathcal{B}_0, \omega}$  is defined in Eq. (5.8), and  $\Omega$  is defined in Eq. (5.11).

- $A_{[0, \geq 1]} \in (\Gamma_{\geq 1}^0)^{2 \times 2}$  is a generic matrix symbol as per Notation 5.1;
- $R_{[\ell, \geq 1]} \in (\mathcal{R}_{\geq 1}^\ell)^{2 \times 2}$  is a generic smoothing remainder as per Notation 5.1.

**Lemma 6.5** (High-frequency diagonalization of the constant-coefficient linear part). *Assume Hypothesis 1. Then  $A_{\text{lo}}^{\mathbb{C}, \text{lin}}$ , defined in Eq. (6.51), belongs to  $(\tilde{\Gamma}^0)^{2 \times 2}$ , and*

$$\mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi) L_0(\xi) \mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi)^{-1} = \mathbf{i} \frac{\mathfrak{a}}{4} (\bar{\Omega}^\perp \cdot \xi) \text{Id}_{\mathbb{C}^2} + (1 - \chi(\xi)) \mathbf{i} \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) J_d + A_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi). \quad (6.61)$$

On the low-frequency core where  $\chi(\xi) = 1$ , one has

$$A_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi) = \frac{\mathbf{i}}{2} \begin{bmatrix} \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) - \frac{|\xi|}{2} & \frac{|\xi|}{2} + \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) \\ -\left(\frac{|\xi|}{2} + \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi)\right) & \frac{|\xi|}{2} - \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi) \end{bmatrix}. \quad (6.62)$$

Consequently, under the change of variables in Eq. (6.49), the linearized equation Eq. (3.1) is equivalent to

$$\begin{aligned} \partial_t U_{\text{lin}} &= \mathbf{i} \frac{\mathfrak{a}}{4} (\bar{\Omega}^\perp \cdot D) U_{\text{lin}} + \text{Op}^{\text{BW}} \left( (1 - \chi(\xi)) \mathbf{i} \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) J_d \right) U_{\text{lin}} \\ &\quad + \text{Op}^{\text{BW}} \left( A_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi) \right) U_{\text{lin}}. \end{aligned} \quad (6.63)$$

*Proof.* By Eqs. (3.4) and (6.39),

$$\mathfrak{d}(\mathfrak{a}, \bar{\Omega}|\xi) = \frac{|\xi|}{2} \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi), \quad \omega_{\frac{3}{2}}^2(\mathfrak{a}, \bar{\Omega}|\xi) = -\frac{|\xi|}{2} \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi).$$

Moreover, Eq. (6.42) gives

$$\mathfrak{q} \mathfrak{m}_{-\frac{1}{4}}^2 = -\omega_{\frac{3}{2}}, \quad \frac{|\xi|}{2} \mathfrak{m}_{-\frac{1}{4}}^{-2} = \omega_{\frac{3}{2}},$$

where all multipliers are evaluated at  $(\mathfrak{a}, \bar{\Omega}|\xi)$ . On the high-frequency region where  $\chi(\xi) = 0$ , one has  $\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} = \mathfrak{m}_{-\frac{1}{4}}$ ; hence ordinary matrix multiplication yields

$$\mathbf{M} \begin{bmatrix} 0 & \frac{|\xi|}{2} \\ \mathfrak{q} & 0 \end{bmatrix} \mathbf{M}^{-1} = i\omega_{\frac{3}{2}} \mathbf{J}_d, \quad \text{on } \text{supp}^{\mathbb{C}}(\chi). \quad (6.64)$$

The transport part  $i\frac{\mathfrak{a}}{4}(\bar{\Omega}^\perp \cdot \xi) \text{Id}_{\mathbb{C}^2}$  is scalar and therefore commutes with  $\mathbf{M}$ . Combining Eqs. (3.3), (6.51) and (6.64) proves Eq. (6.61). The defect vanishes where  $\chi = 0$ , and is therefore supported in a fixed low-frequency ball. Since the symbolic calculus is restricted to  $|\xi| \geq 1/2$ , it belongs to  $(\tilde{\Gamma}^0)^{2 \times 2}$ .

On the core where  $\chi = 1$ , one has  $\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} = 1$ . Direct multiplication of the two constant matrices in Eqs. (6.47) and (6.48) gives Eq. (6.62). Finally, Eq. (6.63) follows from Eqs. (3.1), (6.49) and (6.61).  $\square$

*Proof of Proposition 6.4.* Throughout the proof the physical variables  $\eta(U)$ ,  $\tilde{\psi}(U)$  are those recovered from  $U$  through Eqs. (6.53) and (6.54). We use the notation of Notation 5.1: symbols  $\mathbf{A}_{[m, \geq 1]}$  and smoothing remainders  $\mathbf{R}_{[\rho, \geq 1]}$  are generic and may change from line to line.

**Part 1** (Real order splitting). We first split the nonlinear increments of the real off-diagonal entries according to their symbolic order.

The homogeneous order-one Kelvin–Helmholtz shear symbol is

$$\dot{q}_1^{\text{sh}}(\mathfrak{a}, \bar{\Omega}|\xi) := \frac{4 - \mathfrak{a}^2 (\bar{\Omega}^\perp \cdot \xi)^2}{8 |\xi|} \in \tilde{\Gamma}^1. \quad (6.65)$$

The real order-one and order-zero increments in the upper-right entry are

$$\delta \dot{q}_{12,1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := \lambda_1(\eta(U); x, \xi) - \frac{|\xi|}{2} \in \Gamma_{\geq 1}^1, \quad (6.66)$$

$$\delta \dot{q}_{12,0}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := \mathfrak{a} \lambda_0(\eta(U); x, \xi) \in \Gamma_{\geq 1}^0, \quad (6.67)$$

where  $\lambda_1$  and  $\lambda_0$  are defined in Eqs. (5.4) and (5.5). The real order-two and order-one increments in the lower-left entry are

$$\delta \dot{q}_{21,2}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := a_2^{\pi_0, \eta}(\rho^\pm | \eta(U); x, \xi) + \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2 \in \Gamma_{\geq 1}^2, \quad (6.68)$$

$$\delta \dot{q}_{21,1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := \lambda_1(\eta(U); x, \xi) q_0^{\tilde{\pi}}(\mathfrak{a}, \bar{\Omega} | \eta(U), \tilde{\psi}(U); x, \xi) - \dot{q}_1^{\text{sh}}(\mathfrak{a}, \bar{\Omega}|\xi) \in \Gamma_{\geq 1}^1. \quad (6.69)$$

Here  $a_2^{\pi_0, \eta}$  and  $q_0^{\tilde{\pi}}$  are defined in Eqs. (5.6) and (5.7).

With this notation, the real off-diagonal quasilinear block in Eq. (5.3) decomposes as

$$\mathbf{Q}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) = \begin{bmatrix} 0 & \frac{|\xi|}{2} + \delta \dot{q}_{12,1} + \delta \dot{q}_{12,0} \\ \mathfrak{q} + \delta \dot{q}_{21,2} + \delta \dot{q}_{21,1} & 0 \end{bmatrix} (\mathfrak{a}, \bar{\Omega}|U; x, \xi). \quad (6.70)$$

The flat entries  $|\xi|/2$  and  $\mathfrak{q}(\mathfrak{a}, \bar{\Omega}|\xi)$  are defined in Eq. (3.2).

**Part 2** (Conjugation of the equation). By Eqs. (6.49) and (6.52),

$$U = \mathbf{M}(\mathfrak{a}, \bar{\Omega}|D) Y_{\text{Al}}, \quad Y_{\text{Al}} = \mathbf{M}(\mathfrak{a}, \bar{\Omega}|D)^{-1} U.$$

Since  $\mathbf{M}(\mathfrak{a}, \bar{\Omega}|D)$  is a Fourier multiplier independent of time,

$$\partial_t U = \mathbf{M}(\mathfrak{a}, \bar{\Omega}|D) \partial_t Y_{\text{Al}}. \quad (6.71)$$

Using Eq. (6.38), the Alinhac-reduced real equation can be written as

$$\partial_t Y_{\text{Al}} = \text{Op}^{\text{BW}} \left( \mathbf{Q} + \mathbf{T} + \mathbf{A}_{\text{lo}}^{\mathbb{R}, \text{Al}} \right) Y_{\text{Al}} + \mathbf{R}_{[\varrho, \geq 1]} Y_{\text{Al}}. \quad (6.72)$$

Substituting Eq. (6.72) into Eq. (6.71) gives

$$\partial_t U = \mathbf{M} \text{Op}^{\text{BW}} \left( \mathbf{Q} + \mathbf{T} + \mathbf{A}_{\text{lo}}^{\mathbb{R}, \text{Al}} \right) \mathbf{M}^{-1} U + \mathbf{M} \mathbf{R}_{[\varrho, \geq 1]} \mathbf{M}^{-1} U. \quad (6.73)$$

We study each term separately.

**Part 3** (Paradiifferential formula for the complex conjugation). Let  $\mathbf{A}(U; x, \xi)$  be any matrix symbol whose structure will be specified below. We define

$$\mathbf{C}_{\mathbf{M}}[\mathbf{A}](\mathfrak{a}, \bar{\Omega}|U; x, \xi) := \mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi) \#_{\varrho} \mathbf{A}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \#_{\varrho} \mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi)^{-1}. \quad (6.74)$$

Note that if  $\mathbf{A} = \mathbf{A}(\mathfrak{a}, \bar{\Omega}|\xi)$  is a Fourier multiplier independent of  $U$  and  $x$ , then the relation Eq. (6.74) simplifies into the classical conjugation

$$\mathbf{C}_{\mathbf{M}}[\mathbf{A}](\mathfrak{a}, \bar{\Omega}|\xi) = \mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi) \mathbf{A}(\mathfrak{a}, \bar{\Omega}|\xi) \mathbf{M}(\mathfrak{a}, \bar{\Omega}|\xi)^{-1}$$

and

$$\mathbf{M} \text{Op}^{\text{BW}}(\mathbf{A}) \mathbf{M}^{-1} = \text{Op}^{\text{BW}}(\mathbf{M} \mathbf{A} \mathbf{M}^{-1}). \quad (6.75)$$

In Eq. (6.75), all matrices  $\mathbf{M}^{\pm 1}$  are evaluated at  $(\mathfrak{a}, \bar{\Omega}|D)$  when considered as the conjugating operators and at  $(\mathfrak{a}, \bar{\Omega}|\xi)$  when considered as symbols. Since there is no possible confusion, in what follows, we alleviate the notations removing this dependence. Otherwise, if the matrix symbol  $\mathbf{A}$  belongs to  $(\Gamma_{\geq 1}^m)^{2 \times 2}$ , then, by the Bony–Weyl composition formula Definition 4.22 and Proposition 4.24 and by the composition rules in Lemma 4.25,

$$\mathbf{M} \text{Op}^{\text{BW}}(\mathbf{A}) \mathbf{M}^{-1} = \text{Op}^{\text{BW}}(\mathbf{C}_{\mathbf{M}}[\mathbf{A}]) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.76)$$

Indeed,

$$\mathbf{M} \text{Op}^{\text{BW}}(\mathbf{A}) = \text{Op}^{\text{BW}}(\mathbf{M} \#_{\varrho} \mathbf{A}) + \mathbf{R}_{[\varrho, \geq 1]}, \quad (6.77)$$

$$\text{Op}^{\text{BW}}(\mathbf{M} \#_{\varrho} \mathbf{A}) \mathbf{M}^{-1} = \text{Op}^{\text{BW}}((\mathbf{M} \#_{\varrho} \mathbf{A}) \#_{\varrho} \mathbf{M}^{-1}) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.78)$$

The smoothing remainder produced in the first composition remains smoothing after composition with the Fourier multiplier  $\mathbf{M}^{-1}$ , after decreasing  $\varrho$  by a fixed amount.

**Part 4** (Conjugation of the linear Fourier multiplier and transport component). By Eq. (6.70),

$$\mathbf{Q} = \begin{bmatrix} 0 & \frac{|\xi|}{2} \\ \mathfrak{q} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta \dot{q}_{12,1} \\ \delta \dot{q}_{21,2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta \dot{q}_{12,0} \\ \delta \dot{q}_{21,1} & 0 \end{bmatrix}.$$

Moreover,

$$\mathbf{T} = \mathfrak{i} (\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi) \text{Id}_{\mathbb{C}^2}.$$

By Eqs. (2.93), (3.13) and (5.12), define

$$\mathcal{V}_{\geq 1}(\mathfrak{a}, \bar{\Omega}|U; x) := \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) - \frac{\mathfrak{a}}{4} \bar{\Omega}^{\perp} \in (\Gamma_{\geq 1}^0)^2.$$

Consequently,

$$\mathbf{T} = \mathfrak{i} \frac{\mathfrak{a}}{4} (\bar{\Omega}^{\perp} \cdot \xi) \text{Id}_{\mathbb{C}^2} + \mathfrak{i} (\mathcal{V}_{\geq 1} \cdot \xi) \text{Id}_{\mathbb{C}^2},$$

and, by Eq. (3.3),

$$\mathbf{Q} + \mathbf{T} = \mathbf{L}_0 + \begin{bmatrix} 0 & \delta \dot{q}_{12,1} \\ \delta \dot{q}_{21,2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta \dot{q}_{12,0} \\ \delta \dot{q}_{21,1} & 0 \end{bmatrix} + \mathfrak{i} (\mathcal{V}_{\geq 1} \cdot \xi) \text{Id}_{\mathbb{C}^2}.$$

Applying Lemma 6.5 and Eq. (6.75) gives

$$\mathbf{M} \text{Op}^{\text{BW}}(\mathbf{L}_0(\xi)) \mathbf{M}^{-1} = \text{Op}^{\text{BW}} \left( \mathfrak{i} \frac{\mathfrak{a}}{4} (\bar{\Omega}^{\perp} \cdot \xi) \text{Id}_{\mathbb{C}^2} + (1 - \chi(\xi)) \mathfrak{i} \omega_{\frac{3}{2}} \mathbf{J}_{\text{d}} + \mathbf{A}_{\text{lo}}^{\mathbb{C}, \text{lin}} \right).$$

For the remaining transport component, apply Eqs. (6.74) and (6.76) with  $\mathbf{A} = i(\mathcal{V}'_{\geq 1} \cdot \xi) \text{Id}_{\mathbb{C}^2}$ . Its pointwise conjugation is unchanged because the symbol is scalar in the matrix indices. Every non-pointwise Bony–Weyl correction is of order zero and at least one-homogeneous in  $U$ . Therefore

$$\mathbf{C}_M [i(\mathcal{V}'_{\geq 1} \cdot \xi) \text{Id}_{\mathbb{C}^2}] = i(\mathcal{V}'_{\geq 1} \cdot \xi) \text{Id}_{\mathbb{C}^2} + \mathbf{A}_{[0, \geq 1]}.$$

Recombining the flat and nonlinear transport contributions yields

$$\begin{aligned} \mathbf{M} \text{Op}^{\text{BW}}(\mathbf{Q} + \mathbf{T}) \mathbf{M}^{-1} &= \text{Op}^{\text{BW}} \left( (1 - \chi(\xi)) i\omega_{\frac{3}{2}} \mathbf{J}_d + \mathbf{A}_{\text{lo}}^{\text{C,lin}} + \mathbf{T} + \mathbf{A}_{[0, \geq 1]} \right) + \mathbf{R}_{[\varrho, \geq 1]} \\ &\quad + \text{Op}^{\text{BW}} \left( \mathbf{C}_M \left[ \begin{bmatrix} 0 & \delta \dot{q}_{12,1} \\ \delta \dot{q}_{21,2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \delta \dot{q}_{12,0} \\ \delta \dot{q}_{21,1} & 0 \end{bmatrix} \right] \right). \end{aligned} \quad (6.79)$$

It remains to analyze the last term in Eq. (6.79).

**Part 5** (The high-frequency quasilinear block and the explicit order- $\frac{1}{2}$  block). All positive-order terms are extracted on the high-frequency region  $\text{supp}^{\text{G}}(\chi)$ . On this region

$$\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}(\mathfrak{a}, \bar{\Omega}|\xi) = \mathfrak{m}_{-\frac{1}{4}}(\mathfrak{a}, \bar{\Omega}|\xi).$$

The complementary low-frequency contribution is compactly supported in  $\xi$ , hence it corresponds to bounded operators. Moreover, since  $\delta \dot{q}_{12,1}$ ,  $\delta \dot{q}_{12,0}$ ,  $\delta \dot{q}_{21,2}$  and  $\delta \dot{q}_{21,1}$  are at least 1-homogeneous in  $U$ , such contributions are included in the generic bounded symbol  $\mathbf{A}_{[0, \geq 1]}$ . In what follows, we shall only work within this high-frequency region. Given two scalar symbols  $a, b$ , we define

$$\mathcal{B}(a, b) := \begin{bmatrix} b - a & a + b \\ -(a + b) & -(b - a) \end{bmatrix}.$$

With this notation in hand, one readily has on  $\text{supp}^{\text{G}}(\chi)$ ,

$$\mathbf{C}_M \left[ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right] = \frac{i}{2} \mathcal{B} \left( \mathfrak{m}_{-\frac{1}{4}}^{-1} \#_{\varrho} a \#_{\varrho} \mathfrak{m}_{-\frac{1}{4}}^{-1}, \mathfrak{m}_{-\frac{1}{4}} \#_{\varrho} b \#_{\varrho} \mathfrak{m}_{-\frac{1}{4}} \right).$$

Now, observe the following Weyl cancellation: If  $\mathfrak{m} = \mathfrak{m}(\xi) \in \tilde{\Gamma}^{\mu}$  is a scalar Fourier multiplier, then for any  $c \in \Gamma_{\geq 1}^m$ , we have

$$\mathfrak{m} \#_{\varrho} c \#_{\varrho} \mathfrak{m} = \mathfrak{m}^2 c + \Gamma_{\geq 1}^{2\mu+m-2}. \quad (6.80)$$

Indeed, the first-order Weyl corrections cancel because the same multiplier appears on both sides of  $c$ . Applying Eq. (6.80) with either the couple  $(\mathfrak{m}, c) = \left( \mathfrak{m}_{-\frac{1}{4}}^{-1}, a \right) \in \tilde{\Gamma}^{\frac{1}{4}} \times \Gamma_{\geq 1}^1$  or a couple  $(\mathfrak{m}, c) = \left( \mathfrak{m}_{-\frac{1}{4}}, b \right) \in \tilde{\Gamma}^{-\frac{1}{4}} \times \Gamma_{\geq 1}^2$ , we find on  $\text{supp}^{\text{G}}(\chi)$ ,

$$\mathbf{C}_M \left[ \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right] = \frac{i}{2} \mathcal{B} \left( \mathfrak{m}_{-\frac{1}{4}}^{-2} a, \mathfrak{m}_{-\frac{1}{4}}^2 b \right) + \mathbf{A}_{[0, \geq 1]}. \quad (6.81)$$

According to the explicit formulae Eqs. (6.41) to (6.44), one can check that

$$\mathfrak{m}_{-\frac{1}{4}}^{-2} = \frac{\omega_{\frac{3}{2}}}{\dot{\omega}_{\frac{3}{2}}} \dot{\mathfrak{m}}_{-\frac{1}{4}}^{-2} \quad (6.82)$$

and

$$\mathfrak{m}_{-\frac{1}{4}}^2 - \frac{\omega_{\frac{3}{2}}}{\dot{\omega}_{\frac{3}{2}}} \dot{\mathfrak{m}}_{-\frac{1}{4}}^2 = \dot{\mathfrak{c}}_{-\frac{3}{2}}^{\text{sh}} + \tilde{\Gamma}^{-\frac{5}{2}}, \quad (6.83)$$

with  $\dot{\mathfrak{c}}_{-\frac{3}{2}}^{\text{sh}}$  defined by

$$\dot{\mathfrak{c}}_{-\frac{3}{2}}^{\text{sh}}(\mathfrak{a}, \bar{\Omega}|\xi) := \frac{4 - \mathfrak{a}^2}{32} \left( \frac{2 \langle\langle \rho^{\pm} \rangle\rangle}{\sigma} \right)^{\frac{3}{2}} \frac{(\bar{\Omega}^{\perp} \cdot \xi)^2}{|\xi|^{\frac{7}{2}}} \in \tilde{\Gamma}^{-\frac{3}{2}}. \quad (6.84)$$

Inserting Eqs. (6.82) and (6.83) into Eq. (6.81), we infer for any  $a \in \Gamma_{\geq 1}^1$  and  $b \in \Gamma_{\geq 1}^2$ , on  $\text{supp}^{\mathbb{G}}(\chi)$ ,

$$\mathbf{C}_M \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \frac{i}{2} \frac{\omega_{\frac{3}{2}}}{\dot{\omega}_{\frac{3}{2}}} \mathcal{B} \left( \dot{m}_{-\frac{1}{4}}^{-2} a, \dot{m}_{-\frac{1}{4}}^2 b \right) + \frac{i}{2} \mathcal{B} \left( 0, \dot{c}_{-\frac{3}{2}}^{\text{sh}} b \right) + \mathbf{A}_{[0, \geq 1]}. \quad (6.85)$$

We now extract the quasilinear order-3/2 correction. Applying Eq. (6.85) with the pair

$$(a, b) = (\delta \dot{q}_{12,1}, \delta \dot{q}_{21,2}) \in \Gamma_{\geq 1}^1 \times \Gamma_{\geq 1}^2,$$

we obtain by definition Eq. (6.56) that on  $\text{supp}^{\mathbb{G}}(\chi)$ ,

$$\mathbf{C}_M \begin{bmatrix} 0 & \delta \dot{q}_{12,1} \\ \delta \dot{q}_{21,2} & 0 \end{bmatrix} = i\omega_{\frac{3}{2}} \mathbf{J}_d \dot{\mathbf{E}}_{\text{ql},0}^{\mathbb{C}} + \frac{i}{2} \mathcal{B} \left( 0, \dot{c}_{-\frac{3}{2}}^{\text{sh}} \delta \dot{q}_{21,2} \right) + \mathbf{A}_{[0, \geq 1]}, \quad (6.86)$$

where

$$\dot{\mathbf{E}}_{\text{ql},0}^{\mathbb{C}} := \begin{bmatrix} \dot{e}_0 & \dot{f}_0 \\ \dot{f}_0 & \dot{e}_0 \end{bmatrix},$$

with

$$\dot{e}_0 := - \frac{\dot{m}_{-\frac{1}{4}}^2 \delta \dot{q}_{21,2} - \dot{m}_{-\frac{1}{4}}^{-2} \delta \dot{q}_{12,1}}{2\dot{\omega}_{\frac{3}{2}}}, \quad (6.87)$$

$$\dot{f}_0 := - \frac{\dot{m}_{-\frac{1}{4}}^{-2} \delta \dot{q}_{12,1} + \dot{m}_{-\frac{1}{4}}^2 \delta \dot{q}_{21,2}}{2\dot{\omega}_{\frac{3}{2}}}. \quad (6.88)$$

By Eqs. (5.4), (5.228), (6.43), (6.44), (6.66) and (6.68), we compute explicitly

$$\dot{m}_{-\frac{1}{4}}^{-2} \delta \dot{q}_{12,1} = \dot{\omega}_{\frac{3}{2}} \left( \sqrt{1 + |\nabla^{\perp} \eta \cdot \hat{\xi}|^2} - 1 \right), \quad (6.89)$$

$$\dot{m}_{-\frac{1}{4}}^2 \delta \dot{q}_{21,2} = \dot{\omega}_{\frac{3}{2}} \left( 1 - \frac{1 + |\nabla^{\perp} \eta \cdot \hat{\xi}|^2}{(1 + |\nabla \eta|^2)^{\frac{3}{2}}} \right). \quad (6.90)$$

Inserting Eqs. (6.89) and (6.90) into Eqs. (6.87) and (6.88), we get Eqs. (6.57) and (6.58). Similarly, one can apply Eq. (6.85) to the pair

$$(a, b) = (\delta \dot{q}_{12,0}, \delta \dot{q}_{21,1}) \in \Gamma_{\geq 1}^0 \times \Gamma_{\geq 1}^1$$

and get on  $\text{supp}^{\mathbb{G}}(\chi)$ ,

$$\mathbf{C}_M \begin{bmatrix} 0 & \delta \dot{q}_{12,0} \\ \delta \dot{q}_{21,1} & 0 \end{bmatrix} = \frac{i}{2} \mathcal{B} \left( \dot{m}_{-\frac{1}{4}}^{-2} \delta \dot{q}_{12,0}, \dot{m}_{-\frac{1}{4}}^2 \delta \dot{q}_{21,1} \right) + \mathbf{A}_{[0, \geq 1]}. \quad (6.91)$$

Indeed,  $\frac{\omega_{\frac{3}{2}}}{\dot{\omega}_{\frac{3}{2}}} = 1 + \tilde{\Gamma}^{-\frac{1}{2}}$  so the remaining terms create only bounded symbols. Now observe that the cut-off function being with compact support, for any  $m \in \mathbb{R}$  and any  $\mathbf{A}, \mathbf{B} \in (\Gamma_{\geq 1}^m)^{2 \times 2}$ ,

$$\left( \mathbf{A} = \mathbf{B} \quad \text{on} \quad \text{supp}^{\mathbb{G}}(\chi) \right) \quad \Rightarrow \quad \mathbf{A} = (1 - \chi(\xi)) \mathbf{B} + \mathbf{A}_{[0, \geq 1]}. \quad (6.92)$$

Putting together Eqs. (6.79), (6.86), (6.91) and (6.92), we have obtained

$$\mathbf{M} \text{Op}^{\text{BW}}(\mathbf{Q} + \mathbf{T}) \mathbf{M}^{-1} = \text{Op}^{\text{BW}} \left( (1 - \chi(\xi)) i\omega_{\frac{3}{2}} \mathbf{J}_d \left( \text{Id} + \dot{\mathbf{E}}_{\text{ql},0}^{\mathbb{C}} \right) + \mathbf{A}_{\text{lo}}^{\mathbb{C}, \text{lin}} + \mathbf{T} + (1 - \chi(\xi)) \mathbf{A}_{\frac{1}{2}}^{\mathbb{C}} + \mathbf{A}_{[0, \geq 1]} \right) + \mathbf{R}_{[\varrho, \geq 1]}, \quad (6.93)$$

where

$$\mathbf{A}_{\frac{1}{2}}^{\mathbb{C}} := \frac{i}{2} \left( \mathcal{B} \left( 0, \dot{c}_{-\frac{3}{2}}^{\text{sh}} \delta \dot{q}_{21,2} \right) + \mathcal{B} \left( \dot{m}_{-\frac{1}{4}}^{-2} \delta \dot{q}_{12,0}, \dot{m}_{-\frac{1}{4}}^2 \delta \dot{q}_{21,1} \right) \right). \quad (6.94)$$

We are left with the explicit computation of the symbol  $A_{\frac{1}{2}}^C$  in Eq. (6.94). Note that we can rewrite

$$A_{\frac{1}{2}}^C = \dot{A}_0^C |\xi|^{\frac{1}{2}}, \quad \dot{A}_0^C \triangleq \frac{i}{2} \mathcal{B}(\dot{a}_0^C, \dot{b}_0^C),$$

where

$$\begin{aligned} \dot{a}_0^C &:= |\xi|^{-\frac{1}{2}} \dot{m}_{-\frac{1}{4}}^{-2} \delta \dot{q}_{12,0}, \\ \dot{b}_0^C &:= |\xi|^{-\frac{1}{2}} \left[ \dot{m}_{-\frac{1}{4}}^2 \delta \dot{q}_{21,1} + \dot{c}_{-\frac{3}{2}}^{\text{sh}} \delta \dot{q}_{21,2} \right]. \end{aligned}$$

Combining Eqs. (6.43), (6.44) and (6.67), we immediately obtain the explicit expression Eq. (6.60c) for  $\dot{a}_0^C$ . The explicit computation of  $\dot{b}_0^C$  requires more explanations. We compute the two contributions separately. First, Eqs. (5.4), (5.7), (6.65) and (6.69) give

$$\begin{aligned} \delta \dot{q}_{21,1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) &= \frac{4 - \mathfrak{a}^2}{2} |\xi| \sqrt{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2} \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta(U); x, \xi) \cdot \Omega(\bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \right)^2 \\ &\quad - \frac{4 - \mathfrak{a}^2}{8} |\xi| (\bar{\Omega}^\perp \cdot \hat{\xi})^2. \end{aligned}$$

Moreover,

$$|\xi|^{-\frac{1}{2}} \dot{m}_{-\frac{1}{4}}^2(\mathfrak{a}, \bar{\Omega}|\xi) = \sqrt{\frac{\langle\langle \rho^\pm \rangle\rangle}{2\sigma}} |\xi|^{-1}.$$

Therefore

$$\begin{aligned} &|\xi|^{-\frac{1}{2}} \dot{m}_{-\frac{1}{4}}^2(\mathfrak{a}, \bar{\Omega}|\xi) \delta \dot{q}_{21,1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \\ &= \frac{4 - \mathfrak{a}^2}{16} \sqrt{\frac{2\langle\langle \rho^\pm \rangle\rangle}{\sigma}} \left[ 4 \sqrt{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2} \left( A_0^{\mathcal{B}_0, \omega; \perp}(\eta(U); x, \xi) \cdot \Omega(\bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \right)^2 - (\bar{\Omega}^\perp \cdot \hat{\xi})^2 \right]. \end{aligned} \quad (6.95)$$

Second, by Eqs. (5.6) and (6.68),

$$\delta \dot{q}_{21,2}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) = \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2 \left[ 1 - \frac{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2}{(1 + |\nabla \eta(U)|^2)^{\frac{3}{2}}} \right].$$

Using Eq. (6.84), this gives

$$|\xi|^{-\frac{1}{2}} \dot{c}_{-\frac{3}{2}}^{\text{sh}}(\mathfrak{a}, \bar{\Omega}|\xi) \delta \dot{q}_{21,2}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) = \frac{4 - \mathfrak{a}^2}{16} \sqrt{\frac{2\langle\langle \rho^\pm \rangle\rangle}{\sigma}} (\bar{\Omega}^\perp \cdot \hat{\xi})^2 \left[ 1 - \frac{1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2}{(1 + |\nabla \eta(U)|^2)^{\frac{3}{2}}} \right]. \quad (6.96)$$

Adding Eq. (6.95) and Eq. (6.96) cancels the pure flat shear contribution and yields precisely Eq. (6.60d). The square on  $A_0^{\mathcal{B}_0, \omega; \perp} \cdot \Omega$  comes from  $q_0^{\tilde{\pi}}$  in Eq. (5.7).

**Part 6** (The bounded real lower-order block). We apply Eqs. (6.74) and (6.76) with  $A = A_{\text{lo}}^{\mathbb{R}, \text{Al}}$ . Then

$$M \text{Op}^{\text{BW}}(A_{\text{lo}}^{\mathbb{R}, \text{Al}}) M^{-1} = \text{Op}^{\text{BW}}(A_{\text{conj}}^{\mathbb{C}, \text{lo}}) + R_{[\varrho, \geq 1]}, \quad A_{\text{conj}}^{\mathbb{C}, \text{lo}} := C_M [A_{\text{lo}}^{\mathbb{R}, \text{Al}}].$$

By Eq. (6.8),

$$A_{\text{lo}}^{\mathbb{R}, \text{Al}} \in \begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix}.$$

Moreover, by Eqs. (6.47) and (6.48), the entries of  $M$  and  $M^{-1}$  have orders  $-1/4$  or  $1/4$  at high frequency, and are order zero at low frequency. Therefore the largest order produced by the pointwise products is 0. All non-pointwise Bony–Weyl corrections contain at least one  $\xi$ -derivative and therefore have no larger order. Consequently

$$A_{\text{conj}}^{\mathbb{C}, \text{lo}} \in (\Gamma_{\geq 1}^0)^{2 \times 2}. \quad (6.97)$$

This contribution is included in the generic bounded symbol  $A_{[0, \geq 1]}$ .

**Part 7** (Smoothing remainders and conclusion). The smoothing remainder in Eq. (6.73) is controlled by Lemma 4.25:

$$\mathbf{M}\mathbf{R}_{[\varrho, \geq 1]}\mathbf{M}^{-1} \in (\mathcal{R}_{\geq 1}^{\varrho})^{2 \times 2}, \quad (6.98)$$

after decreasing and relabelling  $\varrho$  and can be renamed  $\mathbf{R}_{[\varrho, \geq 1]}$ . The same convention absorbs all remainders generated by the symbolic compositions in Eqs. (6.77) and (6.78). Combining Eqs. (6.73), (6.93), (6.97) and (6.98), we have obtained Eq. (6.55).  $\square$

**Remark 6.6** (Non-homogeneous Sobolev scale and the explicit order-1/2 block). By Eq. (6.49),

$$u = \frac{1}{\sqrt{2}} \left( \left( \mathbf{m}_{-\frac{1}{4}}^{\text{nh}}(\mathfrak{a}, \bar{\Omega}|D) \right)^{-1} \eta_{\text{Al}} + \mathbf{i} \mathbf{m}_{-\frac{1}{4}}^{\text{nh}}(\mathfrak{a}, \bar{\Omega}|D) \tilde{\psi}_{\text{Al}} \right).$$

Conversely,

$$\eta_{\text{Al}} = \frac{1}{\sqrt{2}} \mathbf{m}_{-\frac{1}{4}}^{\text{nh}}(\mathfrak{a}, \bar{\Omega}|D) (u + \bar{u}), \quad \tilde{\psi}_{\text{Al}} = -\frac{\mathbf{i}}{\sqrt{2}} \left( \mathbf{m}_{-\frac{1}{4}}^{\text{nh}}(\mathfrak{a}, \bar{\Omega}|D) \right)^{-1} (u - \bar{u}).$$

By Eq. (6.46), the multiplier  $\mathbf{m}_{-\frac{1}{4}}^{\text{nh}}$  is equal to 1 at low frequency and coincides with  $\mathbf{m}_{-\frac{1}{4}}$  at high frequency. Hence the complex variable in Eq. (6.49) satisfies the non-homogeneous equivalence

$$\|U\|_{H^s} \sim \|\eta_{\text{Al}}\|_{H^{s+\frac{1}{4}}} + \|\tilde{\psi}_{\text{Al}}\|_{H^{s-\frac{1}{4}}}.$$

In the capillary high-frequency regime,

$$\omega_{\frac{3}{2}} \sim \dot{\omega}_{\frac{3}{2}} = \sqrt{\frac{\sigma}{2\langle\langle \rho^{\pm} \rangle\rangle}} |\xi|^{\frac{3}{2}}, \quad \mathbf{m}_{-\frac{1}{4}} \sim \dot{\mathbf{m}}_{-\frac{1}{4}} \sim C_{\rho^{\pm}, \sigma} |\xi|^{-\frac{1}{4}}.$$

Therefore the explicit order- $\frac{1}{2}$  block

$$(1 - \chi(\xi)) \dot{A}_0^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) |\xi|^{\frac{1}{2}}$$

is displayed only on the high-frequency region. The complementary low-frequency part is a bounded contribution and is included in  $A_{[0, \geq 1]}$ . The Alinhac order- $\frac{1}{2}$  terms are absent because the Alinhac block has already been removed in Proposition 6.2.  $\blacklozenge$

### 6.3 Diagonalization of the positive-order symbols

In this subsection we diagonalize the positive-order part of the complex Alinhac-reduced system obtained in Proposition 6.4. The diagonalization is performed in two steps. First we diagonalize the quasilinear order-3/2 matrix

$$J_{\text{d}} \left( \text{Id}_{\mathbb{C}^2} + \dot{E}_{\text{ql}, 0}^{\mathbb{C}} \right),$$

by a matrix paradifferential operator of order 0. Then we remove the off-diagonal part of the explicit order-1/2 block by a correction of order  $-1$ . The scalar transport part remains diagonal at all positive orders.

Throughout this section we use the conventions of Notation 1.1: scalar symbols are denoted by lower-case letters, two-component variables by upper-case letters, and  $2 \times 2$  matrix symbols and matrix operators by upper-case bold letters. Homogeneous symbols in  $\xi$  are denoted with a dot, and the subscript records the degree of homogeneity.

**Proposition 6.7** (Diagonalization of the positive-order symbols). *Let the assumptions of Proposition 6.4 hold, and let  $U$  solve Eq. (6.55). There exists a bounded and invertible operator*

$$\mathbf{B}(\mathfrak{a}, \bar{\Omega}|U) \in \mathcal{L}(H^s) \quad \forall s \geq s_0,$$

such that, defining

$$W := \mathbf{B}(\mathfrak{a}, \bar{\Omega}|U)^{-1} U, \quad (6.99)$$

the unknown  $W$  solves

$$\partial_t W = \text{Op}^{\text{BW}} \left( A_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi) \right) W + \mathbf{i} \text{Op}^{\text{BW}} \left( D_{>0}^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) + A_{[0, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \right) W + \mathbf{R}_{[\varrho, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U) W. \quad (6.100)$$

The objects entering Eqs. (6.99) and (6.100) are defined as follows.

- $A_{10}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi)$  is the same flat low-frequency linear multiplier introduced in Eq. (6.51);
- $D_{>0}^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \in \left(\Gamma_{\geq 0}^{\frac{3}{2}}\right)^{2 \times 2}$  is the diagonal positive-order matrix of real symbols

$$D_{>0}^{\mathbb{C}} := D_{\frac{3}{2}}^{\mathbb{C}} + \dot{D}_1^{\mathbb{C}} + \dot{D}_{\frac{1}{2}}^{\mathbb{C}}. \quad (6.101)$$

The symbols  $D_{\frac{3}{2}}^{\mathbb{C}}$ ,  $\dot{D}_1^{\mathbb{C}}$ , and  $\dot{D}_{\frac{1}{2}}^{\mathbb{C}}$  are real-valued diagonal matrix symbols for real  $U$  and are explicitly defined as follows:

- the high-frequency diagonalized quasilinear order-3/2 symbol is

$$D_{\frac{3}{2}}^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := (1 - \chi(\xi)) \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \dot{\theta}_0(U; x, \hat{\xi}) J_d \in \left(\Gamma_{\geq 0}^{\frac{3}{2}}\right)^{2 \times 2}; \quad (6.102)$$

here  $J_d$  is defined in Eq. (6.50),  $\omega_{\frac{3}{2}}$  is the order-3/2 linear frequency from Eq. (6.39), and  $\dot{\theta}_0(U; x, \hat{\xi})$  is the real-valued homogeneous degree-zero scalar symbol given by

$$\dot{\ell}_0(U; x, \hat{\xi}) := 1 + \dot{e}_0(U; x, \hat{\xi}), \quad \dot{\theta}_0(U; x, \hat{\xi}) := \sqrt{\dot{\ell}_0^2 - \dot{f}_0^2}(U; x, \hat{\xi}). \quad (6.103)$$

The coefficients  $\dot{e}_0, \dot{f}_0 \in \Gamma_{\geq 1}^0$  are the real-valued homogeneous quasilinear coefficients introduced in Eqs. (6.87) and (6.88);

- the scalar transport symbol is

$$\dot{D}_1^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := (\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi) \text{Id}_{\mathbb{C}^2} \in \left(\Gamma_{\geq 0}^1\right)^{2 \times 2}; \quad (6.104)$$

here  $\mathcal{V}_{\text{can}}$  is the canonical transport velocity defined in Eq. (5.12). This block is scalar in the complex components and is therefore unaffected by the diagonalizing matrices at positive order;

- the high-frequency diagonal order-1/2 symbol is

$$\dot{D}_{\frac{1}{2}}^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := -(1 - \chi(\xi)) \dot{\alpha}_0^{\mathbb{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) |\xi|^{\frac{1}{2}} J_d \in \left(\Gamma_{\geq 1}^{\frac{1}{2}}\right)^{2 \times 2}. \quad (6.105)$$

The coefficient  $\dot{\alpha}_0^{\mathbb{C}, \text{pr}} \in \Gamma_{\geq 1}^0$  is the diagonal component of the full order-1/2 coefficient after the principal diagonalization; it is defined in Eqs. (6.111) and (6.112).

- $A_{[0, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \in \left(\Gamma_{\geq 1}^0\right)^{2 \times 2}$  is a generic bounded matrix symbol in the sense of Notation 5.1.
- $R_{[\varrho, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U) \in \left(\mathcal{R}_{\geq 1}^{\varrho}\right)^{2 \times 2}$  is a generic smoothing remainder in the sense of Notation 5.1.

The operator  $B$  is constructed as a finite composition of paradifferential parametrices, each eliminating a block of off-diagonal symbols at a prescribed order. The diagonalization is carried out in two successive steps, corresponding to the principal and subprincipal orders, in Propositions 6.8 and 6.9 respectively; Proposition 6.7 is then a direct consequence, as detailed in Section 6.3.3.

### 6.3.1 Diagonalization at principal order

Here, we perform the diagonalization of the order- $\frac{3}{2}$  in the complex paradifferential system Eq. (6.55). The result reads as follows.

**Proposition 6.8** (Principal-order diagonalization). *Let the assumptions of Proposition 6.4 hold, and let  $U$  solve Eq. (6.55). There exists a bounded and invertible operator*

$$B_0(\mathfrak{a}, \bar{\Omega}|U) \in \mathcal{L}(H^s) \quad \forall s \geq s_0,$$

such that, defining

$$U_0 := B_0(\mathfrak{a}, \bar{\Omega}|U)^{-1} U, \quad (6.106)$$

the unknown  $U_0$  solves

$$\begin{aligned}
\partial_t U_0 &= \text{Op}^{\text{BW}} \left( A_{10}^{\text{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega} | \xi) \right) U_0 \\
&+ i \text{Op}^{\text{BW}} \left( D_{\frac{3}{2}}^{\text{C}}(\mathfrak{a}, \bar{\Omega} | U; x, \xi) \right) U_0 \\
&+ i \text{Op}^{\text{BW}} \left( \dot{D}_1^{\text{C}}(\mathfrak{a}, \bar{\Omega} | U; x, \xi) \right) U_0 \\
&+ \text{Op}^{\text{BW}} \left( (1 - \chi(\xi)) \dot{A}_0^{\text{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega} | U; x, \hat{\xi}) |\xi|^{\frac{1}{2}} \right) U_0 \\
&+ \text{Op}^{\text{BW}} \left( A_{[0, \geq 1]}^{\text{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega} | U; x, \xi) \right) U_0 + R_{[\varrho, \geq 1]}^{\text{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega} | U) U_0.
\end{aligned} \tag{6.107}$$

The objects entering Eqs. (6.106) and (6.107) are defined as follows.

- $B_0(U)$  is the order-zero paradifferential operator

$$B_0(U) := \text{Op}^{\text{BW}}(P_0(U; x, \xi)),$$

where  $P_0$  is the non-homogeneous principal diagonalizer

$$P_0(U; x, \xi) := \chi(\xi) \text{Id}_{\mathbb{C}^2} + (1 - \chi(\xi)) \dot{P}_0(U; x, \hat{\xi}). \tag{6.108}$$

The high-frequency homogeneous diagonalizer is

$$\dot{P}_0(U; x, \hat{\xi}) := \frac{1}{\sqrt{1 - \dot{\chi}_0^2}} \begin{bmatrix} 1 & \dot{\chi}_0 \\ \dot{\chi}_0 & 1 \end{bmatrix} (U; x, \hat{\xi}), \quad \dot{P}_0 - \text{Id}_{\mathbb{C}^2} \in (\Gamma_{\geq 1}^0)^{2 \times 2}. \tag{6.109}$$

Here  $\dot{\chi}_0 \in \Gamma_{\geq 1}^0$  is the real homogeneous scalar defined by

$$\begin{aligned}
\dot{\chi}_0(U; x, \hat{\xi}) &:= - \frac{\dot{f}_0(U; x, \hat{\xi})}{\dot{\ell}_0(U; x, \hat{\xi}) + \dot{\theta}_0(U; x, \hat{\xi})} \\
&= \frac{\left(1 + |\nabla \eta(U)|^2\right)^{\frac{3}{4}} - \left(1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2\right)^{\frac{1}{4}}}{\left(1 + |\nabla \eta(U)|^2\right)^{\frac{3}{4}} + \left(1 + |\nabla^\perp \eta(U) \cdot \hat{\xi}|^2\right)^{\frac{1}{4}}} \in \Gamma_{\geq 1}^0;
\end{aligned} \tag{6.110}$$

- $A_{10}^{\text{C}, \text{lin}}$  is the flat low-frequency linear multiplier explicitly defined in Eq. (6.51);
- $D_{\frac{3}{2}}^{\text{C}} \in \left(\Gamma_{\geq 0}^{\frac{3}{2}}\right)^{2 \times 2}$  and  $\dot{D}_1^{\text{C}} \in (\Gamma_{\geq 0}^1)^{2 \times 2}$  are the diagonal quasilinear and transport symbols already defined in Eqs. (6.102) and (6.104).
- $\dot{A}_0^{\text{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega} | U; x, \hat{\xi}) \in (\Gamma_{\geq 1}^0)^{2 \times 2}$  is defined as

$$\dot{A}_0^{\text{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega} | U; x, \hat{\xi}) := \left( \dot{P}_0^{-1} \dot{A}_0^{\text{C}} \dot{P}_0 \right) (\mathfrak{a}, \bar{\Omega} | U; x, \hat{\xi}) = i \begin{bmatrix} \dot{\alpha}_0^{\text{C}, \text{pr}} & \dot{\beta}_0^{\text{C}, \text{pr}} \\ -\dot{\beta}_0^{\text{C}, \text{pr}} & -\dot{\alpha}_0^{\text{C}, \text{pr}} \end{bmatrix}, \tag{6.111}$$

where

$$\dot{\alpha}_0^{\text{C}, \text{pr}} := \frac{(1 + \dot{\chi}_0^2) \dot{\alpha}_0^{\text{C}} + 2 \dot{\chi}_0 \dot{\beta}_0^{\text{C}}}{1 - \dot{\chi}_0^2}, \tag{6.112}$$

$$\dot{\beta}_0^{\text{C}, \text{pr}} := \frac{2 \dot{\chi}_0 \dot{\alpha}_0^{\text{C}} + (1 + \dot{\chi}_0^2) \dot{\beta}_0^{\text{C}}}{1 - \dot{\chi}_0^2}. \tag{6.113}$$

Here  $\dot{\alpha}_0^{\text{C}}, \dot{\beta}_0^{\text{C}} \in \Gamma_{\geq 1}^0$  are defined in Eqs. (6.60a) and (6.60b).

- $A_{[0, \geq 1]}^{\text{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega} | U; x, \xi) \in (\Gamma_{\geq 1}^0)^{2 \times 2}$  is a generic bounded matrix symbol in the sense of Notation 5.1.

- $\mathbf{R}_{[\ell, \geq 1]}^{\mathbb{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega}|U) \in (\mathcal{R}_{\geq 1}^{\ell})^{2 \times 2}$  is a generic smoothing remainder in the sense of Notation 5.1.

*Proof.* Throughout the proof all symbols are evaluated at

$$(\mathfrak{a}, \bar{\Omega}|U; x, \xi),$$

unless explicitly stated otherwise. We use the convention of Notation 5.1: the symbols  $\mathbf{A}_{[m, \geq 1]}$  and the remainders  $\mathbf{R}_{[\ell, \geq 1]}$  are generic and may change from line to line. We also use componentwise the symbolic calculus of Definition 4.22 and Proposition 4.24 and the remainder rules Lemma 4.25-i, Lemma 4.25-ii.

**Part 1** (The diagonalizer and its symbolic inverse). We first verify that the high-frequency symbol defining the principal diagonalizer is admissible. By Eqs. (6.57), (6.58) and (6.103), we have the explicit expressions

$$\begin{aligned} \dot{\ell}_0 = 1 + \dot{e}_0 &= \frac{1}{2} \left( \sqrt{1 + |\nabla^\perp \eta \cdot \hat{\xi}|^2} + \frac{1 + |\nabla^\perp \eta \cdot \hat{\xi}|^2}{(1 + |\nabla \eta|^2)^{\frac{3}{2}}} \right), \\ \dot{\theta}_0 &= \left( \frac{1 + |\nabla^\perp \eta \cdot \hat{\xi}|^2}{1 + |\nabla \eta|^2} \right)^{\frac{3}{4}}. \end{aligned} \quad (6.114)$$

In particular,

$$\dot{\ell}_0 - 1 \in \Gamma_{\geq 1}^0, \quad \dot{\theta}_0 - 1 \in \Gamma_{\geq 1}^0.$$

After increasing  $s_0$  and decreasing  $r_0$ , we get

$$\dot{\ell}_0 \geq \frac{3}{4}, \quad \dot{\theta}_0 \geq \frac{3}{4}, \quad \dot{\ell}_0 + \dot{\theta}_0 \geq \frac{3}{2}.$$

This allows to define

$$\dot{\chi}_0 := -\frac{\dot{f}_0}{\dot{\ell}_0 + \dot{\theta}_0} \in \Gamma_{\geq 1}^0, \quad \|\dot{\chi}_0\|_{L^\infty} \leq Cr_0.$$

In particular, decreasing  $r_0$  once more if necessary,

$$1 - \dot{\chi}_0^2 \geq \frac{1}{2}. \quad (6.115)$$

Moreover, after straightforward simplifications, we obtain from Eqs. (6.58) and (6.114), the explicit expression Eq. (6.110). Now, using Eqs. (6.109) and (6.115) and the substitution rule Lemma 4.25-iii, we get

$$\dot{\mathbf{P}}_0 - \text{Id}_{\mathbb{C}^2} \in (\Gamma_{\geq 1}^0)^{2 \times 2}, \quad \dot{\mathbf{P}}_0 \in (\Gamma_{\geq 0}^0)^{2 \times 2}.$$

Moreover its high-frequency pointwise inverse is

$$\dot{\mathbf{P}}_0^{-1} = \frac{1}{\sqrt{1 - \dot{\chi}_0^2}} \begin{bmatrix} 1 & -\dot{\chi}_0 \\ -\dot{\chi}_0 & 1 \end{bmatrix}. \quad (6.116)$$

It satisfies

$$\dot{\mathbf{P}}_0^{-1} - \text{Id}_{\mathbb{C}^2} \in (\Gamma_{\geq 1}^0)^{2 \times 2}, \quad \dot{\mathbf{P}}_0^{-1} \in (\Gamma_{\geq 0}^0)^{2 \times 2}.$$

We next record the differential algebra of the homogeneous diagonalizer, which is the key point both for the symbolic inverse and for the cancellation of the order-1/2 Weyl correction below. Set

$$\mathbf{X}_{\text{od}} := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (6.117)$$

Since

$$\dot{\mathbf{P}}_0 = \exp(\dot{\varphi}_0 \mathbf{X}_{\text{od}}), \quad \dot{\varphi}_0 := \text{arctanh } \dot{\chi}_0,$$

we have, for every  $y \in \{x_1, x_2, \xi_1, \xi_2\}$ ,

$$\dot{\mathbf{A}}_y := \dot{\mathbf{P}}_0^{-1} \partial_y \dot{\mathbf{P}}_0 = (\partial_y \dot{\varphi}_0) \mathbf{X}_{\text{od}}. \quad (6.118)$$

Thus all the matrices  $\dot{\mathbf{A}}_y$  commute. Moreover  $\partial_y \dot{\mathbf{P}}_0^{-1} = -\dot{\mathbf{A}}_y \dot{\mathbf{P}}_0^{-1}$ , and therefore

$$\begin{aligned} \{\dot{\mathbf{P}}_0^{-1}, \dot{\mathbf{P}}_0\} &= \sum_{j=1}^2 \left[ \left( \partial_{\xi_j} \dot{\mathbf{P}}_0^{-1} \right) (\partial_{x_j} \dot{\mathbf{P}}_0) - (\partial_{x_j} \dot{\mathbf{P}}_0^{-1}) (\partial_{\xi_j} \dot{\mathbf{P}}_0) \right] \\ &= \sum_{j=1}^2 \left[ -\dot{\mathbf{A}}_{\xi_j} \dot{\mathbf{A}}_{x_j} + \dot{\mathbf{A}}_{x_j} \dot{\mathbf{A}}_{\xi_j} \right] = 0. \end{aligned} \quad (6.119)$$

Hence the homogeneous symbolic inverse has no order-(-1) defect

$$\dot{\mathbf{P}}_0^{-1} \#_{\varrho} \dot{\mathbf{P}}_0 = \text{Id}_{\mathbb{C}^2} + \mathbf{A}_{[-2, \geq 1]}, \quad \mathbf{A}_{[-2, \geq 1]} \in (\Gamma_{\geq 1}^{-2})^{2 \times 2},$$

and the same identity holds with the two factors exchanged.

We now pass from  $\dot{\mathbf{P}}_0$  to the final non-homogeneous symbol  $\mathbf{P}_0$  of Eq. (6.108). Since  $\dot{\mathbf{P}}_0 - \text{Id}_{\mathbb{C}^2} = O(r_0)$  in the order-zero symbolic seminorms,

$$\mathbf{P}_0 - \text{Id}_{\mathbb{C}^2} = (1 - \chi(\xi)) (\dot{\mathbf{P}}_0 - \text{Id}_{\mathbb{C}^2}) \in (\Gamma_{\geq 1}^0)^{2 \times 2}, \quad \|\mathbf{P}_0 - \text{Id}_{\mathbb{C}^2}\|_{\Gamma^0} \leq Cr_0.$$

Thus  $\mathbf{B}_0 = \text{Op}^{\text{BW}}(\mathbf{P}_0)$  is bounded on  $H^s$ , and after decreasing  $r_0$  it is invertible on  $H^s$  by the Neumann series.

Let  $\mathbf{P}_0^{[-1]}$  be the pointwise inverse of  $\mathbf{P}_0$ . Then

$$\mathbf{P}_0^{[-1]} - \text{Id}_{\mathbb{C}^2} \in (\Gamma_{\geq 1}^0)^{2 \times 2} \quad \text{and} \quad \mathbf{P}_0^{[-1]} = \dot{\mathbf{P}}_0^{-1} \quad \text{on } \text{supp}^{\mathbb{C}}(\chi).$$

On the region  $\chi = 0$ , Eq. (6.119) applies; on the region  $\chi = 1$ , both  $\mathbf{P}_0$  and  $\mathbf{P}_0^{[-1]}$  are the identity. Every discrepancy from the homogeneous calculation is supported in the fixed transition annulus; since the symbols are smooth and compactly supported in  $\xi$  there, it belongs to  $\Gamma^{-N}$  for every  $N$ . Consequently, after absorbing the transition contribution into the order-(-2) symbol,

$$\text{Op}^{\text{BW}}(\mathbf{P}_0^{[-1]}) \mathbf{B}_0 = \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}(\mathbf{A}_{[-2, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}, \quad \mathbf{A}_{[-2, \geq 1]} \in (\Gamma_{\geq 1}^{-2})^{2 \times 2},$$

and analogously on the other side. Since  $\mathbf{B}_0$  is already known to be invertible, its true inverse has the form

$$\mathbf{B}_0^{-1} = \text{Op}^{\text{BW}}(\mathbf{P}_0^{[-1]} + \mathbf{A}_{[-2, \geq 1]}^{\sharp}) + \mathbf{R}_{[\varrho, \geq 1]}, \quad \mathbf{A}_{[-2, \geq 1]}^{\sharp} \in (\Gamma_{\geq 1}^{-2})^{2 \times 2}, \quad (6.120)$$

after decreasing and relabelling  $\varrho$ .

Let now  $\mathbf{A} \in (\Gamma_{\geq 0}^m)^{2 \times 2}$ ,  $m \leq 3/2$ , be one of the symbols treated below. Combining Eq. (6.120) with Definition 4.22 and Proposition 4.24 and Lemma 4.25-i, we obtain

$$\mathbf{B}_0^{-1} \text{Op}^{\text{BW}}(\mathbf{A}) \mathbf{B}_0 = \text{Op}^{\text{BW}}(\mathbf{C}_{\mathbf{P}_0}[\mathbf{A}] + \mathbf{A}_{[m-2, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}, \quad (6.121)$$

where

$$\mathbf{C}_{\mathbf{P}_0}[\mathbf{A}] := \mathbf{P}_0^{[-1]} \#_{\varrho} \mathbf{A} \#_{\varrho} \mathbf{P}_0.$$

The additional term  $\mathbf{A}_{[m-2, \geq 1]}$  is solely due to replacing the true inverse by the symbolic parametrix. Since  $m \leq 3/2$ , it has order at most  $-1/2$ , hence it is bounded and will be absorbed into  $\mathbf{A}_{[0, \geq 1]}$  below.

**Part 2** (Conjugation of the equation). We write

$$U = \mathbf{B}_0 U_0.$$

Differentiating gives

$$\partial_t U_0 = \mathbf{B}_0^{-1} \partial_t U - \mathbf{B}_0^{-1} (\partial_t \mathbf{B}_0) U_0. \quad (6.122)$$

Set

$$\dot{\mathbf{C}}_{\text{ql}, 0} := J_{\text{d}} \left( \text{Id}_{\mathbb{C}^2} + \dot{\mathbf{E}}_{\text{ql}, 0}^{\mathbb{C}} \right).$$

By Eqs. (6.56), (6.87) and (6.88),

$$\dot{\mathbf{C}}_{\text{ql},0} = \begin{bmatrix} -\dot{\ell}_0 & -\dot{f}_0 \\ \dot{f}_0 & \dot{\ell}_0 \end{bmatrix},$$

where  $\dot{\ell}_0, \dot{f}_0$  are defined in Eq. (6.103). We rewrite Eq. (6.55) as

$$\partial_t U = \text{Op}^{\text{BW}} \left( \mathbf{L}_{\text{lo}} + \mathbf{L}_{\frac{3}{2}} + \mathbf{L}_1 + \mathbf{L}_{\frac{1}{2}} + \mathbf{L}_0 \right) U + \mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{C}} U,$$

with

$$\begin{aligned} \mathbf{L}_{\text{lo}} &:= \mathbf{A}_{\text{lo}}^{\mathbb{C}, \text{lin}}(\mathfrak{a}, \bar{\Omega} | \xi), \\ \mathbf{L}_{\frac{3}{2}} &:= (1 - \chi(\xi)) \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0}, \\ \mathbf{L}_1 &:= \text{i} \dot{\mathbf{D}}_1^{\mathbb{C}}, \\ \mathbf{L}_{\frac{1}{2}} &:= (1 - \chi(\xi)) \dot{\mathbf{A}}_0^{\mathbb{C}} |\xi|^{\frac{1}{2}}, \\ \mathbf{L}_0 &:= \mathbf{A}_{[0, \geq 1]}. \end{aligned}$$

The classes are

$$\mathbf{L}_{\text{lo}} \in (\Gamma_{\geq 0}^0)^{2 \times 2}, \quad \mathbf{L}_{\frac{3}{2}} \in \left( \Gamma_{\geq 0}^{\frac{3}{2}} \right)^{2 \times 2}, \quad \mathbf{L}_1 \in (\Gamma_{\geq 0}^1)^{2 \times 2}, \quad \mathbf{L}_{\frac{1}{2}} \in \left( \Gamma_{\geq 1}^{\frac{1}{2}} \right)^{2 \times 2}, \quad \mathbf{L}_0 \in (\Gamma_{\geq 1}^0)^{2 \times 2}.$$

Thus all five blocks are admissible inputs for Eq. (6.121). The factors  $1 - \chi(\xi)$  encode that the positive capillary diagonalization is performed only at high frequency.

**Part 3** (The order-3/2 block). We first work on the homogeneous high-frequency region  $\chi = 0$ , where  $\mathbf{P}_0 = \dot{\mathbf{P}}_0$  and  $\mathbf{L}_{\frac{3}{2}} = \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0}$ . A direct multiplication gives

$$\dot{\mathbf{P}}_0^{-1} \dot{\mathbf{C}}_{\text{ql},0} \dot{\mathbf{P}}_0 = \frac{1}{1 - \dot{\chi}_0^2} \begin{bmatrix} -\dot{\ell}_0(1 + \dot{\chi}_0^2) - 2\dot{f}_0\dot{\chi}_0 & -2\dot{\ell}_0\dot{\chi}_0 - \dot{f}_0(1 + \dot{\chi}_0^2) \\ 2\dot{\ell}_0\dot{\chi}_0 + \dot{f}_0(1 + \dot{\chi}_0^2) & \dot{\ell}_0(1 + \dot{\chi}_0^2) + 2\dot{f}_0\dot{\chi}_0 \end{bmatrix}. \quad (6.123)$$

The choice of  $\dot{\chi}_0$  in Eq. (6.110) gives

$$\begin{aligned} 2\dot{\ell}_0\dot{\chi}_0 + \dot{f}_0(1 + \dot{\chi}_0^2) &= 0, \\ \dot{\ell}_0(1 + \dot{\chi}_0^2) + 2\dot{f}_0\dot{\chi}_0 &= \dot{\theta}_0(1 - \dot{\chi}_0^2). \end{aligned}$$

Inserting these identities into Eq. (6.123) yields

$$\dot{\mathbf{P}}_0^{-1} \dot{\mathbf{C}}_{\text{ql},0} \dot{\mathbf{P}}_0 = \dot{\theta}_0 \mathbf{J}_{\text{d}}.$$

Set

$$\mathbf{D}_{\frac{3}{2}}^{\sharp} := \text{i} \omega_{\frac{3}{2}} \dot{\theta}_0 \mathbf{J}_{\text{d}}.$$

Equivalently,

$$\text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} = \dot{\mathbf{P}}_0 \mathbf{D}_{\frac{3}{2}}^{\sharp} \dot{\mathbf{P}}_0^{-1}, \quad (6.124)$$

and therefore

$$\dot{\mathbf{P}}_0^{-1} \left( \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right) \dot{\mathbf{P}}_0 = \mathbf{D}_{\frac{3}{2}}^{\sharp}. \quad (6.125)$$

We now verify the order-1/2 cancellation in the principal conjugation. On  $\chi = 0$ , the Bony–Weyl expansion gives

$$\begin{aligned} \mathbf{C}_{\dot{\mathbf{P}}_0} \left[ \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right] &= \dot{\mathbf{P}}_0^{-1} \left( \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right) \dot{\mathbf{P}}_0 \\ &\quad + \frac{1}{2\text{i}} \left( \left\{ \dot{\mathbf{P}}_0^{-1}, \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right\} \dot{\mathbf{P}}_0 + \left\{ \dot{\mathbf{P}}_0^{-1} \left( \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right), \dot{\mathbf{P}}_0 \right\} \right) + \mathbf{A}_{[-\frac{1}{2}, \geq 1]}. \end{aligned} \quad (6.126)$$

Differentiating Eq. (6.124) gives, for every  $y \in \{x_1, x_2, \xi_1, \xi_2\}$ ,

$$\dot{\mathbf{P}}_0^{-1} \partial_y \left( \text{i} \omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right) \dot{\mathbf{P}}_0 = \partial_y \mathbf{D}_{\frac{3}{2}}^{\sharp} + \left\| \dot{\mathbf{A}}_y, \mathbf{D}_{\frac{3}{2}}^{\sharp} \right\|.$$

By Eq. (6.118), every  $\dot{\mathbf{A}}_y$  is a scalar multiple of  $\mathbf{X}_{\text{od}}$ , whereas  $\mathbf{D}_{\frac{3}{2}}^\#$  and all its derivatives are scalar multiples of  $\mathbf{J}_d$ . Since

$$\mathbf{X}_{\text{od}}\mathbf{J}_d + \mathbf{J}_d\mathbf{X}_{\text{od}} = 0,$$

we have

$$\dot{\mathbf{A}}_y \partial_z \mathbf{D}_{\frac{3}{2}}^\# + \partial_z \mathbf{D}_{\frac{3}{2}}^\# \dot{\mathbf{A}}_y = 0, \quad \forall y, z \in \{x_1, x_2, \xi_1, \xi_2\}. \quad (6.127)$$

Furthermore,  $\partial_y \dot{\mathbf{P}}_0^{-1} = -\dot{\mathbf{A}}_y \dot{\mathbf{P}}_0^{-1}$ , so that

$$\partial_y \left( \dot{\mathbf{P}}_0^{-1} \left( i\omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right) \right) \partial_z \dot{\mathbf{P}}_0 = \left( \partial_y \mathbf{D}_{\frac{3}{2}}^\# - \mathbf{D}_{\frac{3}{2}}^\# \dot{\mathbf{A}}_y \right) \dot{\mathbf{A}}_z.$$

Consequently, the formally order-1/2 correction in Eq. (6.126) is

$$\begin{aligned} & \left\{ \dot{\mathbf{P}}_0^{-1}, i\omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right\} \dot{\mathbf{P}}_0 + \left\{ \dot{\mathbf{P}}_0^{-1} \left( i\omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right), \dot{\mathbf{P}}_0 \right\} \\ &= \sum_{j=1}^2 \left[ -\dot{\mathbf{A}}_{\xi_j} \left( \partial_{x_j} \mathbf{D}_{\frac{3}{2}}^\# + \left[ \dot{\mathbf{A}}_{x_j}, \mathbf{D}_{\frac{3}{2}}^\# \right] \right) + \dot{\mathbf{A}}_{x_j} \left( \partial_{\xi_j} \mathbf{D}_{\frac{3}{2}}^\# + \left[ \dot{\mathbf{A}}_{\xi_j}, \mathbf{D}_{\frac{3}{2}}^\# \right] \right) \right. \\ & \quad \left. + \left( \partial_{\xi_j} \mathbf{D}_{\frac{3}{2}}^\# - \mathbf{D}_{\frac{3}{2}}^\# \dot{\mathbf{A}}_{\xi_j} \right) \dot{\mathbf{A}}_{x_j} - \left( \partial_{x_j} \mathbf{D}_{\frac{3}{2}}^\# - \mathbf{D}_{\frac{3}{2}}^\# \dot{\mathbf{A}}_{x_j} \right) \dot{\mathbf{A}}_{\xi_j} \right]. \end{aligned}$$

The terms linear in  $\dot{\mathbf{A}}_y$  vanish by Eq. (6.127). The quadratic terms vanish because all  $\dot{\mathbf{A}}_y$  commute with one another and anti-commute with  $\mathbf{D}_{\frac{3}{2}}^\#$ . Hence

$$\left\{ \dot{\mathbf{P}}_0^{-1}, i\omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right\} \dot{\mathbf{P}}_0 + \left\{ \dot{\mathbf{P}}_0^{-1} \left( i\omega_{\frac{3}{2}} \dot{\mathbf{C}}_{\text{ql},0} \right), \dot{\mathbf{P}}_0 \right\} = 0. \quad (6.128)$$

It remains to pass to the non-homogeneous symbols. On  $\chi = 0$  the preceding calculation applies verbatim. On  $\chi = 1$ , one has  $\mathbf{P}_0 = \text{Id}_{\mathbb{C}^2}$  and  $\mathbf{L}_{\frac{3}{2}} = 0$ . Every discrepancy in the transition annulus, including every term in which a  $\xi$ -derivative hits the cut-off, is smooth and compactly supported in  $\xi$ , hence belongs to  $\Gamma^{-N}$  for every  $N$ . Combining Eqs. (6.121), (6.125), (6.126) and (6.128), we obtain

$$\mathbf{B}_0^{-1} \text{Op}^{\text{BW}} \left( \mathbf{L}_{\frac{3}{2}} \right) \mathbf{B}_0 = \text{Op}^{\text{BW}} \left( i\mathbf{D}_{\frac{3}{2}}^{\mathbb{C}} + \mathbf{A}_{[0,\geq 1]} \right) + \mathbf{R}_{[\ell,\geq 1]}. \quad (6.129)$$

In particular, the order-3/2 block produces no homogeneous order-1/2 symbol.

**Part 4** (The flat low-frequency linear block). We treat the flat multiplier

$$\mathbf{L}_{\text{lo}} = \mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi).$$

By Eq. (6.108),

$$\mathbf{P}_0 - \text{Id}_{\mathbb{C}^2} = (1 - \chi(\xi)) (\dot{\mathbf{P}}_0 - \text{Id}_{\mathbb{C}^2}).$$

Since  $\dot{\mathbf{P}}_0 - \text{Id}_{\mathbb{C}^2} \in \Gamma_{\geq 1}^0$  vanishes at least linearly in  $U$ , and since  $\mathbf{L}_{\text{lo}}$  is a constant-coefficient order-zero symbol, the pointwise conjugation gives

$$\mathbf{P}_0^{[-1]} \mathbf{L}_{\text{lo}} \mathbf{P}_0 = \mathbf{L}_{\text{lo}} + \mathbf{A}_{[0,\geq 1]}.$$

The Bony–Weyl correction terms contain either an  $x$ -derivative of  $\mathbf{P}_0$  or an  $x$ -derivative of  $\mathbf{P}_0^{[-1]}$ , hence again at least one nonlinear factor. Since all symbols involved have order 0, these corrections are bounded. Therefore

$$\mathbf{B}_0^{-1} \text{Op}^{\text{BW}}(\mathbf{L}_{\text{lo}}) \mathbf{B}_0 = \text{Op}^{\text{BW}} \left( \mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} + \mathbf{A}_{[0,\geq 1]} \right) + \mathbf{R}_{[\ell,\geq 1]}. \quad (6.130)$$

**Part 5** (The scalar transport block). We apply Eq. (6.121) to  $\mathbf{A} = \mathbf{L}_1$ . By Eq. (6.104),

$$\mathbf{L}_1 = i\dot{\mathbf{D}}_1^{\mathbb{C}} = i \left( \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi \right) \text{Id}_{\mathbb{C}^2}.$$

It is scalar in the matrix indices, so its pointwise conjugation is exactly unchanged on the whole frequency space:

$$\mathbf{P}_0^{[-1]} \mathbf{L}_1 \mathbf{P}_0 = \mathbf{L}_1.$$

Every non-pointwise term in  $\mathbf{C}_{\mathbf{P}_0}[\mathbf{L}_1]$  has order at most 0: the first Bony–Weyl correction contains one  $\xi$ -derivative of an order-one symbol, and all higher corrections are lower order. Terms in which a  $\xi$ -derivative hits the cut-off are supported in the fixed transition annulus and are smoothing. Therefore

$$\mathbf{B}_0^{-1} \text{Op}^{\text{BW}}(\mathbf{L}_1) \mathbf{B}_0 = \text{Op}^{\text{BW}}(\mathbf{i} \dot{\mathbf{D}}_1^{\mathbb{C}} + \mathbf{A}_{[0, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.131)$$

Thus the scalar transport block is unchanged at every positive order.

**Part 6** (The explicit order-1/2 block). By Eq. (6.128), the order-3/2 block produces no homogeneous order-1/2 correction. Hence the only surviving source at order 1/2 is

$$\mathbf{L}_{\frac{1}{2}} = (1 - \chi(\xi)) \dot{\mathbf{A}}_0^{\mathbb{C}} |\xi|^{\frac{1}{2}}.$$

On the high-frequency region  $\chi = 0$ , one has  $\mathbf{P}_0 = \dot{\mathbf{P}}_0$ . Since  $\dot{\mathbf{P}}_0^{\pm 1} \in (\Gamma_{\geq 0}^0)^{2 \times 2}$ , the first non-pointwise Bony–Weyl correction in the conjugation of  $\mathbf{L}_{\frac{1}{2}}$  has order  $-1/2$ . Every derivative of the cut-off is supported in the fixed transition annulus and is smoothing. Therefore

$$\mathbf{C}_{\mathbf{P}_0}[\mathbf{L}_{\frac{1}{2}}] = (1 - \chi(\xi)) \dot{\mathbf{P}}_0^{-1} \dot{\mathbf{A}}_0^{\mathbb{C}} \dot{\mathbf{P}}_0 |\xi|^{\frac{1}{2}} + \mathbf{A}_{[0, \geq 1]}.$$

By Eq. (6.59),

$$\dot{\mathbf{A}}_0^{\mathbb{C}} = \mathbf{i} \begin{bmatrix} \dot{\alpha}_0^{\mathbb{C}} & \dot{\beta}_0^{\mathbb{C}} \\ -\dot{\beta}_0^{\mathbb{C}} & -\dot{\alpha}_0^{\mathbb{C}} \end{bmatrix}.$$

Using Eqs. (6.109) and (6.116), we obtain

$$\begin{aligned} \dot{\mathbf{P}}_0^{-1} \dot{\mathbf{A}}_0^{\mathbb{C}} \dot{\mathbf{P}}_0 &= \frac{\mathbf{i}}{1 - \chi_0^2} \begin{bmatrix} 1 & -\chi_0 \\ -\chi_0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha}_0^{\mathbb{C}} & \dot{\beta}_0^{\mathbb{C}} \\ -\dot{\beta}_0^{\mathbb{C}} & -\dot{\alpha}_0^{\mathbb{C}} \end{bmatrix} \begin{bmatrix} 1 & \chi_0 \\ \chi_0 & 1 \end{bmatrix} \\ &= \mathbf{i} \begin{bmatrix} \dot{\alpha}_0^{\mathbb{C}, \text{pr}} & \dot{\beta}_0^{\mathbb{C}, \text{pr}} \\ -\dot{\beta}_0^{\mathbb{C}, \text{pr}} & -\dot{\alpha}_0^{\mathbb{C}, \text{pr}} \end{bmatrix}. \end{aligned}$$

where  $\dot{\alpha}_0^{\mathbb{C}, \text{pr}}$  and  $\dot{\beta}_0^{\mathbb{C}, \text{pr}}$  are defined in Eqs. (6.112) and (6.113). Thus

$$\mathbf{B}_0^{-1} \text{Op}^{\text{BW}}(\mathbf{L}_{\frac{1}{2}}) \mathbf{B}_0 = \text{Op}^{\text{BW}}\left((1 - \chi(\xi)) \dot{\mathbf{A}}_0^{\mathbb{C}, \text{pr}} |\xi|^{\frac{1}{2}} + \mathbf{A}_{[0, \geq 1]}\right) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.132)$$

This is the unique homogeneous order-1/2 block in the principal-diagonalized system.

**Part 7** (Bounded symbols, time derivative, and smoothing remainders). For  $\mathbf{L}_0 = \mathbf{A}_{[0, \geq 1]}^{\mathbb{C}}$ , Eq. (6.121) gives

$$\mathbf{B}_0^{-1} \text{Op}^{\text{BW}}(\mathbf{L}_0) \mathbf{B}_0 = \text{Op}^{\text{BW}}(\mathbf{A}_{[0, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.133)$$

Indeed the three factors are order-zero operators and all symbolic corrections are of order at most 0.

We now treat the time derivative in Eq. (6.122). Since  $\mathbf{B}_0 = \text{Op}^{\text{BW}}(\mathbf{P}_0)$ , the symbolic composition formula gives

$$\mathbf{B}_0^{-1} (\partial_t \mathbf{B}_0) = \text{Op}^{\text{BW}}\left(\mathbf{P}_0^{[-1]} \partial_t \mathbf{P}_0 + \mathbf{A}_{[-1, \geq 1]}\right) + \mathbf{R}_{[\varrho, \geq 1]}.$$

The cut-off is time-independent and  $\partial_t \mathbf{P}_0 = (1 - \chi) \partial_t \dot{\mathbf{P}}_0$ . Hence  $\partial_t \mathbf{P}_0$  is of order zero in  $\xi$ : the time derivative acts only on the coefficients depending on  $U$  and does not alter their high-frequency homogeneity. Its tame bounds follow, after increasing  $s_0$ , from the equation for  $U$ , from Eqs. (6.87), (6.88), (6.103) and (6.110), and from Lemma 4.25-iii. Thus

$$\mathbf{B}_0^{-1} (\partial_t \mathbf{B}_0) = \text{Op}^{\text{BW}}(\mathbf{A}_{[0, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}, \quad (6.134)$$

and this contribution is absorbed into  $\mathbf{A}_{[0, \geq 1]}^{\mathbb{C}, \text{pr}}$ .

Finally, by Lemma 4.25-i,

$$\mathbf{B}_0^{-1} \mathbf{R}_{[\varrho, \geq 1]}^{\mathbb{C}} \mathbf{B}_0 \in (\mathcal{R}^{\varrho})^{2 \times 2}, \quad (6.135)$$

after decreasing and relabelling  $\varrho$ .

Collecting Eqs. (6.129) to (6.135) inside Eq. (6.122), and using Eq. (6.128), the only homogeneous order-1/2 contribution is the pointwise-conjugated block defined in Eq. (6.111). Absorbing all bounded symbols into  $A_{[0,\geq 1]}^{\mathbb{C},\text{pr}}$  and all smoothing contributions into  $R_{[\ell,\geq 1]}^{\mathbb{C},\text{pr}}$ , we obtain

$$\begin{aligned}\partial_t U_0 &= \text{Op}^{\text{BW}}\left(A_{\text{lo}}^{\mathbb{C},\text{lin}}\right) U_0 + \text{Op}^{\text{BW}}\left(iD_{\frac{3}{2}}^{\mathbb{C}}\right) U_0 + \text{Op}^{\text{BW}}\left(iD_1^{\mathbb{C}}\right) U_0 \\ &\quad + \text{Op}^{\text{BW}}\left((1-\chi(\xi)) \dot{A}_0^{\mathbb{C},\text{pr}} |\xi|^{\frac{1}{2}}\right) U_0 + \text{Op}^{\text{BW}}\left(A_{[0,\geq 1]}^{\mathbb{C},\text{pr}}\right) U_0 + R_{[\ell,\geq 1]}^{\mathbb{C},\text{pr}} U_0.\end{aligned}$$

Restoring the common symbol arguments

$$(\mathfrak{a}, \bar{\Omega}|U; x, \xi),$$

and the multiplier arguments  $(\mathfrak{a}, \bar{\Omega}|\xi)$ , gives precisely Eq. (6.107).  $\square$

### 6.3.2 Diagonalization at subprincipal order

Here we perform the diagonalization of the lower positive order terms. The result reads as follows.

**Proposition 6.9** (Subprincipal diagonalization). *Let the assumptions of Proposition 6.7 hold, and let  $U_0$  solve Eq. (6.107). There exists a bounded and invertible operator*

$$B_{-1}(\mathfrak{a}, \bar{\Omega}|U) \in \mathcal{L}(H^s) \quad \forall s \geq s_0,$$

such that, defining

$$W := B_{-1}(\mathfrak{a}, \bar{\Omega}|U)^{-1} U_0, \quad (6.136)$$

the unknown  $W$  solves

$$\begin{aligned}\partial_t W &= \text{Op}^{\text{BW}}\left(A_{\text{lo}}^{\mathbb{C},\text{lin}}(\mathfrak{a}, \bar{\Omega}|\xi)\right) W \\ &\quad + i \text{Op}^{\text{BW}}\left(D_{\frac{3}{2}}^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) + \dot{D}_1^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) + \dot{D}_{\frac{1}{2}}^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) + A_{[0,\geq 1]}^{\mathbb{C},\text{d}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi)\right) W \\ &\quad + R_{[\ell,\geq 1]}^{\mathbb{C},\text{d}}(\mathfrak{a}, \bar{\Omega}|U) W.\end{aligned} \quad (6.137)$$

The objects entering Eqs. (6.136) and (6.137) are defined as follows.

- $B_{-1}(\mathfrak{a}, \bar{\Omega}|U)$  is the order-zero perturbation of the identity

$$B_{-1}(\mathfrak{a}, \bar{\Omega}|U) := \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}\left(S_{-1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi)\right).$$

We use the fixed off-diagonal matrix  $X_{\text{od}}$  introduced in Eq. (6.117). The final non-homogeneous generator is

$$S_{-1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := (1 - \chi(\xi)) \dot{S}_{-1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}, \quad (6.138)$$

where the high-frequency homogeneous generator is

$$\dot{S}_{-1}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := \frac{\dot{\beta}_0^{\mathbb{C},\text{pr}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) |\xi|^{\frac{1}{2}}}{2\dot{\omega}_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \dot{\theta}_0(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi})} X_{\text{od}}. \quad (6.139)$$

Here  $X_{\text{od}}$  is the off-diagonal matrix defined in Eq. (6.117),  $\dot{\omega}_{\frac{3}{2}}$  is defined in Eq. (6.43),  $\dot{\theta}_0$  is defined in Eq. (6.103), and  $\dot{\beta}_0^{\mathbb{C},\text{pr}}$  is defined in Eq. (6.113);

- $A_{\text{lo}}^{\mathbb{C},\text{lin}}$  is the same flat low-frequency linear multiplier from Eq. (6.51);
- the diagonal symbols

$$D_{\frac{3}{2}}^{\mathbb{C}}, \quad \dot{D}_1^{\mathbb{C}}, \quad \dot{D}_{\frac{1}{2}}^{\mathbb{C}}$$

are those defined in Eqs. (6.102), (6.104) and (6.105). They belong respectively to

$$\left(\Gamma_{\geq 0}^{\frac{3}{2}}\right)^{2 \times 2}, \quad (\Gamma_{\geq 0}^1)^{2 \times 2}, \quad \left(\Gamma_{\geq 1}^{\frac{1}{2}}\right)^{2 \times 2},$$

and are real-valued diagonal matrix symbols for real  $U$ ;

- $A_{[0,\geq 1]}^{\mathbb{C},d}(\alpha, \bar{\Omega}|U; x, \xi) \in (\Gamma_{\geq 1}^0)^{2 \times 2}$  is a generic bounded matrix symbol in the sense of Notation 5.1;
- $R_{[\varrho, \geq 1]}^{\mathbb{C},d}(\alpha, \bar{\Omega}|U) \in (\mathcal{R}_{\geq 1}^\varrho)^{2 \times 2}$  is a generic smoothing remainder in the sense of Notation 5.1.

*Proof.* Throughout the proof all symbols are evaluated at

$$(\alpha, \bar{\Omega}|U; x, \xi),$$

unless explicitly stated otherwise; Fourier multipliers are evaluated at  $(\alpha, \bar{\Omega}|\xi)$ . We use the convention of Notation 5.1: the symbols  $A_{[m,\geq 1]}$  and the remainders  $R_{[\varrho,\geq 1]}$  are generic and may change from line to line. The symbolic calculus is used componentwise, through Definition 4.22 and Proposition 4.24, and the smoothing remainder calculus is used through Lemma 4.25-i and Lemma 4.25-ii.

**Part 1** (The subprincipal diagonalizer and its inverse). We first check that the symbol defining the change of variables is of order  $-1$  on the high-frequency region. By Eq. (6.113),

$$\dot{\beta}_0^{\mathbb{C},\text{pr}} \in \Gamma_{\geq 1}^0.$$

Moreover, by Eq. (6.103), after increasing  $s_0$  and shrinking  $r_0$ ,

$$\dot{\theta}_0 - 1 \in \Gamma_{\geq 1}^0, \quad \dot{\theta}_0 \geq \frac{3}{4}, \quad \dot{\theta}_0^{-1} \in \Gamma_{\geq 0}^0.$$

Since

$$\dot{\omega}_{\frac{3}{2}}^{-1} |\xi|^{\frac{1}{2}} \in \tilde{\Gamma}^{-1}$$

by Eq. (6.43), the product rule for  $\Gamma_{\geq 0}$ -symbols and Lemma 4.25-iii give

$$\dot{s}_{-1} := \frac{\dot{\beta}_0^{\mathbb{C},\text{pr}} |\xi|^{\frac{1}{2}}}{2\dot{\omega}_{\frac{3}{2}} \dot{\theta}_0} \in \Gamma_{\geq 1}^{-1}.$$

The coefficient  $\dot{s}_{-1}$  is real-valued for real  $U$ . Therefore

$$\dot{\mathbf{S}}_{-1} = \dot{s}_{-1} \mathbf{X}_{\text{od}} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}.$$

Multiplying by the fixed high-frequency cut-off gives

$$\mathbf{S}_{-1} = (1 - \chi(\xi)) \dot{\mathbf{S}}_{-1} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}.$$

In particular,  $\mathbf{S}_{-1}$  is a non-homogeneous order- $(-1)$  symbol: it vanishes at low frequency and coincides with the homogeneous generator  $\dot{\mathbf{S}}_{-1}$  where  $1 - \chi(\xi) = 1$ .

By the action estimate for paradifferential operators,  $\text{Op}^{\text{BW}}(\mathbf{S}_{-1})$  is bounded on  $H^s$ , for  $s \geq s_0$ . Since  $\mathbf{S}_{-1}$  vanishes at least linearly in  $U$ , its low Sobolev seminorm is  $O(r_0)$ . By Theorem 4.15, after decreasing  $r_0$ , we get

$$\|\text{Op}^{\text{BW}}(\mathbf{S}_{-1})\|_{\mathcal{L}(H^s)} < 1, \quad s \geq s_0.$$

Hence

$$\mathbf{B}_{-1} = \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}(\mathbf{S}_{-1}) \tag{6.140}$$

is bounded and invertible on  $H^s$  by the Neumann series.

We next record the symbolic expansion of the inverse. Since we have that  $\mathbf{S}_{-1} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}$ , then by virtue of Definition 4.22 and Proposition 4.24, we infer

$$\begin{aligned} (\text{Id}_{\mathbb{C}^2} - \text{Op}^{\text{BW}}(\mathbf{S}_{-1})) (\text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}(\mathbf{S}_{-1})) &= \text{Id}_{\mathbb{C}^2} - \text{Op}^{\text{BW}}(\mathbf{S}_{-1} \# \mathbf{S}_{-1}) + \mathbf{R}_{[\varrho, \geq 1]} \\ &= \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}(\mathbf{A}_{[-2, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}. \end{aligned} \tag{6.141}$$

The same identity holds with the two factors exchanged. Combining the invertibility statement Eq. (6.140) with the parametrix identity Eq. (6.141), and using Lemma 4.25-i, we may write the true inverse as

$$\mathbf{B}_{-1}^{-1} = \text{Id}_{\mathbb{C}^2} - \text{Op}^{\text{BW}}(\mathbf{S}_{-1}) + \text{Op}^{\text{BW}}(\mathbf{A}_{[-2, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}, \tag{6.142}$$

after decreasing and relabelling  $\varrho$ .

**Part 2** (Symbolic conjugation formula). Let  $A \in (\Gamma_{\geq 0}^m)^{2 \times 2}$ ,  $m \leq 3/2$ , be any of the symbols to be conjugated below. By Eq. (6.142), Definition 4.22 and Proposition 4.24, and Lemma 4.25-i,

$$\mathbf{B}_{-1}^{-1} \text{Op}^{\text{BW}}(A) \mathbf{B}_{-1} = \text{Op}^{\text{BW}}(A + A \#_{\varrho} \mathbf{S}_{-1} - \mathbf{S}_{-1} \#_{\varrho} A) + \text{Op}^{\text{BW}}(A_{[m-2, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.143)$$

The pointwise products  $A \mathbf{S}_{-1}$  and  $\mathbf{S}_{-1} A$  have order  $m - 1$ . The first non-pointwise Bony–Weyl correction contains one  $\xi$ -derivative falling either on  $A$  or on  $\mathbf{S}_{-1}$ , and therefore has order  $m - 2$ . Consequently, at positive order,

$$\mathbf{B}_{-1}^{-1} \text{Op}^{\text{BW}}(A) \mathbf{B}_{-1} = \text{Op}^{\text{BW}}(A + A \mathbf{S}_{-1} - \mathbf{S}_{-1} A + A_{[m-2, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.144)$$

For  $m = 3/2$  the discarded terms have order at most  $-1/2$ , and for  $m \leq 1$  they have order at most  $-1$ . All such contributions are bounded in the non-homogeneous scale and are absorbed into a generic  $A_{[0, \geq 1]}$ . If a  $\xi$ -derivative hits the cut-off  $1 - \chi(\xi)$ , the resulting contribution is compactly supported in frequency and is again absorbed into  $A_{[0, \geq 1]}$ .

**Part 3** (Conjugation of the equation). Set

$$U_0 = \mathbf{B}_{-1} W.$$

Differentiating this identity gives

$$\partial_t W = \mathbf{B}_{-1}^{-1} \partial_t U_0 - \mathbf{B}_{-1}^{-1} (\partial_t \mathbf{B}_{-1}) W. \quad (6.145)$$

We write Eq. (6.107) as

$$\partial_t U_0 = \text{Op}^{\text{BW}}(L_{\text{lo}} + L_{\frac{3}{2}} + L_1 + L_{\frac{1}{2}} + L_0) U_0 + \mathbf{R}_{[\varrho, \geq 1]}^{\text{C, pr}} U_0,$$

where

$$\begin{aligned} L_{\text{lo}} &:= A_{\text{lo}}^{\text{C, lin}}(\mathfrak{a}, \bar{\Omega}|\xi), \\ L_{\frac{3}{2}} &:= iD_{\frac{3}{2}}^{\text{C}}, \\ L_1 &:= iD_1^{\text{C}}, \\ L_{\frac{1}{2}} &:= (1 - \chi(\xi)) \dot{A}_0^{\text{C, pr}} |\xi|^{\frac{1}{2}}, \\ L_0 &:= A_{[0, \geq 1]}^{\text{C, pr}}. \end{aligned} \quad (6.146)$$

Moreover  $L_{\text{lo}} \in (\Gamma_{\geq 0}^0)^{2 \times 2}$  is constant-coefficient and does not vanish at  $U = 0$ . It will be kept explicitly through the conjugation. The bounded nonlinear defects generated by its commutators with the diagonalizer are absorbed into  $A_{[0, \geq 1]}$ . The transport symbol  $\dot{D}_1^{\text{C}}$  is the scalar diagonal symbol defined in Eq. (6.104). The factors  $1 - \chi(\xi)$  in  $D_{\frac{3}{2}}^{\text{C}}$ ,  $L_{\frac{1}{2}}$ , and  $\dot{D}_{\frac{1}{2}}^{\text{C}}$  encode that the subprincipal diagonalization is performed only at high frequency.

**Part 4** (The flat low-frequency linear block). The subprincipal diagonalizer has the form

$$\mathbf{B}_{-1} = \text{Id}_{\mathbb{C}^2} + \text{Op}^{\text{BW}}(\mathbf{S}_{-1}), \quad \mathbf{S}_{-1} = (1 - \chi(\xi)) \dot{\mathbf{S}}_{-1}.$$

In particular  $\mathbf{S}_{-1}$  vanishes at least linearly in  $U$ . Since  $L_{\text{lo}} = A_{\text{lo}}^{\text{C, lin}}$  is a constant-coefficient order-zero multiplier, the conjugation formula gives

$$L_{\text{lo}} + L_{\text{lo}} \mathbf{S}_{-1} - \mathbf{S}_{-1} L_{\text{lo}} + A_{[-1, \geq 1]}.$$

The commutator term has order  $-1$  and vanishes at least linearly in  $U$ . It is therefore included in the bounded nonlinear symbol  $A_{[0, \geq 1]}$ . Thus

$$\mathbf{B}_{-1}^{-1} \text{Op}^{\text{BW}}(L_{\text{lo}}) \mathbf{B}_{-1} = \text{Op}^{\text{BW}}(A_{\text{lo}}^{\text{C, lin}} + A_{[0, \geq 1]}) + \mathbf{R}_{[\varrho, \geq 1]}. \quad (6.147)$$

**Part 5** (The order-3/2 block and the subprincipal correction). We compute the order-1/2 term produced by conjugating the already diagonal order-3/2 block with the subprincipal diagonalizer. This is the term designed to cancel the off-diagonal part of the explicit order-1/2 block. First,

$$J_d X_{od} - X_{od} J_d = 2J.$$

By Eqs. (6.138), (6.139) and (6.146),

$$L_{\frac{3}{2}} = i(1 - \chi(\xi)) \omega_{\frac{3}{2}} \dot{\theta}_0 J_d, \quad S_{-1} = (1 - \chi(\xi)) \dot{s}_{-1} X_{od},$$

where

$$\dot{s}_{-1} = \frac{\dot{\beta}_0^{C,pr} |\xi|^{\frac{1}{2}}}{2\omega_{\frac{3}{2}} \dot{\theta}_0}.$$

The pointwise commutator in Eq. (6.144) is therefore

$$\begin{aligned} L_{\frac{3}{2}} S_{-1} - S_{-1} L_{\frac{3}{2}} &= i(1 - \chi(\xi))^2 \omega_{\frac{3}{2}} \dot{\theta}_0 \dot{s}_{-1} (J_d X_{od} - X_{od} J_d) \\ &= i(1 - \chi(\xi))^2 \frac{\omega_{\frac{3}{2}}}{\dot{\omega}_{\frac{3}{2}}} \dot{\beta}_0^{C,pr} |\xi|^{\frac{1}{2}} J. \end{aligned}$$

By Eq. (6.45),  $\omega_{\frac{3}{2}}/\dot{\omega}_{\frac{3}{2}} - 1 \in \tilde{\Gamma}^{-1}$ . Hence its contribution after multiplication by  $\dot{\beta}_0^{C,pr} |\xi|^{\frac{1}{2}}$  has order  $-1/2$ . Moreover,  $(1 - \chi)^2 - (1 - \chi)$  is supported in the fixed transition annulus and is smoothing. Therefore

$$L_{\frac{3}{2}} S_{-1} - S_{-1} L_{\frac{3}{2}} = i(1 - \chi(\xi)) \dot{\beta}_0^{C,pr} |\xi|^{\frac{1}{2}} J + A_{[-\frac{1}{2}, \geq 1]}. \quad (6.148)$$

The first non-pointwise Bony–Weyl term in  $L_{\frac{3}{2}} \#_{\varrho} S_{-1} - S_{-1} \#_{\varrho} L_{\frac{3}{2}}$  also has order  $3/2 - 1 - 1 = -1/2$ , and all subsequent terms are lower order. Combining Eqs. (6.144) and (6.148), we obtain

$$B_{-1}^{-1} \text{Op}^{\text{BW}}(L_{\frac{3}{2}}) B_{-1} = \text{Op}^{\text{BW}}(iD_{\frac{3}{2}}^C + i(1 - \chi(\xi)) \dot{\beta}_0^{C,pr} |\xi|^{\frac{1}{2}} J + A_{[0, \geq 1]}) + R_{[\varrho, \geq 1]}. \quad (6.149)$$

Here  $A_{[0, \geq 1]}$  absorbs all order  $-1/2$  and lower terms.

**Part 6** (The explicit order-1/2 block). We now treat the explicit order-1/2 block in the principal-diagonalized system. By Eqs. (6.105) and (6.111),

$$L_{\frac{1}{2}} = (1 - \chi(\xi)) \dot{A}_0^{C,pr} |\xi|^{\frac{1}{2}} = i\dot{D}_{\frac{1}{2}}^C - i(1 - \chi(\xi)) \dot{\beta}_0^{C,pr} |\xi|^{\frac{1}{2}} J.$$

Here the coefficient  $\dot{\beta}_0^{C,pr}$  comes from the pointwise conjugation of the explicit order-1/2 block. The cancellation in Eq. (6.128) shows that the order-3/2 block adds no homogeneous order-1/2 contribution.

Since  $L_{\frac{1}{2}} \in (\Gamma_{\geq 1}^{1/2})^{2 \times 2}$  and  $S_{-1} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2}$ , the pointwise commutator

$$L_{\frac{1}{2}} S_{-1} - S_{-1} L_{\frac{1}{2}}$$

has order  $-1/2$ , and the first non-pointwise Bony–Weyl corrections have even lower order. Therefore the subprincipal diagonalizer does not modify the order-1/2 part of  $L_{\frac{1}{2}}$ . Using Eq. (6.144), we get

$$B_{-1}^{-1} \text{Op}^{\text{BW}}(L_{\frac{1}{2}}) B_{-1} = \text{Op}^{\text{BW}}(i\dot{D}_{\frac{1}{2}}^C - i(1 - \chi(\xi)) \dot{\beta}_0^{C,pr} |\xi|^{\frac{1}{2}} J + A_{[0, \geq 1]}) + R_{[\varrho, \geq 1]}. \quad (6.150)$$

The off-diagonal term in Eq. (6.150) cancels exactly with the off-diagonal order-1/2 term generated by the conjugation of  $L_{\frac{3}{2}}$ , computed in Eq. (6.149). Therefore

$$B_{-1}^{-1} \text{Op}^{\text{BW}}(L_{\frac{3}{2}} + L_{\frac{1}{2}}) B_{-1} = \text{Op}^{\text{BW}}(iD_{\frac{3}{2}}^C + i\dot{D}_{\frac{1}{2}}^C + A_{[0, \geq 1]}) + R_{[\varrho, \geq 1]}. \quad (6.151)$$

**Part 7** (The scalar transport block). We now conjugate  $L_1 = i\dot{D}_1^C$ . By Eq. (6.104),

$$\dot{D}_1^C = (\mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}| \eta(U), \tilde{\psi}(U); x) \cdot \xi) \text{Id}_{\mathbb{C}^2}.$$

Thus the pointwise commutator with  $S_{-1}$  vanishes. The first non-pointwise Bony–Weyl commutator has order  $1 + (-1) - 1 = -1$ , and the defects produced by the  $A_{[-2, \geq 1]}$ -part of the inverse have order  $1 - 2 = -1$ . Hence

$$B_{-1}^{-1} \text{Op}^{\text{BW}}(L_1) B_{-1} = \text{Op}^{\text{BW}}(i\dot{D}_1^C + A_{[0, \geq 1]}) + R_{[\varrho, \geq 1]}. \quad (6.152)$$

Thus the subprincipal diagonalizer does not change the scalar transport block at positive order.

**Part 8** (Bounded terms, time derivative, and smoothing remainders). Since  $L_0 = A_{[0, \geq 1]}^{\text{C, pr}} \in (\Gamma_{\geq 1}^0)^{2 \times 2}$ , Eq. (6.143) gives

$$B_{-1}^{-1} \text{Op}^{\text{BW}}(L_0) B_{-1} = \text{Op}^{\text{BW}}(A_{[0, \geq 1]}) + R_{[\varrho, \geq 1]}. \quad (6.153)$$

Indeed, all additional products with  $S_{-1}$  have order at most  $-1$ , hence are bounded.

We next treat the time derivative in Eq. (6.145). Since the Bony–Weyl cut-off is time-independent,

$$\partial_t B_{-1} = \text{Op}^{\text{BW}}(\partial_t S_{-1}).$$

The factor  $1 - \chi(\xi)$  is independent of time, and differentiating the high-frequency homogeneous coefficient  $\dot{S}_{-1}$  along the solution does not change the  $\xi$ -order. After replacing  $\partial_t U$  by the vector field in Eq. (6.55), and increasing  $s_0$  if necessary to absorb the fixed loss of derivatives in the tame bounds, we may regard

$$\partial_t S_{-1} \in (\Gamma_{\geq 1}^{-1})^{2 \times 2} \subset (\Gamma_{\geq 1}^0)^{2 \times 2}$$

as a bounded matrix symbol depending on  $U$  and on the fixed parameters. Therefore

$$B_{-1}^{-1} (\partial_t B_{-1}) = \text{Op}^{\text{BW}}(A_{[0, \geq 1]}) + R_{[\varrho, \geq 1]}. \quad (6.154)$$

Finally, by Lemma 4.25-i and Lemma 4.25-ii,

$$B_{-1}^{-1} R_{[\varrho, \geq 1]}^{\text{C, pr}} B_{-1} \in (\mathcal{R}_{\geq 1}^{\varrho})^{2 \times 2}, \quad (6.155)$$

after decreasing and relabeling  $\varrho$ . Indeed, the identity parts of  $B_{-1}^{\pm 1}$  preserve the smoothing order, whereas every factor  $\text{Op}^{\text{BW}}(S_{-1})$  improves it, because  $S_{-1}$  has order  $-1$ .

**Part 9** (Conclusion). Substituting Eqs. (6.147) and (6.151) to (6.155) into Eq. (6.145), we obtain

$$\partial_t W = \text{Op}^{\text{BW}}(A_{\text{lo}}^{\text{C, lin}}) W + \text{Op}^{\text{BW}}\left(iD_{\frac{3}{2}}^C + i\dot{D}_1^C + i\dot{D}_{\frac{1}{2}}^C + A_{[0, \geq 1]}\right) W + R_{[\varrho, \geq 1]}^{\text{C, d}} W.$$

Since  $A_{[0, \geq 1]} \in (\Gamma_{\geq 1}^0)^{2 \times 2}$ , the symbol  $-iA_{[0, \geq 1]}$  is again a generic bounded matrix symbol. Renaming it as  $A_{[0, \geq 1]}^{\text{C, d}}$ , and renaming the smoothing remainder as  $R_{[\varrho, \geq 1]}^{\text{C, d}}$ , gives exactly Eq. (6.137). □

### 6.3.3 Proof of Proposition 6.7

**Part 1** (Construction of the complete diagonalizer). Let  $B_0$  and  $B_{-1}$  be the principal and subprincipal diagonalizing operators constructed in Propositions 6.8 and 6.9, respectively, and define

$$B(\alpha, \bar{\Omega}|U) := B_0(\alpha, \bar{\Omega}|U) B_{-1}(\alpha, \bar{\Omega}|U). \quad (6.156)$$

Both factors in Eq. (6.156) are bounded and invertible on  $H^s$ , for every  $s \geq s_0$ . Hence  $B$  has the same properties. Moreover, by Eqs. (6.106) and (6.136),

$$W = B_{-1}^{-1} U_0 = B_{-1}^{-1} B_0^{-1} U = B^{-1} U,$$

which proves Eq. (6.99).

**Part 2** (Identification of the positive-order symbol). By Eq. (6.137), the variable  $W$  satisfies

$$\begin{aligned} \partial_t W &= \text{Op}^{\text{BW}} \left( A_{10}^{\text{C,lin}}(\mathfrak{a}, \bar{\Omega}|\xi) \right) W \\ &+ i \text{Op}^{\text{BW}} \left( D_{\frac{3}{2}}^{\text{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) + \dot{D}_1^{\text{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) + \dot{D}_{\frac{1}{2}}^{\text{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) + A_{[0, \geq 1]}^{\text{C,d}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) \right) W \\ &+ R_{[\varrho, \geq 1]}^{\text{C,d}}(\mathfrak{a}, \bar{\Omega}|U) W. \end{aligned} \quad (6.157)$$

The three positive-order symbols in Eq. (6.157) are

$$\begin{aligned} D_{\frac{3}{2}}^{\text{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) &:= (1 - \chi(\xi)) \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \dot{\theta}_0(U; x, \hat{\xi}) J_{\text{d}}, \\ \dot{D}_1^{\text{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) &:= (\mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi) \text{Id}_{\mathbb{C}^2}, \\ \dot{D}_{\frac{1}{2}}^{\text{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) &:= -(1 - \chi(\xi)) \dot{\alpha}_0^{\text{C,pr}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) |\xi|^{\frac{1}{2}} J_{\text{d}}. \end{aligned}$$

These formulas are precisely Eqs. (6.102), (6.104) and (6.105). In particular, the symbols are real-valued and diagonal for real  $U$ , and have orders  $3/2$ ,  $1$ , and  $1/2$ , respectively. Therefore, by Eq. (6.101),

$$D_{>0}^{\text{C}} = D_{\frac{3}{2}}^{\text{C}} + \dot{D}_1^{\text{C}} + \dot{D}_{\frac{1}{2}}^{\text{C}} \in \left( \Gamma_{\geq 0}^{\frac{3}{2}} \right)^{2 \times 2}. \quad (6.158)$$

**Part 3** (Bounded symbols and smoothing remainders). Set

$$A_{[0, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) := A_{[0, \geq 1]}^{\text{C,d}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi), \quad R_{[\varrho, \geq 1]}(\mathfrak{a}, \bar{\Omega}|U) := R_{[\varrho, \geq 1]}^{\text{C,d}}(\mathfrak{a}, \bar{\Omega}|U). \quad (6.159)$$

By Proposition 6.9,

$$A_{[0, \geq 1]} \in \left( \Gamma_{\geq 1}^0 \right)^{2 \times 2}, \quad R_{[\varrho, \geq 1]} \in \left( \mathcal{R}_{\geq 1}^{\varrho} \right)^{2 \times 2}.$$

The flat multiplier  $A_{10}^{\text{C,lin}}$  remains explicit in Eq. (6.157) and is not included in  $A_{[0, \geq 1]}$ . Substituting Eqs. (6.158) and (6.159) into Eq. (6.157) gives Eq. (6.100). □

## 6.4 Uniform energy estimates for the diagonalized system

The purpose of this subsection is to prove a uniform a priori estimate for classical solutions in the standard Sobolev phase space  $\mathcal{X}^s$  of Eq. (1.27). The proof is carried out in three steps. First we prove a closed estimate for a modified energy adapted to the diagonalized variables. Then we compare this modified energy with the Sobolev norm, keeping track of the purely low-frequency component of  $\tilde{\psi}$ . Finally we combine the two estimates.

**Proposition 6.10** (Uniform inverse-size a priori estimate in the Sobolev scale). *Let  $K \in \mathcal{S}_{\text{st}}$ . There exist  $s_0, r_0 > 0$  such that, for every  $s \geq s_0$ , the following holds uniformly for  $(\mathfrak{a}, \bar{\Omega}) \in K$ . Let*

$$Y(t) := \begin{bmatrix} \eta(t) \\ \tilde{\psi}(t) \end{bmatrix}$$

*be a real classical solution on an interval  $I \ni 0$ . Assume that the reductions of Propositions 6.2, 6.4 and 6.7 are defined on  $I$ , and that*

$$\sup_{t \in I} \|Y(t)\|_{\mathcal{X}^{s_0}} \leq r_0.$$

*Then there exist constants*

$$\varepsilon_{s,K} > 0, \quad c_{s,K} > 0, \quad C_{s,K} > 0,$$

*such that, if*

$$\|Y(0)\|_{\mathcal{X}^s} \leq \varepsilon \leq \varepsilon_{s,K},$$

*then, for every  $t \in I$  such that*

$$|t| \leq c_{s,K} \varepsilon^{-1},$$

one has

$$\|Y(t)\|_{\mathcal{X}^s} \leq C_{s,K} (\varepsilon + c_{s,K}). \quad (6.160)$$

Moreover, the Hamiltonian component satisfies the sharper bound

$$\|Y(t)\|_{\mathcal{X}_{\text{Ham}}^s(\mathfrak{a}, \bar{\Omega})} \leq C_{s,K} \varepsilon. \quad (6.161)$$

In particular, after choosing  $c_{s,K}$  sufficiently small, the solution remains in a fixed small Sobolev neighbourhood of the origin on a time interval of size comparable to  $\varepsilon^{-1}$ . All constants are uniform for  $(\mathfrak{a}, \bar{\Omega}) \in K$ .

**Roadmap of the proof.** The proof of Proposition 6.10 is organized in four steps.

1. The first step makes the dependence on the physical parameters quantitative. For every  $K \subseteq \mathcal{S}_{\text{st}}$ , compactness and the strict Atwood–Weber inequality provide a uniform positive distance from the stability threshold. In Part 1 this is converted into the uniform ellipticity estimate Eq. (6.163) for the flat restoring symbol. Consequently, the flat diagonalizing multipliers, their inverses, and the corresponding symbolic estimates are uniform for  $(\mathfrak{a}, \bar{\Omega}) \in K$ . This is the source of the uniformity in the parameter-dependent constants used throughout the argument.
2. The second step works in the diagonalized complex variable constructed in Propositions 6.2, 6.4 and 6.7. At high frequency the positive-order part is diagonal at principal level. The modified weight is chosen so that the potentially derivative-losing commutator is cancelled by the Poisson-bracket identity Eq. (6.179). At low frequency the energy is completed with the Hamiltonian low-frequency symmetrizer of Lemma 6.12. The crucial point is the exact flat cancellation Eq. (6.174): the isolated low-frequency defect is not skew-adjoint, but the complete flat linear symbol is. These two cancellations give the closed modified energy estimate of Proposition 6.11 and the inverse-size modified energy bound of Corollary 6.14.
3. The third step compares the modified energy with the physical Sobolev variables. This is done in Section 6.4.2. The modified energy is equivalent to the Hamiltonian Sobolev norm  $\mathcal{X}_{\text{Ham}}^s(\mathfrak{a}, \bar{\Omega})$ , which controls all high-frequency components and, at low frequency, the Hamiltonian variables

$$\eta, \quad |D|^{\frac{1}{2}} \tilde{\psi}.$$

The only missing component of the full inhomogeneous Sobolev norm is the undifferentiated low-frequency part of  $\tilde{\psi}$ , isolated in Eq. (6.185).

4. The fourth step propagates this missing component by the rough estimate of Lemma 6.18. Since the Hamiltonian energy remains  $O(\varepsilon)$ , this component grows at most like  $O(\varepsilon |t|)$ . Therefore, on the time interval  $|t| \leq c_{s,K} \varepsilon^{-1}$ , it is bounded by  $O(\varepsilon + c_{s,K})$ . Choosing  $c_{s,K}$  sufficiently small keeps the full Sobolev norm inside the perturbative regime, while the Hamiltonian component remains  $O(\varepsilon)$ . This proves the inverse-size a priori estimate stated in Proposition 6.10.

#### 6.4.1 Uniform modified energy estimates

**Part 1** (Uniform control on compact stable parameter sets). Fix  $K \subseteq \mathcal{S}_{\text{st}}$ . Since  $W_{\mathfrak{a}, \bar{\Omega}}$  is continuous in the physical parameters and  $W_{\mathfrak{a}, \bar{\Omega}} < 1$  on  $\mathcal{S}_{\text{st}}$ , compactness gives

$$\delta_K := 1 - \sup_{(\mathfrak{a}, \bar{\Omega}) \in K} W_{\mathfrak{a}, \bar{\Omega}} > 0. \quad (6.162)$$

Moreover,

$$\inf_{(\mathfrak{a}, \bar{\Omega}) \in K} g\mathfrak{a} > 0.$$

Let  $\xi \neq 0$ . From Eqs. (1.26) and (3.2),

$$-q(\mathfrak{a}, \bar{\Omega}|\xi) \geq g\mathfrak{a} + \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2 - 2W_{\mathfrak{a}, \bar{\Omega}} \sqrt{\frac{\sigma g\mathfrak{a}}{\langle\langle \rho^\pm \rangle\rangle}} |\xi|.$$

Using

$$2\sqrt{\frac{\sigma \mathfrak{g} \mathfrak{a}}{\langle\langle \rho^\pm \rangle\rangle}} |\xi| \leq \mathfrak{g} \mathfrak{a} + \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2$$

and Eq. (6.162), we obtain

$$-q(\mathfrak{a}, \bar{\Omega}|\xi) \geq \delta_K \left( \mathfrak{g} \mathfrak{a} + \frac{\sigma}{\langle\langle \rho^\pm \rangle\rangle} |\xi|^2 \right).$$

Since the coefficients in the last expression vary continuously on  $K$ , there exist  $0 < c_K \leq C_K$  such that, uniformly for  $(\mathfrak{a}, \bar{\Omega}) \in K$ ,

$$c_K \langle \xi \rangle^2 \leq -q(\mathfrak{a}, \bar{\Omega}|\xi) \leq C_K \langle \xi \rangle^2, \quad \xi \in \mathbb{R}^2. \quad (6.163)$$

By Eqs. (6.40) and (6.163), there exist  $0 < c'_K \leq C'_K$  such that, uniformly for  $(\mathfrak{a}, \bar{\Omega}) \in K$ ,

$$c'_K |\xi|^{\frac{1}{2}} \langle \xi \rangle \leq \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \leq C'_K |\xi|^{\frac{1}{2}} \langle \xi \rangle, \quad \xi \neq 0.$$

Hence, by Eq. (6.41),

$$\mathfrak{m}_{-\frac{1}{4}}(\mathfrak{a}, \bar{\Omega}|\xi) \sim_K |\xi|^{\frac{1}{4}} \langle \xi \rangle^{-\frac{1}{2}}, \quad \mathfrak{m}_{-\frac{1}{4}}^{-1}(\mathfrak{a}, \bar{\Omega}|\xi) \sim_K |\xi|^{-\frac{1}{4}} \langle \xi \rangle^{\frac{1}{2}}. \quad (6.164)$$

In particular, on  $\text{supp}(1 - \chi)$ , where  $|\xi|$  is bounded away from zero,

$$\mathfrak{m}_{-\frac{1}{4}} \sim_K \langle \xi \rangle^{-\frac{1}{4}}, \quad \mathfrak{m}_{-\frac{1}{4}}^{-1} \sim_K \langle \xi \rangle^{\frac{1}{4}}.$$

The corresponding symbol seminorms are uniform on  $K$ : indeed,  $q$  is given explicitly by Eq. (3.2), while  $\omega_{\frac{3}{2}}$  and  $\mathfrak{m}_{-\frac{1}{4}}$  are obtained from it through the smooth operations in Eqs. (6.40) and (6.41); on  $\text{supp}(1 - \chi)$  their denominators are uniformly bounded away from zero by the estimates above.

It remains only to check the low-frequency regularization. By Eq. (6.46),  $\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}} = 1$  on the core where  $\chi = 1$ , while it coincides with  $\mathfrak{m}_{-\frac{1}{4}}$  where  $\chi = 0$ . On the fixed transition annulus  $0 < \chi < 1$ , Eq. (6.164) and compactness of  $K$  give uniform upper and positive lower bounds for  $\mathfrak{m}_{-\frac{1}{4}}$ . Therefore  $\mathfrak{m}_{-\frac{1}{4}}^{\text{nh}}$  and its inverse have uniform symbol seminorms for  $(\mathfrak{a}, \bar{\Omega}) \in K$ . By Eqs. (6.47) and (6.48), the same is true for the flat complex change of variables and its inverse.

Thus the flat multipliers which enter the complex reduction are uniformly controlled on  $K$ . Together with the convention that the symbol and smoothing classes have uniform seminorm bounds when the external parameters range in a fixed compact set, this allows all constants arising in the subsequent diagonalization, symmetrization, and energy estimates to be chosen uniformly for  $(\mathfrak{a}, \bar{\Omega}) \in K$ .

Recall the cutoff function  $\chi$  fixed in Eq. (4.7). We set

$$\chi_{\text{lo}}(\xi) := \chi\left(\frac{\xi}{2}\right), \quad \chi_{\text{hi}}(\xi) := 1 - \chi_{\text{lo}}(\xi).$$

In particular, recalling Eqs. (6.51) and (6.64),

$$\chi_{\text{lo}} = 1 \quad \text{on } \text{supp}(\chi), \quad \chi_{\text{hi}} A_{\text{lo}}^{\mathbb{C}, \text{lin}} = 0. \quad (6.165)$$

Reminding Eq. (6.101), we can write

$$\mathbf{D}_{>0}^{\mathbb{C}}(\mathfrak{a}, \bar{\Omega}|U; x, \xi) = \text{diag}(d_1(\mathfrak{a}, \bar{\Omega}|U; x, \xi), d_2(\mathfrak{a}, \bar{\Omega}|U; x, \xi)),$$

where

$$\begin{aligned} d_1 &:= \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi - (1 - \chi(\xi)) \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \dot{\theta}_0(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) \\ &\quad + (1 - \chi(\xi)) \dot{\alpha}_0^{\mathbb{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) |\xi|^{\frac{1}{2}}, \end{aligned} \quad (6.166)$$

$$\begin{aligned} d_2 &:= \mathcal{V}_{\text{can}}(\mathfrak{a}, \bar{\Omega}|\eta(U), \tilde{\psi}(U); x) \cdot \xi + (1 - \chi(\xi)) \omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) \dot{\theta}_0(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) \\ &\quad - (1 - \chi(\xi)) \dot{\alpha}_0^{\mathbb{C}, \text{pr}}(\mathfrak{a}, \bar{\Omega}|U; x, \hat{\xi}) |\xi|^{\frac{1}{2}}. \end{aligned} \quad (6.167)$$

Here  $\hat{\xi} := \xi / |\xi|$ , and every symbol depending on  $\hat{\xi}$  is multiplied by  $1 - \chi(\xi)$ . We split

$$\mathbf{D}_{>0}^{\mathbb{C}} = \mathbf{D}_{>0,\text{lin}}^{\mathbb{C}} + \mathbf{D}_{>0,\text{nl}}^{\mathbb{C}},$$

where

$$\mathbf{D}_{>0,\text{lin}}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|\xi) := \mathbf{D}_{>0}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|0; x, \xi),$$

and

$$\mathbf{D}_{>0,\text{nl}}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|U; x, \xi) := \mathbf{D}_{>0}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|U; x, \xi) - \mathbf{D}_{>0,\text{lin}}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|\xi).$$

Thus the diagonalized equation for  $W$  can be written as

$$\begin{aligned} \partial_t W = & \text{Op}^{\text{BW}} \left( \mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}}(\mathbf{a}, \bar{\Omega}|\xi) + i \mathbf{D}_{>0,\text{lin}}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|\xi) \right) W \\ & + i \text{Op}^{\text{BW}} \left( \mathbf{D}_{>0,\text{nl}}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|U; x, \xi) + \mathbf{A}_{[0,\geq 1]}(\mathbf{a}, \bar{\Omega}|U; x, \xi) \right) W + \mathbf{R}_{[\varrho,\geq 1]}(\mathbf{a}, \bar{\Omega}|U) W. \end{aligned}$$

We next introduce the flat low-frequency symmetrizer. Let

$$\mathbf{S}_{\text{lo}}^{\mathbb{R}}(\mathbf{a}, \bar{\Omega}|\xi) := \begin{bmatrix} -\mathbf{q}(\mathbf{a}, \bar{\Omega}|\xi) & 0 \\ 0 & \frac{|\xi|}{2} \end{bmatrix},$$

and define

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|\xi) := \left( \mathbf{M}(\mathbf{a}, \bar{\Omega}|\xi)^{-1} \right)^* \mathbf{S}_{\text{lo}}^{\mathbb{R}}(\mathbf{a}, \bar{\Omega}|\xi) \mathbf{M}(\mathbf{a}, \bar{\Omega}|\xi)^{-1}. \quad (6.168)$$

For  $s \geq 0$ , let

$$\lambda_s(z) := (1 + z^2)^{\frac{s}{3}}, \quad \Lambda_{s,\text{hi}}(\mathbf{a}, \bar{\Omega}|U; x, \xi) := \chi_{\text{hi}}(\xi) \text{diag}(\lambda_s(d_1), \lambda_s(d_2)).$$

Since  $d_j$  is of order  $3/2$ , the weight  $\lambda_s(d_j)$  is of order  $s$  on the high-frequency region. For the unknowns  $U$  and  $W$ , define

$$W_{s,\text{hi}} := \text{Op}^{\text{BW}}(\Lambda_{s,\text{hi}}(\mathbf{a}, \bar{\Omega}|U)) W, \quad U_{s,\text{lo}} := \chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{3}{4}} U.$$

The modified energy is

$$E_s(t) := \frac{1}{2} \|W_{s,\text{hi}}(t)\|_{L^2}^2 + \frac{1}{2} \langle \mathbf{S}_{\text{lo}}^{\mathbb{C}}(\mathbf{a}, \bar{\Omega}|D) U_{s,\text{lo}}(t) | U_{s,\text{lo}}(t) \rangle. \quad (6.169)$$

Our goal is to prove the following modified energy estimate of the diagonalized system.

**Proposition 6.11** (Modified energy estimate for the diagonalized system). *Let the assumptions of Proposition 6.7 hold. There exist  $s_0, r_0 > 0$  such that, for every  $s \geq s_0$ , every classical real solution satisfying*

$$\sup_{t \in I} E_{s_0}^{\frac{1}{2}}(t) \leq r_0$$

*obeys*

$$\left| \frac{d}{dt} E_s(t) \right| \leq C_s E_{s_0}^{\frac{1}{2}}(t) E_s(t). \quad (6.170)$$

*The constant  $C_s$  is uniform when  $(\mathbf{a}, \bar{\Omega})$  ranges in a compact subset of  $\mathcal{S}_{\text{st}}$ .*

To prove Proposition 6.11, we start by proving the following two lemmas about flat low-frequency Hamiltonian symmetrizer and high-low frequency modified energy bounds.

**Lemma 6.12** (Flat low-frequency Hamiltonian symmetrizer). *Assume Hypothesis 1. Set*

$$\mathbf{Q}_0(\mathbf{a}, \bar{\Omega}|\xi) := \begin{bmatrix} 0 & \frac{|\xi|}{2} \\ \mathbf{q}(\mathbf{a}, \bar{\Omega}|\xi) & 0 \end{bmatrix}, \quad \nu_1(\mathbf{a}, \bar{\Omega}|\xi) := \frac{\mathbf{a}}{4} (\bar{\Omega}^\perp \cdot \xi).$$

*Then the following identities hold pointwise in  $\xi$ .*

1. The complex low-frequency symmetrizer has the explicit form

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}} = \frac{1}{2} \begin{bmatrix} -q \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^2 + \frac{|\xi|}{2} \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^{-2} & -q \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^2 - \frac{|\xi|}{2} \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^{-2} \\ -q \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^2 - \frac{|\xi|}{2} \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^{-2} & -q \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^2 + \frac{|\xi|}{2} \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^{-2} \end{bmatrix} (\alpha, \bar{\Omega}|\xi). \quad (6.171)$$

In particular  $\mathbf{S}_{\text{lo}}^{\mathbb{C}}$  is Hermitian and nonnegative. It is positive on the Hamiltonian low-frequency variables

$$\eta, \quad |D|^{\frac{1}{2}} \tilde{\psi},$$

and it is degenerate only on the purely undifferentiated zero-frequency component of  $\tilde{\psi}$ .

2. The real Hamiltonian symmetrizer cancels the flat off-diagonal real symbol:

$$\mathbf{S}_{\text{lo}}^{\mathbb{R}} \mathbf{Q}_0 + \mathbf{Q}_0^* \mathbf{S}_{\text{lo}}^{\mathbb{R}} = 0. \quad (6.172)$$

Equivalently, in the complex coordinate,

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}} (\mathbf{M} \mathbf{Q}_0 \mathbf{M}^{-1}) + (\mathbf{M} \mathbf{Q}_0 \mathbf{M}^{-1})^* \mathbf{S}_{\text{lo}}^{\mathbb{C}} = 0. \quad (6.173)$$

3. The complete flat complex linear symbol satisfies

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}} \left( \mathbf{A}_{\text{lo}}^{\mathbb{C}, \text{lin}} + i \mathbf{D}_{>0, \text{lin}}^{\mathbb{C}} \right) + \left( \mathbf{A}_{\text{lo}}^{\mathbb{C}, \text{lin}} + i \mathbf{D}_{>0, \text{lin}}^{\mathbb{C}} \right)^* \mathbf{S}_{\text{lo}}^{\mathbb{C}} = 0. \quad (6.174)$$

4. The low-frequency block has the diagonalization defect

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}} \mathbf{A}_{\text{lo}}^{\mathbb{C}, \text{lin}} + \left( \mathbf{A}_{\text{lo}}^{\mathbb{C}, \text{lin}} \right)^* \mathbf{S}_{\text{lo}}^{\mathbb{C}} = -i (1 - \chi(\xi)) \omega_{\frac{3}{2}} (\alpha, \bar{\Omega}|\xi) (\mathbf{S}_{\text{lo}}^{\mathbb{C}} \mathbf{J}_d - \mathbf{J}_d \mathbf{S}_{\text{lo}}^{\mathbb{C}}). \quad (6.175)$$

*Proof.* The explicit formula Eq. (6.171) follows from Eq. (6.168). Indeed,

$$\mathbf{M}^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} m_{-\frac{1}{4}}^{\text{nh}} & m_{-\frac{1}{4}}^{\text{nh}} \\ -i \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^{-1} & i \left( m_{-\frac{1}{4}}^{\text{nh}} \right)^{-1} \end{bmatrix}, \quad \mathbf{S}_{\text{lo}}^{\mathbb{R}} = \begin{bmatrix} -q & 0 \\ 0 & \frac{|\xi|}{2} \end{bmatrix},$$

and

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}} = (\mathbf{M}^{-1})^* \mathbf{S}_{\text{lo}}^{\mathbb{R}} \mathbf{M}^{-1}.$$

A direct multiplication gives Eq. (6.171). Since Hypothesis 1 gives  $-q(\alpha, \bar{\Omega}|\xi) > 0$ , the real quadratic form associated with  $\mathbf{S}_{\text{lo}}^{\mathbb{R}}$  is

$$-q |\eta|^2 + \frac{|\xi|}{2} |\tilde{\psi}|^2.$$

It is nonnegative and degenerates only at  $\xi = 0$  on the undifferentiated  $\tilde{\psi}$ -component. Since  $\mathbf{S}_{\text{lo}}^{\mathbb{C}}$  is the pull-back of  $\mathbf{S}_{\text{lo}}^{\mathbb{R}}$  by the invertible matrix  $\mathbf{M}$ , the same positivity and degeneracy properties hold in the complex coordinate.

We now prove the symmetrizer identities. Direct multiplication gives

$$\mathbf{S}_{\text{lo}}^{\mathbb{R}} \mathbf{Q}_0 = \begin{bmatrix} 0 & -\frac{|\xi|}{2} q \\ \frac{|\xi|}{2} q & 0 \end{bmatrix}, \quad \mathbf{Q}_0^* \mathbf{S}_{\text{lo}}^{\mathbb{R}} = \begin{bmatrix} 0 & \frac{|\xi|}{2} q \\ -\frac{|\xi|}{2} q & 0 \end{bmatrix}.$$

This proves Eq. (6.172). Conjugating this identity by  $\mathbf{M}^{-1}$ , and using the definition of  $\mathbf{S}_{\text{lo}}^{\mathbb{C}}$ , gives Eq. (6.173).

At the flat state the nonlinear principal corrections vanish:

$$\dot{\theta}_0 = 1, \quad \mathcal{V}_{\text{can}}(\alpha, \bar{\Omega}|0, 0) = \frac{\alpha}{4} \bar{\Omega}^\perp, \quad \dot{\alpha}_0^{\mathbb{C}, \text{pr}} = 0.$$

Hence the flat diagonal positive-order part is

$$\mathbf{D}_{>0,\text{lin}}^{\mathbb{C}} = \nu_1 \text{Id}_{\mathbb{C}^2} + (1 - \chi) \omega_{\frac{3}{2}} \mathbf{J}_d.$$

On the other hand, by Eq. (6.51),

$$\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} = \mathbf{M} \mathbf{Q}_0 \mathbf{M}^{-1} - i(1 - \chi) \omega_{\frac{3}{2}} \mathbf{J}_d.$$

Therefore the two transition terms cancel inside

$$\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} + i \mathbf{D}_{>0,\text{lin}}^{\mathbb{C}}.$$

Consequently, the left-hand side of Eq. (6.174) reduces to

$$\begin{aligned} & \mathbf{S}_{\text{lo}}^{\mathbb{C}} (\mathbf{M} \mathbf{Q}_0 \mathbf{M}^{-1}) + (\mathbf{M} \mathbf{Q}_0 \mathbf{M}^{-1})^* \mathbf{S}_{\text{lo}}^{\mathbb{C}} \\ & + \mathbf{S}_{\text{lo}}^{\mathbb{C}} i \nu_1 \text{Id}_{\mathbb{C}^2} + (i \nu_1 \text{Id}_{\mathbb{C}^2})^* \mathbf{S}_{\text{lo}}^{\mathbb{C}}. \end{aligned}$$

The first line vanishes by Eq. (6.173). The second line also vanishes, since  $\nu_1$  is real and the term  $i \nu_1 \text{Id}_{\mathbb{C}^2}$  is scalar and skew-adjoint:

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}} i \nu_1 \text{Id}_{\mathbb{C}^2} + (i \nu_1 \text{Id}_{\mathbb{C}^2})^* \mathbf{S}_{\text{lo}}^{\mathbb{C}} = i \nu_1 \mathbf{S}_{\text{lo}}^{\mathbb{C}} - i \nu_1 \mathbf{S}_{\text{lo}}^{\mathbb{C}} = 0.$$

This proves Eq. (6.174).

It remains to record what happens if the diagonal positive-order flat part is removed. Using again

$$\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} = \mathbf{M} \mathbf{Q}_0 \mathbf{M}^{-1} - i(1 - \chi) \omega_{\frac{3}{2}} \mathbf{J}_d,$$

and cancelling the off-diagonal term by Eq. (6.173), one obtains

$$\mathbf{S}_{\text{lo}}^{\mathbb{C}} \mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} + (\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}})^* \mathbf{S}_{\text{lo}}^{\mathbb{C}} = -i(1 - \chi) \omega_{\frac{3}{2}} \mathbf{S}_{\text{lo}}^{\mathbb{C}} \mathbf{J}_d + i(1 - \chi) \omega_{\frac{3}{2}} \mathbf{J}_d \mathbf{S}_{\text{lo}}^{\mathbb{C}}.$$

This is exactly Eq. (6.175). □

**Lemma 6.13** (High- and low-frequency modified energy bounds). *Under the assumptions of Proposition 6.11, one has*

$$\left| \frac{1}{2} \frac{d}{dt} \|W_{s,\text{hi}}\|_{L^2}^2 \right| \leq C_s E_{s_0}^{\frac{1}{2}} E_s, \quad (6.176)$$

$$\left| \frac{1}{2} \frac{d}{dt} \langle \mathbf{S}_{\text{lo}}^{\mathbb{C}} U_{s,\text{lo}} \mid U_{s,\text{lo}} \rangle \right| \leq C_s E_{s_0}^{\frac{1}{2}} E_s. \quad (6.177)$$

*Proof.* We omit the parameters  $\mathbf{a}, \bar{\Omega}$  and the time variable whenever no confusion is possible.

**Part 1** (High-frequency contribution). Differentiating the first term in Eq. (6.169) and using Eq. (6.100), we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|W_{s,\text{hi}}\|_{L^2}^2 &= \text{Re} \langle \text{Op}^{\text{BW}}(\partial_t \Lambda_{s,\text{hi}}) W \mid W_{s,\text{hi}} \rangle \\ &+ \text{Re} \langle \text{Op}^{\text{BW}}(\Lambda_{s,\text{hi}}) \text{Op}^{\text{BW}}(\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}}) W \mid W_{s,\text{hi}} \rangle \\ &+ \text{Re} \langle i \text{Op}^{\text{BW}}(\Lambda_{s,\text{hi}}) \text{Op}^{\text{BW}}(\mathbf{D}_{>0}^{\mathbb{C}}) W \mid W_{s,\text{hi}} \rangle \\ &+ \text{Re} \langle i \text{Op}^{\text{BW}}(\Lambda_{s,\text{hi}}) \text{Op}^{\text{BW}}(\mathbf{A}_{[0,\geq 1]}) W \mid W_{s,\text{hi}} \rangle \\ &+ \text{Re} \langle \text{Op}^{\text{BW}}(\Lambda_{s,\text{hi}}) \mathbf{R}_{[\ell,\geq 1]} W \mid W_{s,\text{hi}} \rangle. \end{aligned} \quad (6.178)$$

At  $U = 0$ , the weight  $\Lambda_{s,\text{hi}}(0)$  is a Fourier multiplier supported away from  $\text{supp } \mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}}$ . Hence its composition with the flat low-frequency multiplier vanishes. The remaining part of  $\Lambda_{s,\text{hi}}$  vanishes at least linearly in  $U$ . The spectral localization of the Bony–Weyl quantization and Eq. (6.165) therefore give

$$\left\| \text{Op}^{\text{BW}}(\Lambda_{s,\text{hi}}) \text{Op}^{\text{BW}}(\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}}) W \right\|_{L^2} \leq C_s E_{s_0}^{\frac{1}{2}} E_s^{\frac{1}{2}}.$$

We now consider the positive-order diagonal contribution. Since  $\mathbf{D}_{>0}^{\mathbb{C}}$  is real and diagonal,

$$\operatorname{Re} \langle \operatorname{Op}^{\text{BW}}(\mathbf{D}_{>0}^{\mathbb{C}}) W_{s,\text{hi}} \mid W_{s,\text{hi}} \rangle = 0.$$

Thus only the commutator remains. For  $j = 1, 2$ , its potentially order-losing symbol is

$$\frac{1}{i} \{ \chi_{\text{hi}}(\xi) \lambda_s(d_j), d_j \}.$$

Where  $\chi_{\text{hi}}$  is constant, one has the crucial scalar cancellation

$$\{ \lambda_s(d_j), d_j \} = \lambda'_s(d_j) \{ d_j, d_j \} = 0. \quad (6.179)$$

When a  $\xi$ -derivative hits  $\chi_{\text{hi}}$ , the resulting symbol is supported in a fixed annulus and contains  $\nabla_x d_j$ . It therefore vanishes at  $U = 0$ . After the cancellation Eq. (6.179), the first remaining high-frequency Weyl term has order

$$s + \frac{3}{2} - 3 = s - \frac{3}{2}.$$

Consequently,

$$\| [\operatorname{Op}^{\text{BW}}(\mathbf{A}_{s,\text{hi}}), \operatorname{Op}^{\text{BW}}(\mathbf{D}_{>0}^{\mathbb{C}})] W \|_{L^2} \leq C_s E_{s_0}^{\frac{1}{2}} E_s^{\frac{1}{2}}.$$

Similarly,  $\partial_t \mathbf{A}_{s,\text{hi}}(0) = 0$ , and differentiating the nonlinear coefficients produces one lower norm of  $\partial_t U$ . Using the equation, the tame estimates, and Eq. (6.188), we get

$$\| \operatorname{Op}^{\text{BW}}(\partial_t \mathbf{A}_{s,\text{hi}}) W \|_{L^2} \leq C_s E_{s_0}^{\frac{1}{2}} E_s^{\frac{1}{2}}. \quad (6.180)$$

Finally, the bounded symbol and the smoothing remainder vanish at  $U = 0$ . Their tame estimates give the same right-hand side. Substitution in Eq. (6.178) proves Eq. (6.176).

**Part 2** (Low-frequency contribution). We differentiate the low-frequency part of Eq. (6.169) using the complex equation Eq. (6.55). The flat contribution is

$$\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} + i \mathbf{D}_{>0,\text{lin}}^{\mathbb{C}}.$$

Since all low-frequency weights are constant-coefficient Fourier multipliers, Eq. (6.174) gives

$$\operatorname{Re} \langle \mathbf{S}_{\text{lo}}^{\mathbb{C}} (\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} + i \mathbf{D}_{>0,\text{lin}}^{\mathbb{C}}) U_{s,\text{lo}} \mid U_{s,\text{lo}} \rangle = 0. \quad (6.181)$$

This cancellation acts on the whole multiplier  $\mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}}$ , including its transition-frequency part.

It remains to estimate the nonlinear terms. Conjugating the low-frequency quadratic form by  $\mathbf{M}^{-1}$ , these estimates reduce to the real Alinhac system Eq. (6.7). The off-diagonal quasilinear block has the same Hamiltonian order structure as its flat part, while the lower-order block satisfies

$$\mathbf{A}_{\text{lo}}^{\mathbb{R},\text{Al}} \in \begin{bmatrix} \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^{-1} \\ \Gamma_{\geq 1}^0 & \Gamma_{\geq 1}^0 \end{bmatrix}$$

by Eq. (6.8). Together with the spectral localization of the Bony–Weyl quantization, this structure is bounded on the low-frequency Hamiltonian energy space

$$L^2 \times H^{\frac{1}{2}}.$$

The nonlinear coefficients vanish at the flat state. Since the CDE coefficients depend on  $\eta$  and on differentiated components of  $\tilde{\psi}$ , their lower seminorms are controlled by the Hamiltonian smallness

$$\| Y \|_{\mathcal{X}_{\text{Ham}}^{s_0}(\mathfrak{a}, \tilde{\Omega})} \lesssim E_{s_0}^{\frac{1}{2}}.$$

The undifferentiated low-frequency component  $\| Y \|_{\mathcal{Z}_{\text{lo}}^{s_0}}$  is therefore not needed in this closed symmetrized estimate. The smoothing remainder satisfies the same bound after fixing  $\varrho$  sufficiently large. Hence

$$\left| \operatorname{Re} \left\langle \mathbf{S}_{\text{lo}}^{\mathbb{C}} \chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{3}{4}} \left( \partial_t U - \left( \mathbf{A}_{\text{lo}}^{\mathbb{C},\text{lin}} + i \mathbf{D}_{>0,\text{lin}}^{\mathbb{C}} \right) U \right) \mid U_{s,\text{lo}} \right\rangle \right| \leq C_s E_{s_0}^{\frac{1}{2}} E_s.$$

Combining this bound with Eq. (6.181) proves Eq. (6.177).

□

We are now ready to prove Proposition 6.11.

*Proof of Proposition 6.11.* By Eq. (6.169),

$$E_s = \frac{1}{2} \|W_{s,hi}\|_{L^2}^2 + \frac{1}{2} \langle \mathbf{S}_{lo}^c U_{s,lo} \mid U_{s,lo} \rangle.$$

Adding Eqs. (6.176) and (6.177) gives

$$\left| \frac{d}{dt} E_s \right| \leq C_s E_{s_0}^{\frac{1}{2}} E_s,$$

which is Eq. (6.170). □

**Corollary 6.14** (Uniform a priori estimate in the modified energy). *Under the assumptions of Proposition 6.11, set*

$$e_0(t) := E_{s_0}^{\frac{1}{2}}(t).$$

*Then*

$$e_0(t) \leq \frac{e_0(0)}{1 - C_{s_0} e_0(0) |t|} \quad (6.182)$$

*whenever the denominator is positive. Moreover, for every  $s \geq s_0$ ,*

$$E_s(t) \leq E_s(0) (1 - C_{s_0} e_0(0) |t|)^{-\frac{C_s}{C_{s_0}}}. \quad (6.183)$$

*Proof.* Taking  $s = s_0$  in Eq. (6.170) gives

$$\left| \frac{d}{dt} e_0 \right| \leq C_{s_0} e_0^2.$$

Integrating forward and backward from  $t = 0$  proves Eq. (6.182). For  $s \geq s_0$ ,

$$\left| \frac{d}{dt} E_s \right| \leq C_s e_0 E_s.$$

Integrating again and using Eq. (6.182) proves Eq. (6.183). □

#### 6.4.2 Equivalence between the modified and Sobolev energies

For  $s \in \mathbb{R}$  and  $Y = \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}$ , define the Hamiltonian low-high energy norm

$$\begin{aligned} \|Y\|_{\mathcal{X}_{\text{Ham}}^s(\mathbf{a}, \tilde{\Omega})}^2 &:= \|\chi_{hi}(D) \eta\|_{H^{s+\frac{1}{4}}}^2 + \|\chi_{hi}(D) \tilde{\psi}\|_{H^{s-\frac{1}{4}}}^2 \\ &\quad + \left\| \chi_{lo}(D) \langle D \rangle^{s-\frac{3}{4}} (-\mathbf{q}(\mathbf{a}, \tilde{\Omega}|D))^{\frac{1}{2}} \eta \right\|_{L^2}^2 \\ &\quad + \frac{1}{2} \left\| \chi_{lo}(D) \langle D \rangle^{s-\frac{3}{4}} |D|^{\frac{1}{2}} \tilde{\psi} \right\|_{L^2}^2. \end{aligned} \quad (6.184)$$

We also isolate the purely inhomogeneous low-frequency component of  $\tilde{\psi}$ :

$$\|Y\|_{\mathcal{Z}_{lo}^s}^2 := \left\| \chi_{lo}(D) \langle D \rangle^{s-\frac{1}{4}} \tilde{\psi} \right\|_{L^2}^2. \quad (6.185)$$

**Remark 6.15** (Hamiltonian scale versus the inhomogeneous Sobolev scale). The norm in Eq. (6.184) is the scale naturally produced by the flat Hamiltonian symmetrizer. It is fully inhomogeneous on  $\eta$  and on the high-frequency part of  $\tilde{\psi}$ , but at low frequencies it measures  $|D|^{1/2} \tilde{\psi}$ , not  $\tilde{\psi}$  itself. This is already present in the real flat Hamiltonian energy through the block

$$\begin{bmatrix} -\mathbf{q}(\mathbf{a}, \tilde{\Omega}|\xi) & 0 \\ 0 & |\xi|/2 \end{bmatrix}.$$

Thus the missing component in the standard Sobolev norm  $\mathcal{X}^s$  is precisely  $\|Y\|_{\mathcal{Z}_{lo}^s}$ . The symmetrized energy gives a closed estimate in the Hamiltonian scale; the missing low-frequency Sobolev component is propagated by a rough estimate. ♦

**Remark 6.16** (Why the enlarged low cutoff is used). The choice  $\chi_{\text{lo}}(\xi) = \chi(\xi/2)$  guarantees that  $\chi_{\text{lo}} = 1$  on the whole support of the low-frequency linear defect  $\mathbf{A}_{\text{lo}}^{\text{C,lin}}$ . Therefore the low-frequency Hamiltonian symmetrizer sees the entire flat low block. The transition annulus between  $\chi$  and  $\chi_{\text{lo}}$  is compact in frequency; all symbols supported there are treated as bounded symbols in the energy estimate.  $\blacklozenge$

**Lemma 6.17** (Coercivity of the modified energy). *Let  $K \Subset \mathcal{S}_{\text{st}}$ . There exists  $r_0, s_0 > 0$ , such that for every  $s \geq s_0$  and every real classical solution satisfying*

$$E_{s_0}^{\frac{1}{2}}(t) \leq r_0,$$

*one has*

$$E_s(t) \sim_{s,K} \|(\eta(t), \tilde{\psi}(t))\|_{\mathcal{X}_{\text{Ham}}^s(\mathbf{a}, \bar{\Omega})}^2, \quad (6.186)$$

$$\|(\eta(t), \tilde{\psi}(t))\|_{\mathcal{X}^s}^2 \sim_{s,K} E_s(t) + \|(\eta(t), \tilde{\psi}(t))\|_{\mathcal{Z}_{\text{lo}}^s}^2. \quad (6.187)$$

*In particular,*

$$\|(\eta(t), \tilde{\psi}(t))\|_{\mathcal{X}_{\text{Ham}}^{s_0}(\mathbf{a}, \bar{\Omega})} \leq C_K E_{s_0}^{\frac{1}{2}}(t). \quad (6.188)$$

*Proof.* On  $\text{supp } \chi_{\text{hi}}$ , one has  $\chi = 0$ , hence  $\mathbf{m}_{-\frac{1}{4}}^{\text{nh}} = \mathbf{m}_{-\frac{1}{4}}$ . The ellipticity of the diagonal entries gives

$$\lambda_s(d_j) \sim_{s,K} \langle \xi \rangle^s, \quad j = 1, 2.$$

Using the high-frequency form of  $\mathbf{M}$ , the tame invertibility of  $\mathbf{B}$ , and the real Alinhac transformation, we obtain

$$\|W_{s,\text{hi}}\|_{L^2}^2 \sim_{s,K} \|\chi_{\text{hi}}(D)\eta\|_{H^{s+\frac{1}{4}}}^2 + \|\chi_{\text{hi}}(D)\tilde{\psi}\|_{H^{s-\frac{1}{4}}}^2. \quad (6.189)$$

For the low-frequency part, using  $U = \mathbf{M}Y_{\text{Al}}$  and Eq. (6.168), one has

$$\langle \mathbf{S}_{\text{lo}}^{\text{C}} U_{s,\text{lo}} | U_{s,\text{lo}} \rangle = \|\chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{3}{4}} (-\mathbf{q}(\mathbf{a}, \bar{\Omega}|D))^\frac{1}{2} \eta_{\text{Al}}\|_{L^2}^2 + \frac{1}{2} \|\chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{3}{4}} |D|^\frac{1}{2} \tilde{\psi}_{\text{Al}}\|_{L^2}^2. \quad (6.190)$$

By Eq. (6.163), Eq. (6.190) is uniformly coercive for  $(\mathbf{a}, \bar{\Omega}) \in K$ . The Alinhac transformation is triangular and tame on the Hamiltonian low-frequency scale. More explicitly, the correction to  $\tilde{\psi}$  is an order-zero coefficient times  $\eta$ , and therefore

$$\chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{3}{4}} |D|^\frac{1}{2} (\tilde{\psi}_{\text{Al}} - \tilde{\psi})$$

is controlled by the corresponding low-frequency  $\eta$ -term. Hence Eq. (6.190) is equivalent to the low-frequency part of  $\|Y\|_{\mathcal{X}_{\text{Ham}}^s(\mathbf{a}, \bar{\Omega})}$ . Together with Eq. (6.189), this proves Eq. (6.186).

Finally, by the definition of the standard inhomogeneous space  $\mathcal{X}^s$  in Eq. (1.27), the only missing term in the Hamiltonian norm is the low-frequency inhomogeneous component of  $\tilde{\psi}$ . Therefore

$$\|Y\|_{\mathcal{X}^s}^2 \sim_{s,K} \|Y\|_{\mathcal{X}_{\text{Ham}}^s(\mathbf{a}, \bar{\Omega})}^2 + \|Y\|_{\mathcal{Z}_{\text{lo}}^s}^2.$$

Using Eq. (6.186) gives Eq. (6.187), and Eq. (6.188) follows immediately.  $\square$

**Lemma 6.18** (Rough propagation of the missing low-frequency component). *Let  $K \Subset \mathcal{S}_{\text{st}}$ . There exist  $s_0, r_0 > 0$  such that, for every  $s \geq s_0$ , every real classical solution satisfying*

$$\sup_{t \in I} E_{s_0}^{\frac{1}{2}}(t) \leq r_0$$

*obeys*

$$\frac{d}{dt} \|Y(t)\|_{\mathcal{Z}_{\text{lo}}^s} \leq C_s E_s^{\frac{1}{2}}(t) + C_s E_{s_0}^{\frac{1}{2}}(t) \|Y(t)\|_{\mathcal{Z}_{\text{lo}}^s}. \quad (6.191)$$

*Consequently, if*

$$\sup_{|t| \leq T} E_{s_0}^{\frac{1}{2}}(t) \leq M_0 \varepsilon, \quad \sup_{|t| \leq T} E_s^{\frac{1}{2}}(t) \leq M_s \varepsilon,$$

*then, for  $|t| \leq T$ ,*

$$\|Y(t)\|_{\mathcal{Z}_{\text{lo}}^s} \leq \exp(C_s M_0 \varepsilon |t|) \left( \|Y_0\|_{\mathcal{Z}_{\text{lo}}^s} + C_s M_s \varepsilon |t| \right). \quad (6.192)$$

*Proof.* The CDE vector field depends on  $\tilde{\psi}$  only through differentiated quantities. At the level of the second equation, its flat low-frequency part has the form

$$\partial_t \tilde{\psi} = \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|D) \eta + \text{terms at least linear in } Y \text{ and involving differentiated } \tilde{\psi}.$$

Applying  $\chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{1}{4}}$  gives

$$\left\| \chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{1}{4}} \mathfrak{q}(\mathfrak{a}, \bar{\Omega}|D) \eta \right\|_{L^2} \leq C_{s,K} \|Y\|_{\mathcal{X}_{\text{Ham}}^s(\mathfrak{a}, \bar{\Omega})},$$

because  $\mathfrak{q}$  is a smooth bounded multiplier on the support of  $\chi_{\text{lo}}$ , while the Hamiltonian low-frequency norm controls

$$\chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{3}{4}} \left( -\mathfrak{q}(\mathfrak{a}, \bar{\Omega}|D) \right)^{\frac{1}{2}} \eta.$$

For the nonlinear terms, the tame parilinearization estimates of Theorem 5.2 and the real Alinhac structure give

$$\left\| \chi_{\text{lo}}(D) \langle D \rangle^{s-\frac{1}{4}} \mathfrak{N}(Y) \right\|_{L^2} \leq C_s \|Y\|_{\mathcal{X}_{\text{Ham}}^{s_0}(\mathfrak{a}, \bar{\Omega})} \left( \|Y\|_{\mathcal{X}_{\text{Ham}}^s(\mathfrak{a}, \bar{\Omega})} + \|Y\|_{Z_{\text{lo}}^s} \right).$$

The appearance of  $Z_{\text{lo}}^s$  here accounts only for bounded zero-order low-frequency contributions. Using Eq. (6.186), we obtain Eq. (6.191). The integrated bound Eq. (6.192) follows by Gronwall's lemma.  $\square$

### 6.4.3 Proof of the uniform Sobolev estimate

*Proof of Proposition 6.10.* By the bootstrap smallness assumption in  $\mathcal{X}^{s_0}$  and the coercivity estimate Eq. (6.187), after decreasing  $r_0$  we may apply Proposition 6.11. Set

$$e_s(t) := E_s^{\frac{1}{2}}(t), \quad z_s(t) := \|Y(t)\|_{Z_{\text{lo}}^s}.$$

The initial bound in  $\mathcal{X}^s$ , together with Eq. (6.187), gives

$$e_s(0) + z_s(0) \leq C_{s,K} \varepsilon, \quad e_{s_0}(0) \leq C_{s_0,K} \varepsilon. \quad (6.193)$$

Choose  $c_{s,K} > 0$  and  $\varepsilon_{s,K} > 0$  so small that

$$C_{s_0,K} e_{s_0}(0) |t| \leq \frac{1}{2} \quad \text{whenever } |t| \leq c_{s,K} \varepsilon^{-1}.$$

Then Corollary 6.14 gives, for every  $t \in I$  with  $|t| \leq c_{s,K} \varepsilon^{-1}$ ,

$$e_s(t) \leq C_{s,K} \varepsilon, \quad e_{s_0}(t) \leq C_{s_0,K} \varepsilon. \quad (6.194)$$

In particular, by Eq. (6.186), Eq. (6.194) gives the Hamiltonian bound Eq. (6.161).

It remains to control the undifferentiated low-frequency component of  $\tilde{\psi}$ . Applying Eq. (6.192) with

$$T = c_{s,K} \varepsilon^{-1}, \quad M_0 = C_{s_0,K}, \quad M_s = C_{s,K},$$

and using Eq. (6.193), we obtain

$$z_s(t) \leq C_{s,K} (\varepsilon + \varepsilon |t|) \leq C_{s,K} (\varepsilon + c_{s,K}) \quad (6.195)$$

for every  $t \in I$  with  $|t| \leq c_{s,K} \varepsilon^{-1}$ , after possibly decreasing  $c_{s,K}$  once more to keep the exponential factor uniformly bounded. Combining Eqs. (6.187), (6.194) and (6.195) proves Eq. (6.160). The final small-neighbourhood statement follows by choosing  $c_{s,K}$  and  $\varepsilon_{s,K}$  sufficiently small.  $\square$

Under Hypothesis 1, the two branches of the flat linearized system are simple and purely imaginary. We denote the positive dispersive frequency by

$$\omega_{\frac{3}{2}}(\mathfrak{a}, \bar{\Omega}|\xi) := \sqrt{-\mathfrak{d}_{\mathfrak{a}, \bar{\Omega}}(\xi)}.$$

For every  $\xi \neq 0$ , the corresponding flat eigenvectors suggest the complex coordinate

$$u_{\text{hom}} := \frac{1}{\sqrt{2}} \left( m_{-\frac{1}{4}}^{-1}(D) \eta_{\text{Al}} + i m_{-\frac{1}{4}}(D) \tilde{\psi}_{\text{Al}} \right), \quad m_{-\frac{1}{4}}(\alpha, \bar{\Omega}|\xi) := \sqrt{\frac{|\xi|}{2\omega_{\frac{3}{2}}(\alpha, \bar{\Omega}|\xi)}}.$$

This choice diagonalizes the flat system exactly away from  $\xi = 0$  and has the correct high-frequency orders. The obstruction is confined to low frequencies. Indeed, by Eqs. (3.2) and (3.4),

$$q(\alpha, \bar{\Omega}|\xi) \longrightarrow -g\alpha, \quad \omega_{\frac{3}{2}}(\alpha, \bar{\Omega}|\xi) \sim \sqrt{\frac{g\alpha}{2}} |\xi|^{\frac{1}{2}}, \quad m_{-\frac{1}{4}}(\alpha, \bar{\Omega}|\xi) \sim \frac{|\xi|^{\frac{1}{4}}}{(2g\alpha)^{\frac{1}{4}}}.$$

Hence  $m_{-\frac{1}{4}}^{-1}(\alpha, \bar{\Omega}|\xi)$  grows like  $|\xi|^{-\frac{1}{4}}$  as  $\xi \rightarrow 0$ . The exact eigenvector normalization therefore makes the map  $\eta_{\text{Al}} \mapsto m_{-\frac{1}{4}}^{-1}(D) \eta_{\text{Al}}$  singular at the origin; in particular, it does not define a bounded Fourier multiplier from  $H^{s+\frac{1}{4}}$  to  $H^s$ . Equivalently, the squared Fourier weight carried by the  $\eta_{\text{Al}}$ -component behaves like  $|\xi|^{-\frac{1}{2}}$ .

This singularity is not present in the physical quadratic energy. At the flat state the latter contains

$$\int_{\mathbb{R}^2} \left( -q(\alpha, \bar{\Omega}|\xi) |\hat{\eta}(\xi)|^2 + \frac{|\xi|}{2} |\hat{\tilde{\psi}}(\xi)|^2 \right) d\xi.$$

Since  $-q(\alpha, \bar{\Omega}|\xi) \rightarrow g\alpha > 0$ , the gravitational term controls the low-frequency part of  $\eta$  with an ordinary  $L^2$  weight. Thus the factor  $|\xi|^{-\frac{1}{4}}$  is an artifact of the exact homogeneous eigenvector normalization, rather than a degeneracy of the flat Hamiltonian. The homogeneous coordinate is therefore well adapted to the high-frequency diagonalization, but not to the natural inhomogeneous Sobolev phase space.

## 6.5 Proof of Theorem 1.12

*Proof.* Fix  $K$  a compact subset of stable parameter set  $\mathcal{S}_{\text{st}}$  introduced in Definition 1.10. Write

$$p := (\alpha, \bar{\Omega}) \in K.$$

We construct the proof of Theorem 1.12 following a standard approximation procedure for quasilinear-hyperbolic systems.

**Part 1 (Analyticity).** Recall that we use the notation  $Y := \begin{bmatrix} \eta \\ \tilde{\psi} \end{bmatrix}$ . Denote by  $\mathfrak{F}_p(Y)$  the right-hand side of Eq. (2.9). Its formulas combine pointwise analytic functions, Fourier multipliers, the kernels in Eqs. (2.3a) and (2.3b), analytic in the normalized differences of  $\eta$ , and the analytic inverse  $(\text{Id} + \alpha\tau_0(\eta))^{-1}$  of Lemma 5.23. Thus, for  $s_0$  sufficiently large,  $r_* > 0$  sufficiently small, and  $q \geq s_0$ ,

$$\mathfrak{F}_p \in C^\omega(B_{\mathcal{X}^q}(0, r_*); \mathcal{X}^{q-\frac{3}{2}}), \quad (6.196)$$

with uniform tame estimates for  $p \in K$ . The smallness of  $\|\eta\|_{H^{s_0+\frac{1}{4}}}$  keeps the normalized kernel factors and the inverse controlled when differentiating these formulas.

Define

$$\begin{aligned} U_p(Y) &:= \mathbf{M}(p|D) \mathbf{G}(p|Y) Y, \\ \Phi_p(Y) &:= \mathbf{B}(p|U_p(Y))^{-1} U_p(Y). \end{aligned} \quad (6.197)$$

By Eqs. (6.4), (6.49) and (6.99), the maps entering Eq. (6.197) are real-analytic and tame for  $\|Y\|_{\mathcal{X}^{s_0}} < r_*$ , uniformly for  $p \in K$ . Hence, for  $q \geq s_0$ ,

$$\Phi_p \in C^\omega(B_{\mathcal{X}^q}(0, r_*); H^q)$$

on the real subspace, with uniform tame bounds. Moreover,

$$\Phi_p(0) = 0, \quad d\Phi_p(0) = \mathbf{M}(p|D),$$

and  $M(p|D)$  is a uniform isomorphism from  $\mathcal{X}^q$  onto the real subspace of  $H^q(\mathbb{R}^2; \mathbb{C}^2)$ . After decreasing  $r_*$ , the analytic inverse function theorem therefore gives a real-analytic tame inverse

$$\Psi_p := \Phi_p^{-1},$$

uniformly for  $p \in K$ .

We use  $\Phi_p$  as a nonlinear change of coordinates and set

$$W := \Phi_p(Y), \quad Y = \Psi_p(W).$$

Accordingly, the vector field of the canonical CDE Eq. (2.9) in the  $W$ -coordinates is

$$\mathfrak{S}_p(W) := d\Phi_p(\Psi_p(W)) [\mathfrak{F}_p(\Psi_p(W))].$$

Since  $\mathfrak{F}_p$ ,  $\Phi_p$ , and  $\Psi_p$  are real-analytic and tame, for every  $q \geq s_0$ ,

$$\mathfrak{S}_p \in C^\omega \left( B_{H^q}(0, r_*) ; H^{q-\frac{3}{2}} \right), \quad (6.198)$$

on the real subspace, with uniform tame bounds.

If  $Y$  solves the canonical CDE

$$\partial_t Y = \mathfrak{F}_p(Y),$$

then, using  $W = \Phi_p(Y)$  and the chain rule,

$$\partial_t W = d\Phi_p(Y) [\partial_t Y] = d\Phi_p(Y) [\mathfrak{F}_p(Y)] = \mathfrak{S}_p(\Phi_p(Y)) = \mathfrak{S}_p(W). \quad (6.199)$$

Conversely, since  $Y = \Psi_p(W)$ , the same argument with the inverse change of coordinates shows that

$$\partial_t Y = \mathfrak{F}_p(Y) \iff \partial_t W = \mathfrak{S}_p(W).$$

We now write explicitly the quasilinear operator associated with  $\mathfrak{S}_p$ . Set

$$\mathcal{U}_p(W) := U_p(\Psi_p(W)),$$

and, for a sufficiently regular  $Z$ , define

$$\begin{aligned} \mathcal{L}_p(W) Z &:= \text{Op}^{\text{BW}} \left( A_{\text{lo}}^{\text{C,lin}}(p|\xi) \right) Z \\ &\quad + i \text{Op}^{\text{BW}} \left( D_{>0}^{\text{C}}(p|\mathcal{U}_p(W); x, \xi) + A_{[0, \geq 1]}(p|\mathcal{U}_p(W); x, \xi) \right) Z \\ &\quad + R_{[\varrho, \geq 1]}(p|\mathcal{U}_p(W)) Z. \end{aligned}$$

The paradifferential representation of Eq. (6.199) is precisely Eq. (6.100). By Eq. (6.198),  $\mathcal{L}_p(W)$  depends real-analytically and tamely on  $W$ , and

$$\mathfrak{S}_p(W) = \mathcal{L}_p(W)[W].$$

**Part 2** (The regularized problem). We now construct the approximate solutions. Let  $\tilde{\chi} \in C_c^\infty(\mathbb{R}; [0, 1])$  be real and even, with  $\tilde{\chi} = 1$  on  $[-1, 1]$ , and set

$$P_N := \tilde{\chi} \left( \frac{\langle D \rangle}{2^N} \right).$$

The operators  $P_N$  are real, self-adjoint smoothing Fourier multipliers and commute with Fourier multipliers. Let

$$D_p(W) := \text{diag}(d_1(W; x, \xi), d_2(W; x, \xi))$$

denote the real diagonal order-3/2 part of  $\mathcal{L}_p(W)$  in the sense of Eqs. (6.166) and (6.167), and write

$$\mathcal{B}_p(W)[Z] := \mathcal{L}_p(W)[Z] - i \text{Op}^{\text{BW}}(D_p(W)) Z.$$

Thus  $\mathcal{B}_p(W)$  contains the bounded part of the linearized transformed vector field. For  $N \geq 0$ , define the regularized diagonal symbols

$$\begin{aligned} f_{N+1}(z) &:= z \tilde{\chi}^2 \left( \frac{z}{2^{\frac{3}{2}(N+1)}} \right), \\ \widetilde{d}_{j,N+1}(W; x, \xi) &:= f_{N+1}(d_j(W; x, \xi)), \quad j = 1, 2, \\ \widetilde{\mathbf{D}}_{p,N+1}(W) &:= \text{diag}(\widetilde{d}_{1,N+1}(W), \widetilde{d}_{2,N+1}(W)). \end{aligned}$$

Since the cutoff is real and even, the reality relation between the two diagonal branches is preserved. Set

$$W_{\text{in}} := \Phi_p(Y_0), \quad W_0(t) \equiv 0. \quad (6.200)$$

Having constructed  $W_N$ , define  $W_{N+1}$  as the solution of

$$\begin{cases} \partial_t W_{N+1} = i \text{Op}^{\text{BW}}(\widetilde{\mathbf{D}}_{p,N+1}(W_N)) W_{N+1} + P_{N+1} \mathcal{B}_p(W_N)[W_{N+1}], \\ W_{N+1}(0) = P_{N+1} W_{\text{in}}. \end{cases} \quad (6.201)$$

For fixed  $N$ , Eq. (6.201) is linear in  $W_{N+1}$ . The symbol  $\widetilde{\mathbf{D}}_{p,N+1}(W_N)$  has compact frequency support, so its Bony–Weyl quantization is bounded on every fixed Sobolev space, with a bound which may depend on  $N$ . The second term in Eq. (6.201) is bounded on the same space by the Friedrichs regularization.

Since  $W_N$  is known and continuous in time, the right-hand side of Eq. (6.201) is a continuous family of bounded linear operators on  $H^s$ . Hence the linear Cauchy–Lipschitz theorem gives a unique solution  $W_{N+1}$  on a possibly  $N$ -dependent time interval. The real subspace is invariant, and compatibility of the same equation at higher Sobolev indices gives smooth approximate solutions for the smoothed initial data.

**Part 3** (The uniform bounds). We prove that the solutions constructed in Part 2 exist on a common time interval, independent of  $N$ .

For  $N \geq 0$ , set

$$U_N := \mathcal{U}_p(W_N).$$

For a real function  $Z$ , let  $\mathcal{E}_{q,N}[Z]$  denote the quadratic energy obtained from Eq. (6.169) by freezing the state-dependent coefficients at  $U_N$  and replacing  $W$  by  $Z$ . In particular, its high-frequency weight is

$$\Lambda_{q,\text{hi}}^N := \chi_{\text{hi}}(\xi) \text{diag}(\lambda_q(d_1(U_N; x, \xi)), \lambda_q(d_2(U_N; x, \xi))). \quad (6.202)$$

The low-frequency part is obtained by freezing the change  $W \mapsto U$  at  $U_N$  and using the flat symmetrizer  $\mathbf{S}_{\text{lo}}^{\mathbb{C}}$  of Eq. (6.168). The coercivity argument of Lemma 6.17 is uniform with respect to  $N$ .

We verify that the regularization of Part 2 preserves the estimates used in the proof of Lemma 6.13.

The crucial point is that the principal high-frequency cancellation is preserved *exactly*. We use in Eq. (6.202) the unregularized weight  $\lambda_q(d_j)$ . For  $j = 1, 2$ , the potentially order-losing symbol in the commutator with the regularized diagonal operator is

$$\frac{1}{i} \left\{ \chi_{\text{hi}}(\xi) \lambda_q(d_j), \widetilde{d}_{j,N+1} \right\}.$$

Where  $\chi_{\text{hi}}$  is constant, one has

$$\left\{ \lambda_q(d_j), \widetilde{d}_{j,N+1} \right\} = \left\{ \lambda_q(d_j), f_{N+1}(d_j) \right\} = \lambda'_q(d_j) f'_{N+1}(d_j) \{d_j, d_j\} = 0.$$

Thus the regularization preserves exactly the cancellation in Eq. (6.179). When a  $\xi$ -derivative hits  $\chi_{\text{hi}}$ , the resulting symbol is supported in its fixed transition annulus and contains  $\nabla_x d_j$ . In particular, it vanishes at  $U = 0$  and is estimated exactly as in the proof of Lemma 6.13.

Finally, the bounded part of Eq. (6.201) is harmless:  $P_{N+1}$  is real, self-adjoint and uniformly bounded on every Sobolev space. We choose once and for all the initial regularization scale in Part 2 so that, for every  $N \geq 0$ , both  $P_{N+1}$  and  $d_j \mapsto \widetilde{d}_{j,N+1}$  are the identity on the fixed low-frequency region entering Eq. (6.168). Hence the flat low-frequency cancellation Eq. (6.181) is unchanged.

We may therefore repeat the proof of Lemma 6.13 with the coefficients frozen at  $U_N$ . The only difference is that the coefficients depend on  $W_N$ , whereas the equation is solved by  $W_{N+1}$ . When the time derivative

falls on a frozen coefficient it produces  $\partial_t W_N$ ; using the equation at the Part 2 and the same tame estimates as in Eq. (6.180) gives, for every  $q \geq s_0$ ,

$$\frac{d}{dt} \mathcal{E}_{q,N}^{\frac{1}{2}} [W_{N+1}] \leq C_{q,K} \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_N] \mathcal{E}_{q,N}^{\frac{1}{2}} [W_{N+1}] + C_{q,K} \mathcal{E}_{q,N}^{\frac{1}{2}} [W_N] \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_{N+1}]. \quad (6.203)$$

In particular, for  $q = s_0$ ,

$$\frac{d}{dt} \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_{N+1}] \leq C_{s_0,K} \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_N] \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_{N+1}]. \quad (6.204)$$

The initial data satisfy, uniformly in  $N$ ,

$$\mathcal{E}_{s,N}^{\frac{1}{2}} [W_{N+1} (0)] \leq C_{s,K} \varepsilon,$$

by the tame bounds for  $\Phi_p$  and the uniform boundedness of  $P_{N+1}$ . We set

$$T_\varepsilon := c_{s,K} \varepsilon^{-1}, \quad I_\varepsilon := [-T_\varepsilon, T_\varepsilon].$$

We first apply Eq. (6.204). Assuming inductively that

$$\sup_{t \in I_\varepsilon} \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_N (t)] \leq M_0 \varepsilon,$$

Gronwall's inequality gives

$$\sup_{t \in I_\varepsilon} \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_{N+1} (t)] \leq C_K \varepsilon \exp (C_K M_0 \varepsilon T_\varepsilon).$$

Choosing first  $M_0$  and then  $c_{s,K}$  sufficiently small closes the induction. With this lower bound available, Eq. (6.203) gives in the same way, at the index  $s$ ,

$$\sup_{N \geq 0} \sup_{t \in I_\varepsilon} \mathcal{E}_{s,N}^{\frac{1}{2}} [W_{N+1} (t)] \leq C_{s,K} \varepsilon. \quad (6.205)$$

It remains only to propagate the undifferentiated low-frequency potential, which is not contained in the Hamiltonian energy. The proof of Lemma 6.18 applies to the frozen equation in the same way. If  $z_{q,N+1}$  denotes the corresponding frozen version of Eq. (6.185), its tame polarization gives

$$\begin{aligned} \frac{d}{dt} z_{q,N+1} &\leq C_{q,K} \mathcal{E}_{q,N}^{\frac{1}{2}} [W_{N+1}] + C_{q,K} \mathcal{E}_{s_0,N}^{\frac{1}{2}} [W_N] z_{q,N+1} \\ &\quad + C_{q,K} \mathcal{E}_{q,N}^{\frac{1}{2}} [W_N] z_{s_0,N+1}. \end{aligned}$$

Using first  $q = s_0$ , and then  $q = s$ , together with Eq. (6.205), yields

$$\sup_{N \geq 0} z_{s,N+1} (t) \leq C_{s,K} (\varepsilon + \varepsilon |t|), \quad t \in I_\varepsilon.$$

The frozen analogue of Eq. (6.187) therefore gives

$$\sup_{N \geq 0} \sup_{t \in I_\varepsilon} \|W_N (t)\|_{H^s} \leq C_{s,K} (\varepsilon + c_{s,K}). \quad (6.206)$$

After decreasing  $c_{s,K}$  and  $\varepsilon_0$ , the right-hand side of Eq. (6.206) remains inside the coordinate neighbourhood on which all the transformations and coefficients are defined. The continuation criterion for the linear problems of Part 2 therefore extends every  $W_N$  to the whole interval  $I_\varepsilon$ .

**Part 4** (Convergence of approximate solutions). We prove that the sequence  $(W_N)_N$  is Cauchy in  $\mathcal{C}(I_\varepsilon; L^2)$ . For convenience, denote by

$$\mathcal{L}_{p,N} (W) [Z] := \text{iOp}^{\text{BW}} (\tilde{\mathcal{D}}_{p,N} (W)) Z + P_N \mathcal{B}_p (W) [P_N Z]$$

the regularized linear operator introduced in Part 2. Thus

$$\partial_t W_{N+1} = \mathcal{L}_{p,N+1} (W_N) [W_{N+1}].$$

Set

$$Z_N := W_{N+1} - W_N.$$

Subtracting the equations at the two consecutive levels gives

$$\begin{aligned} \partial_t Z_N &= \mathcal{L}_{p,N+1}(W_N)[Z_N] \\ &\quad + (\mathcal{L}_{p,N+1}(W_N) - \mathcal{L}_{p,N+1}(W_{N-1}))[W_N] \\ &\quad + (\mathcal{L}_{p,N+1}(W_{N-1}) - \mathcal{L}_{p,N}(W_{N-1}))[W_N]. \end{aligned} \quad (6.207)$$

The first line is the homogeneous linearized equation. Since the diagonal symbols  $\widetilde{d}_{j,N+1}$  are real, their Bony–Weyl quantizations are self-adjoint and hence

$$\operatorname{Re} \langle i \operatorname{Op}^{\text{BW}}(\widetilde{\mathcal{D}}_{p,N+1}(W_N)) Z_N, Z_N \rangle_{L^2} = 0.$$

The bounded part contributes at most  $C \|Z_N\|_{L^2}^2$ , uniformly in  $N$ .

By the tame Lipschitz estimates for the coefficients and the uniform bounds of Part 3, for  $W, V$  in the structural ball and  $X \in H^s$ ,

$$\|(\mathcal{L}_{p,N+1}(W) - \mathcal{L}_{p,N+1}(V))[X]\|_{L^2} \leq C_{s,K} \|W - V\|_{L^2} \|X\|_{H^s}.$$

For the regularization error, set

$$a_{j,N}(W) := f_{N+1}(d_j(W)) - f_N(d_j(W)), \quad j = 1, 2.$$

Since  $f_{N+1} - f_N$  vanishes for  $|z| \lesssim 2^{\frac{3}{2}N}$ , the high-frequency ellipticity of  $d_j$  on the  $B_{s_0, \mathbb{R}}(I; r_0)$  implies

$$\operatorname{supp} a_{j,N}(W) \subset \{\langle \xi \rangle \gtrsim_K 2^N\}.$$

Together with the uniform order-3/2 symbol bounds proved in Part 3, this gives

$$\|\operatorname{Op}^{\text{BW}}(a_{j,N}(W))X\|_{L^2} \leq C_{s,K} 2^{-(s-\frac{3}{2})N} \|X\|_{H^s}, \quad j = 1, 2.$$

For the bounded part, one has

$$\begin{aligned} &P_{N+1} \mathcal{B}_p(W) P_{N+1} - P_N \mathcal{B}_p(W) P_N \\ &= (P_{N+1} - P_N) \mathcal{B}_p(W) P_{N+1} + P_N \mathcal{B}_p(W) (P_{N+1} - P_N). \end{aligned}$$

Since

$$\|(P_{N+1} - P_N)X\|_{L^2} \leq C_s 2^{-sN} \|X\|_{H^s},$$

and  $\mathcal{B}_p(W)$  is uniformly bounded on the Sobolev scale,

$$\|(P_{N+1} \mathcal{B}_p(W) P_{N+1} - P_N \mathcal{B}_p(W) P_N)X\|_{L^2} \leq C_{s,K} 2^{-sN} \|X\|_{H^s}.$$

Consequently,

$$\|(\mathcal{L}_{p,N+1}(W) - \mathcal{L}_{p,N}(W))[X]\|_{L^2} \leq C_{s,K} 2^{-(s-\frac{3}{2})N} \|X\|_{H^s}.$$

Using Part 3 in Eq. (6.207) therefore gives for some  $\sigma > 0$ ,

$$\frac{d}{dt} \|Z_N\|_{L^2} \leq C_{s,K} \|Z_N\|_{L^2} + C_{s,K} \|Z_{N-1}\|_{L^2} + C_{s,K} 2^{-\sigma N}. \quad (6.208)$$

Moreover,

$$Z_N(0) = (P_{N+1} - P_N) W_{\text{in}},$$

and hence, after decreasing  $\sigma > 0$  if necessary,

$$\|Z_N(0)\|_{L^2} \leq C_{s,K} 2^{-\sigma N}. \quad (6.209)$$

Integrating Eq. (6.208) and using Eq. (6.209), for  $0 \leq t \leq T_\varepsilon$ , we obtain

$$A_N(t) \leq C_{s,K} 2^{-\sigma N} t + C_{s,K} \int_0^t A_N(\tau) d\tau + C_{s,K} \int_0^t A_{N-1}(\tau) d\tau,$$

where

$$A_N(t) := \sup_{0 \leq \tau \leq t} \|Z_N(\tau)\|_{L^2}.$$

Summing in  $N \geq 1$  and applying Gronwall's inequality yields

$$\sum_{N \geq 1} \sup_{0 \leq t \leq T_\varepsilon} \|Z_N(t)\|_{L^2} < +\infty.$$

The same argument on  $[-T_\varepsilon, 0]$  gives, recalling the Notation 1.3-3,

$$\sum_{N \geq 1} \|W_{N+1} - W_N\|_{0, I_\varepsilon} < +\infty.$$

Consequently there exists a real-valued

$$W \in \mathcal{C}(I_\varepsilon; L^2)$$

such that

$$W_N \longrightarrow W \quad \text{in} \quad \mathcal{C}(I_\varepsilon; L^2).$$

By Part 3 the sequence is uniformly bounded in  $L^\infty(I_\varepsilon; H^s)$ . Therefore, for every  $0 < q < s$ , Sobolev interpolation gives

$$\|W_N - W_M\|_{q, I_\varepsilon} \leq C \|W_N - W_M\|_{0, I_\varepsilon}^{1-\frac{q}{s}} (\|W_N\|_{L^\infty H^s} + \|W_M\|_{L^\infty H^s})^{\frac{q}{s}}.$$

Hence

$$W_N \longrightarrow W \quad \text{in} \quad \mathcal{C}(I_\varepsilon; H^q), \quad 0 < q < s.$$

In particular, choosing  $s_0 \leq q < s$  and using the local Lipschitz continuity of  $\Psi_p : H^q \rightarrow \mathcal{X}^q$ , we obtain, with

$$Y_N := \Psi_p(W_N), \quad Y := \Psi_p(W),$$

that

$$Y_N \longrightarrow Y \quad \text{in} \quad \mathcal{C}(I_\varepsilon; \mathcal{X}^q), \quad s_0 \leq q < s. \quad (6.210)$$

**Part 5** (The limit is a solution of the original equation). Fix  $q$  with  $s_0 \leq q < s$ . By Part 4,

$$W_N \longrightarrow W \quad \text{in} \quad \mathcal{C}(I_\varepsilon; H^q). \quad (6.211)$$

We pass to the limit in the integral form of Eq. (6.201),

$$W_{N+1}(t) = P_{N+1} W_{\text{in}} + \int_0^t \mathcal{L}_{p, N+1}(W_N(\tau)) [W_{N+1}(\tau)] d\tau. \quad (6.212)$$

We first remove the regularization. The uniform  $H^s$ -bounds of Part 3, the Fourier localization of  $P_N - \text{Id}$ , and the definition  $\widetilde{d}_{j,N} = f_N(d_j)$  give

$$\|\mathcal{L}_{p, N+1}(W_N) [W_{N+1}] - \mathcal{L}_p(W_N) [W_{N+1}]\|_{q-\frac{3}{2}, I_\varepsilon} \leq C_{q,s,K} 2^{-(s-q)N} \|W_{N+1}(t)\|_{s, I_\varepsilon} \xrightarrow{N \rightarrow \infty} 0. \quad (6.213)$$

For the diagonal part this follows from the fact that  $f_N(d_j) - d_j$  is supported where  $\langle \xi \rangle \gtrsim 2^N$ , since  $|d_j| \sim \langle \xi \rangle^{3/2}$ ; the bounded part is treated directly with  $P_N - \text{Id}$ .

By Proposition 6.7, the concrete operator  $\mathcal{L}_p(V)$  consists of a symbol of order  $3/2$ , bounded symbols, the fixed low-frequency multiplier, and a smoothing remainder. The symbol estimates in Definition 4.7 and Eq. (4.4), the Bony–Weyl boundedness Theorem 4.15 and Eq. (4.11), and the smoothing estimate Definition 4.5 and Eq. (4.3) therefore give, uniformly for  $V$  in  $B_{s_0, \mathbb{R}}(I_\varepsilon, r_0)$ ,

$$\|\mathcal{L}_p(V) [Z]\|_{H^{q-\frac{3}{2}}} \leq C_{q,K} \|Z\|_{H^q}. \quad (6.214)$$

Here the lower-order paradifferential terms take values in spaces of higher regularity and are then embedded into  $H^{q-\frac{3}{2}}$ .

We also use the dependence of the concrete coefficients on the state. The same analytic dependence of the CDE and of the changes of variables used in Part 1 to obtain Eqs. (6.196) and (6.198), together with the polarized versions of Eqs. (4.3) and (4.4), gives

$$\|\mathrm{d}\mathcal{L}_p(V)[H][Z]\|_{H^{q-\frac{3}{2}}} \leq C_{q,s,K} \|H\|_{H^q} \|Z\|_{H^s},$$

for  $V$  in the same ball. In particular, since

$$\mathcal{L}_p(V_1) - \mathcal{L}_p(V_2) = \int_0^1 \mathrm{d}\mathcal{L}_p(V_2 + \theta(V_1 - V_2))[V_1 - V_2] \mathrm{d}\theta,$$

we have, thanks to the analyticity of the nonlinear vector field

$$\|(\mathcal{L}_p(V_1) - \mathcal{L}_p(V_2))[Z]\|_{H^{q-\frac{3}{2}}} \leq C_{q,s,K} \|V_1 - V_2\|_{H^q} \|Z\|_{H^s}. \quad (6.215)$$

We now apply these estimates to

$$\begin{aligned} & \mathcal{L}_p(W_N)[W_{N+1}] - \mathcal{L}_p(W)[W] \\ &= \mathcal{L}_p(W_N)[W_{N+1} - W_N] + (\mathcal{L}_p(W_N) - \mathcal{L}_p(W))[W_N] + \mathcal{L}_p(W)[W_N - W]. \end{aligned}$$

By Eq. (6.214),

$$\begin{aligned} \|\mathcal{L}_p(W_N)[W_{N+1} - W_N]\|_{q-\frac{3}{2}, I_\varepsilon} &\leq C_{q,K} \|W_{N+1} - W_N\|_{q, I_\varepsilon} \longrightarrow 0, \\ \|\mathcal{L}_p(W)[W_N - W]\|_{q-\frac{3}{2}, I_\varepsilon} &\leq C_{q,K} \|W_N - W\|_{q, I_\varepsilon} \longrightarrow 0. \end{aligned}$$

For the middle term, Eq. (6.215) and the uniform  $H^s$ -bound of Part 3 give

$$\|(\mathcal{L}_p(W_N) - \mathcal{L}_p(W))[W_N]\|_{q-\frac{3}{2}, I_\varepsilon} \leq C_{q,s,K} \|W_N - W\|_{q, I_\varepsilon} \|W_N\|_{s, I_\varepsilon} \longrightarrow 0$$

by Eq. (6.211). Hence

$$\|\mathcal{L}_p(W_N)[W_{N+1}] - \mathcal{L}_p(W)[W]\|_{q-\frac{3}{2}, I_\varepsilon} \longrightarrow 0. \quad (6.216)$$

Since

$$\mathcal{L}_p(W)[W] = \mathfrak{S}_p(W),$$

Eqs. (6.213) and (6.216) give

$$\mathcal{L}_{p,N+1}(W_N)[W_{N+1}] \longrightarrow \mathfrak{S}_p(W) \quad \text{in } \mathcal{C}\left(I_\varepsilon; H^{q-\frac{3}{2}}\right).$$

Moreover,

$$P_{N+1} W_{\text{in}} \longrightarrow W_{\text{in}} \quad \text{in } H^q.$$

Passing to the limit in Eq. (6.212) therefore gives

$$W(t) = W_{\text{in}} + \int_0^t \mathfrak{S}_p(W(\tau)) \mathrm{d}\tau.$$

Thus

$$W \in \mathcal{C}(I_\varepsilon; H^q) \cap \mathcal{C}^1\left(I_\varepsilon; H^{q-\frac{3}{2}}\right)$$

and solves the exact transformed equation

$$\partial_t W = \mathfrak{S}_p(W).$$

Finally set

$$Y := \Psi_p(W).$$

Since  $W_{\text{in}} = \Phi_p(Y_0)$ , one has  $Y(0) = Y_0$ . By the exact equivalence established in Part 1,

$$\begin{cases} \partial_t W = \mathfrak{S}_p(W) \\ W(0) = W_{\text{in}} \end{cases} \iff \begin{cases} \partial_t Y = \mathfrak{F}_p(Y) \\ Y(0) = Y_0. \end{cases}$$

Hence

$$Y(t) = Y_0 + \int_0^t \mathfrak{F}_p(Y(\tau)) \mathrm{d}\tau, \quad (6.217)$$

and the limit is a solution of the original canonical CDE.

**Part 6** (Uniqueness of the solution). Let  $Y_1, Y_2$  be two solutions in the class constructed above with initial data  $Y_{1,0}, Y_{2,0}$ , and set

$$W_j := \Phi_p(Y_j), \quad Z := W_1 - W_2.$$

Since

$$\partial_t W_j = \mathcal{L}_p(W_j)[W_j],$$

one has

$$\partial_t Z = \mathcal{L}_p(W_1)[Z] + (\mathcal{L}_p(W_1) - \mathcal{L}_p(W_2))[W_2].$$

The  $L^2$ -comparison estimate of Part 4 applies directly, now with no regularization error, and gives

$$\|Z(t)\|_{L^2} \leq C_{s,K} \|Z(0)\|_{L^2} + C_{s,K} \int_0^{|t|} \|Z(\tau)\|_{L^2} d\tau, \quad t \in I_\varepsilon.$$

Hence, by Gronwall's inequality,

$$\|W_1 - W_2\|_{0,I_\varepsilon} \leq C_{s,K,I_\varepsilon} \|W_1(0) - W_2(0)\|_{L^2}.$$

In particular, if  $Y_{1,0} = Y_{2,0}$ , then  $W_1 = W_2$ , and therefore  $Y_1 = Y_2$  since  $\Phi_p$  is invertible. This proves uniqueness.

**Part 7** (Regularity of the limit). By Part 3 and Eq. (6.210), for every  $t \in I_\varepsilon$  weak lower semicontinuity gives

$$\|Y(t)\|_{\mathcal{X}^s} \leq \liminf_{N \rightarrow \infty} \|Y_N(t)\|_{\mathcal{X}^s} \leq C_{s,K}(\varepsilon + \varepsilon|t|).$$

Hence

$$Y \in L^\infty(I_\varepsilon; \mathcal{X}^s).$$

Moreover, by Part 4,

$$Y \in \mathcal{C}(I_\varepsilon; \mathcal{X}^q), \quad s_0 \leq q < s.$$

It remains to prove strong continuity with values in  $\mathcal{X}^s$ . Fix a lower Sobolev index  $q$  such that

$$s_0 \leq q < s - \frac{3}{2}, \quad \delta := s - q > \frac{3}{2}.$$

For  $N \geq 1$ , recall from Eqs. (6.200) and (6.201) that

$$W_N(0) = P_N W_{\text{in}}, \quad W_{\text{in}} = \Phi_p(Y_0).$$

Since  $W_N(0)$  is smooth, the solution construction of Parts 2–4 may be performed at the index  $s + \frac{3}{2}$ . Its associated lifespan is still  $I_\varepsilon$  up to slightly modify the value of the constant  $c_{s,K}$ , since

$$\|W_N(0)\|_{H^s} \leq C \|W_{\text{in}}\|_{H^s} \leq C_{s,K} \|Y_0\|_{\mathcal{X}^s},$$

by the uniform boundedness of  $P_N$  and the tame bounds for  $\Phi_p$  following Eq. (6.197). Moreover, the Fourier localization gives

$$\|W_N(0)\|_{H^{s+\frac{3}{2}}} \leq C 2^{\frac{3}{2}N} \|W_{\text{in}}\|_{H^s} \leq C_{s,K} 2^{\frac{3}{2}N} \|Y_0\|_{\mathcal{X}^s}.$$

Hence the higher-order estimate of Part 3 gives

$$\sup_{t \in I_\varepsilon} \|W_N(t)\|_{H^{s+\frac{3}{2}}} \leq C_{s,K,I_\varepsilon} 2^{\frac{3}{2}N} \|Y_0\|_{\mathcal{X}^s}.$$

Take  $M, N \in \mathbb{N}^*$  with  $M > N$  and set

$$\mathcal{Z}_{M,N} := W_M - W_N, \quad \tau_{M,N} := \|(P_M - P_N)W_{\text{in}}\|_{H^s}.$$

Then

$$\tau_{M,N} \longrightarrow 0, \quad \text{as } N \rightarrow \infty.$$

The frequency localization of  $P_M - P_N$  gives

$$\begin{aligned}\|\mathcal{Z}_{M,N}(0)\|_{H^s} &= \tau_{M,N}, \\ \|\mathcal{Z}_{M,N}(0)\|_{H^q} &\leq C_{s,K} 2^{-\delta N} \tau_{M,N}.\end{aligned}\tag{6.218}$$

We first compare the two solutions at the lower index  $q$ . The difference  $\mathcal{Z}_{M,N}$  satisfies the same equation used in Part 5. Since

$$q + \frac{3}{2} < s,$$

the order- $\frac{3}{2}$  coefficient-difference term is controlled by the uniform  $H^s$ -bounds of Part 3, and the corresponding energy estimate gives

$$\|\mathcal{Z}_{M,N}\|_{q,I_\varepsilon} \leq C_{s,K,I_\varepsilon} \|\mathcal{Z}_{M,N}(0)\|_{H^q} \leq C_{s,K,I_\varepsilon} 2^{-\delta N} \tau_{M,N}.\tag{6.219}$$

We now repeat the same comparison at the index  $s$ , using the symmetrized energy estimate of Lemma 6.13. The homogeneous part is controlled exactly as in Eqs. (6.176) and (6.177). For the coefficient-difference term, recall from Proposition 6.7 that the only positive-order contribution has order  $3/2$ . The tame symbol estimate Eq. (4.4), the Bony–Weyl boundedness Theorem 4.15 and Eq. (4.11), and the smoothing estimate Eq. (4.3), applied in their polarized form as in Eq. (6.215), give

$$\|(\mathcal{L}_{p,M}(W_M) - \mathcal{L}_{p,N}(W_N))[W_N]\|_{H^s} \leq C_{s,K} \|\mathcal{Z}_{M,N}\|_{H^s} + C_{s,K} \|\mathcal{Z}_{M,N}\|_{H^{s-2}} \left( \|W_M\|_{H^{s+\frac{3}{2}}} + \|W_N\|_{H^{s+\frac{3}{2}}} \right).$$

Here the first term is the high–low contribution and is absorbed in the Gronwall estimate, while in the second one the high derivatives are placed on the smoother solution. This is the same tame splitting used in Eq. (6.203). Recalling that

$$\sup_{t \in I_\varepsilon} \|W_N(t)\|_{H^{s+\frac{3}{2}}} \leq C_{s,K,I_\varepsilon} 2^{\frac{3}{2}N} \|Y_0\|_{\mathcal{X}^s},$$

the coefficient-difference term is controlled by the lower-order comparison multiplied by  $2^{\frac{3}{2}N}$ .

$$\|\mathcal{Z}_{M,N}\|_{s,I_\varepsilon} \leq C_{s,K,I_\varepsilon} \left( \|\mathcal{Z}_{M,N}(0)\|_{H^s} + \|\mathcal{Z}_{M,N}\|_{q,I_\varepsilon} \left( \|W_M\|_{s+\frac{3}{2},I_\varepsilon} + \|W_N\|_{s+\frac{3}{2},I_\varepsilon} \right) \right).$$

Using Eqs. (6.218) and (6.219), we obtain

$$\|\mathcal{Z}_{M,N}\|_{s,I_\varepsilon} \leq C_{s,K,I_\varepsilon} \left( 1 + 2^{-(\delta-\frac{3}{2})N} \right) \tau_{M,N}.\tag{6.220}$$

Since  $\delta > \frac{3}{2}$  and  $\tau_N \rightarrow 0$ , Eq. (6.220) implies that the sequence  $(W_N)_N$  is Cauchy in  $L^\infty(I_\varepsilon; H^s)$ . By uniqueness of the limit, Part 4 gives

$$W_N \longrightarrow W \quad \text{in} \quad L^\infty(I_\varepsilon; H^s).$$

The local Lipschitz continuity of  $\Psi_p : H^s \rightarrow \mathcal{X}^s$  yields

$$Y_N \longrightarrow Y \quad \text{in} \quad L^\infty(I_\varepsilon; \mathcal{X}^s).$$

Each  $Y_N$  belongs to  $\mathcal{C}(I_\varepsilon; \mathcal{X}^s)$ ; hence, as a uniform limit of continuous functions,

$$Y \in \mathcal{C}(I_\varepsilon; \mathcal{X}^s).$$

Finally, by Eq. (6.196),

$$\mathfrak{F}_p : \mathcal{X}^s \longrightarrow \mathcal{X}^{s-\frac{3}{2}}$$

is continuous. Therefore

$$\mathfrak{F}_p(Y) \in \mathcal{C}(I_\varepsilon; \mathcal{X}^{s-\frac{3}{2}}),$$

and the integral equation Eq. (6.217) gives

$$Y \in \mathcal{C}(I_\varepsilon; \mathcal{X}^s) \cap \mathcal{C}^1(I_\varepsilon; \mathcal{X}^{s-\frac{3}{2}}).$$

Together with Part 3 and Part 6, this completes the proof. □

## A Technical miscellanea

**Lemma A.1** (Radial distribution identities). *The following identities hold true in  $\mathcal{S}'(\mathbb{R})$ , one has*

$$\int_0^\infty \frac{\sin(rt)}{r} dr = \frac{\pi}{2} \operatorname{sgn}(t), \quad \int_0^\infty e^{-ir t} dr = \pi \delta(t) - \text{i.p.v.} \frac{1}{t}, \quad \int_0^\infty \frac{1 - \cos(rt)}{r^2} dr = \frac{\pi}{2} |t|. \quad (\text{A.1})$$

*Proof.* We prove the three identities separately.

**Part 1.** For  $\epsilon > 0$ , define

$$I_\epsilon(t) := \int_0^\infty e^{-\epsilon r} \frac{\sin(rt)}{r} dr.$$

The integral is absolutely convergent and may be differentiated with respect to  $t$ . Thus

$$\partial_t I_\epsilon(t) = \int_0^\infty e^{-\epsilon r} \cos(rt) dr = \frac{\epsilon}{\epsilon^2 + t^2}.$$

Since  $I_\epsilon(0) = 0$ , integration in  $t$  gives

$$I_\epsilon(t) = \arctan(t/\epsilon).$$

Hence

$$I_\epsilon(t) \longrightarrow \frac{\pi}{2} \operatorname{sgn}(t) \quad \text{for } t \neq 0.$$

Moreover,  $|I_\epsilon(t)| \leq \pi/2$  uniformly in  $\epsilon > 0$  and  $t \in \mathbb{R}$ . Therefore, for every  $\varphi \in \mathcal{S}(\mathbb{R})$ , dominated convergence yields

$$\lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} I_\epsilon(t) \varphi(t) dt = \frac{\pi}{2} \int_{\mathbb{R}} \operatorname{sgn}(t) \varphi(t) dt.$$

This proves the first identity in Eq. (A.1) in  $\mathcal{S}'(\mathbb{R})$ .

**Part 2.** We next consider

$$J_\epsilon(t) := \int_0^\infty e^{-\epsilon r - ir t} dr = \frac{1}{\epsilon + it} = \frac{\epsilon}{\epsilon^2 + t^2} - i \frac{t}{\epsilon^2 + t^2}.$$

We treat the real and imaginary parts separately. Let  $\varphi \in \mathcal{S}(\mathbb{R})$ . For the real part, the change of variables  $t = \epsilon s$  gives

$$\begin{aligned} \int_{\mathbb{R}} \frac{\epsilon}{\epsilon^2 + t^2} \varphi(t) dt &= \int_{\mathbb{R}} \frac{\varphi(\epsilon s)}{1 + s^2} ds \\ &\longrightarrow \varphi(0) \int_{\mathbb{R}} \frac{1}{1 + s^2} ds = \pi \varphi(0). \end{aligned}$$

Thus

$$\frac{\epsilon}{\epsilon^2 + t^2} \longrightarrow \pi \delta(t) \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

For the imaginary part, fix an even function  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  such that  $\chi \equiv 1$  in a neighborhood of the origin. Since  $t(\epsilon^2 + t^2)^{-1} \chi(t)$  is odd,

$$\begin{aligned} \int_{\mathbb{R}} \frac{t}{\epsilon^2 + t^2} \varphi(t) dt &= \int_{\mathbb{R}} \frac{t}{\epsilon^2 + t^2} [\varphi(t) - \varphi(0) \chi(t)] dt \\ &= \int_{\mathbb{R}} \frac{t^2}{\epsilon^2 + t^2} \frac{\varphi(t) - \varphi(0) \chi(t)}{t} dt. \end{aligned}$$

The function

$$t \longmapsto \frac{\varphi(t) - \varphi(0) \chi(t)}{t}$$

extends smoothly through  $t = 0$  and belongs to  $L^1(\mathbb{R})$ . Since  $0 \leq \frac{t^2}{\epsilon^2 + t^2} \leq 1$ , dominated convergence gives

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}} \frac{t}{\epsilon^2 + t^2} \varphi(t) dt &= \int_{\mathbb{R}} \frac{\varphi(t) - \varphi(0) \chi(t)}{t} dt \\ &= \text{p.v.} \int_{\mathbb{R}} \frac{\varphi(t)}{t} dt, \end{aligned}$$

where the last equality uses the evenness of  $\chi$ . Consequently,

$$\frac{t}{\epsilon^2 + t^2} \longrightarrow \text{p.v.} \frac{1}{t} \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Combining the real and imaginary parts proves the second identity in Eq. (A.1).

**Part 3.** Finally, for every  $t \in \mathbb{R}$ , the integral

$$K(t) := \int_0^\infty \frac{1 - \cos(rt)}{r^2} dr$$

is absolutely convergent. Indeed,

$$1 - \cos(rt) = \begin{cases} O(r^2 t^2) & \text{as } r \rightarrow 0, \\ O(1) & \text{as } r \rightarrow \infty. \end{cases}$$

For  $t \neq 0$ , the change of variables  $s = |t| r$  gives

$$K(t) = |t| \int_0^\infty \frac{1 - \cos s}{s^2} ds.$$

It remains to evaluate the constant on the right-hand side. For  $0 < \delta < R$ , integration by parts gives

$$\int_\delta^R \frac{1 - \cos r}{r^2} dr = \left[ -\frac{1 - \cos r}{r} \right]_\delta^R + \int_\delta^R \frac{\sin r}{r} dr.$$

The boundary term tends to zero as  $\delta \downarrow 0$  and  $R \rightarrow \infty$ . The remaining improper integral converges by the Dirichlet criterion. Denoting

$$F(R) := \int_0^R \frac{\sin r}{r} dr,$$

let  $L := \lim_{R \rightarrow \infty} F(R)$ . Since  $F$  is bounded, integration by parts gives, for every  $\epsilon > 0$ ,

$$\begin{aligned} \int_0^\infty e^{-\epsilon r} \frac{\sin r}{r} dr &= \epsilon \int_0^\infty e^{-\epsilon r} F(r) dr \\ &= \int_0^\infty e^{-s} F(s/\epsilon) ds. \end{aligned}$$

Dominated convergence therefore implies that the last expression tends to  $L$  as  $\epsilon \downarrow 0$ . On the other hand, by the calculation for  $I_\epsilon$ ,

$$\int_0^\infty e^{-\epsilon r} \frac{\sin r}{r} dr = \arctan(1/\epsilon) \longrightarrow \frac{\pi}{2}.$$

Hence  $L = \pi/2$ , and therefore

$$\int_0^\infty \frac{1 - \cos r}{r^2} dr = \frac{\pi}{2}.$$

It follows that

$$K(t) = \frac{\pi}{2} |t|,$$

including  $t = 0$ , where both sides vanish. This proves the third identity in Eq. (A.1) and concludes the proof.  $\square$

## B Abstract parilinearization for subcritical homogeneous graph kernels

Let  $0 < n < d$  and let  $(h, k)$  be an admissible homogeneous pair of order  $-n$  and vanishing order  $\ell$  in the sense of Definition 5.4. Throughout this appendix we assume

$$\beta := d - n > 0, \quad k(0) = 0, \quad \ell > \beta. \quad (\text{B.1})$$

The condition  $k(0) = 0$  removes the flat homogeneous convolution, while  $\ell > \beta$  makes the nonlinear far-field contribution integrable.

For every real  $\eta$  satisfying  $\sup_{z \neq 0} \|\Delta_z \eta\|_{L^\infty} < r_k$ , define

$$\begin{aligned} t_{h,k}^{(n)}(\eta)[\omega](x) &:= \int_{\mathbb{R}^d} \frac{h(\hat{z})}{|z|^n} k(\Delta_z \eta(x)) \omega(x-z) dz, \\ \Delta_z \eta(x) &:= \frac{\eta(x) - \eta(x-z)}{|z|}, \quad \hat{z} := \frac{z}{|z|}. \end{aligned} \quad (\text{B.2})$$

The integral in Eq. (B.2) is ordinary both at the origin and at infinity: local integrability follows from  $n < d$ , while Eq. (B.1) and the vanishing order of  $k$  give the required far-field decay. Our main goal is to prove the following theorem.

**Theorem B.1** (Abstract subcritical parilinearization). *Let  $0 < n < d$ , and let  $(h, k)$  be an admissible homogeneous pair of order  $-n$  in the sense of Definition 5.4 and satisfy Eq. (B.1). There exists  $\varrho_0 > 0$  such that, for every  $\varrho \geq \varrho_0$ , there exist  $s_0 > d/2 + 1$ ,  $\mu \geq 0$ , and  $r_0 > 0$  such that, for every  $s \geq s_0$ , every real*

$$\eta \in H^{s+1+\mu}(\mathbb{R}^d; \mathbb{R}), \quad \omega \in H^{s+\mu}(\mathbb{R}^d; \mathbb{R}), \quad \|\eta\|_{H^{s_0+1+\mu}} < r_0,$$

the operator in Eq. (B.2) can be recasted as

$$t_{h,k}^{(n)}(\eta)[\omega] = \text{Op}^{\text{BW}} \left( A_{1,-\beta}^{h,k}[\omega](\eta; x, \xi) + \left[ \begin{array}{c} a_{[0, \geq \ell]}[\omega](\eta; x, \xi) \\ a_{[-\beta-2, \geq \ell]}(\eta; x, \xi) \end{array} \right] \cdot \left[ \begin{array}{c} \eta \\ \omega \end{array} \right] + R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega) \cdot \left[ \begin{array}{c} \eta \\ \omega \end{array} \right] \right). \quad (\text{B.3})$$

The objects in Eq. (B.3) are as follows.

- The vector symbol

$$A_{1,-\beta}^{h,k}[\omega](\eta; x, \xi) := \left[ \begin{array}{c} \omega(x) a_1^{h,k;\eta}(\nabla \eta(x), \xi) \\ a_{-\beta}^{h,k;\omega}(\nabla \eta(x), \xi) \end{array} \right] \quad (\text{B.4})$$

has the following components.

- $a_1^{h,k;\eta}(\nabla \eta, \xi) \in \Gamma_{\geq \ell-1}^1$  is the symbol

$$a_1^{h,k;\eta}(p, \xi) := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n+1}} k'(p \cdot \hat{z}) (1 - e^{-iz \cdot \xi}) dz; \quad (\text{B.5})$$

- $a_{-\beta}^{h,k;\omega}(\nabla \eta, \xi) \in \Gamma_{\geq \ell}^{-\beta}$  is defined by

$$a_{-\beta}^{h,k;\omega}(p, \xi) := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} k(p \cdot \hat{z}) e^{-iz \cdot \xi} dz. \quad (\text{B.6})$$

Both components satisfy

$$a_1^{h,k;\eta}(p, -\xi) = \overline{a_1^{h,k;\eta}(p, \xi)}, \quad a_{-\beta}^{h,k;\omega}(p, -\xi) = \overline{a_{-\beta}^{h,k;\omega}(p, \xi)}.$$

- The generic vector symbol

$$\left[ \begin{array}{c} a_{[0, \geq \ell]}[\omega](\eta; x, \xi) \\ a_{[-\beta-2, \geq \ell]}(\eta; x, \xi) \end{array} \right]$$

has the following components.

- $a_{[0, \geq \ell]}[\omega] \in \mathcal{L}(H^{s_0}; \Gamma^0)$ , and, for every real  $\omega$ ,

$$a_{[0, \geq \ell]}[\omega](\eta; x, -\xi) = \overline{a_{[0, \geq \ell]}[\omega](\eta; x, \xi)}.$$

If  $\ell \geq 2$ , the range of  $a_{[0, \geq \ell]}[\omega]$  improves to  $\Gamma_{\geq \ell-1}^0$ ;

–  $a_{[-\beta-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-\beta-2}$ , and

$$a_{[-\beta-2, \geq \ell]}(\eta; x, -\xi) = \overline{a_{[-\beta-2, \geq \ell]}(\eta; x, \xi)}.$$

- $R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega)$  is a real-to-real linear smoothing remainder of the form

$$R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega) := \begin{bmatrix} r_{[\ell, \geq \ell]}^{\mathbb{R}}[\omega](\eta) \\ r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta) \end{bmatrix}, \quad (\text{B.7})$$

where

$$r_{[\ell, \geq \ell]}^{\mathbb{R}}[\cdot] \in \mathcal{L}(H^{s_0}; \mathcal{R}^\ell), \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^\ell.$$

Moreover, if  $\ell \geq 2$ , then

$$r_{[\ell, \geq \ell]}^{\mathbb{R}}[\cdot] \in \mathcal{L}(H^{s_0}; \mathcal{R}_{\geq \ell-1}^\ell).$$

## B.1 Local-tail and Bony reduction

For fixed  $z \neq 0$ , write  $\omega_z(x) := \omega(x - z)$ . Since  $k(0) = 0$ , one has  $\tilde{k} = k$  in the notation of Eq. (5.13).

**Lemma B.2** (Local-tail and paraproduct reduction). *For every prescribed  $\rho \geq 0$ , one has*

$$t_{h,k}^{(n)}(\eta)[\omega] = t_{h,k}^{\text{QL}}(\eta)[\omega] + t_{h,k}^{\text{tr}}(\eta)[\omega] + t_{h,k}^R(\eta)[\omega] + \text{Op}^{\text{BW}}\left(f_{\geq \ell}^{\text{tail}, \mathbb{R}}[\omega](\eta; x)\right)\eta + R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega) \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix}. \quad (\text{B.8})$$

The objects in Eq. (B.8) are as follows.

- The quasilinear branch is

$$t_{h,k}^{\text{QL}}(\eta)[\omega] := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} \text{Op}^{\text{BW}}(k(\Delta_z \eta)) \omega_z dz. \quad (\text{B.9})$$

- The transported branch is

$$t_{h,k}^{\text{tr}}(\eta)[\omega] := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} \text{Op}^{\text{BW}}(\omega_z)[k(\Delta_z \eta)] dz. \quad (\text{B.10})$$

- The localized Bony remainder is

$$t_{h,k}^R(\eta)[\omega] := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} R^{\text{BW}}[k(\Delta_z \eta), \omega_z] dz.$$

- The tail function is

$$f_{\geq \ell}^{\text{tail}, \mathbb{R}}[\omega](\eta; x) := \int_{\mathbb{R}^d} (1 - \chi(z)) \frac{h(\hat{z})}{|z|^{n+1}} k'(\Delta_z \eta(x)) \omega_z(x) dz.$$

- The smoothing vector  $R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega)$  is as in Eq. (B.7).

*Proof.* The localized contribution is decomposed as in the proof of Lemma 5.9 by applying Lemma 4.18 to  $k(\Delta_z \eta) \omega_z$ .

For the tail contribution, by Eq. (B.1),

$$n + \ell = d - \beta + \ell > d.$$

Moreover,

$$b_h(z) := \frac{h(\hat{z})}{|z|^n}$$

satisfies the far-field bounds required in the definition of  $K\mathcal{F}_{\text{tail}, \geq \ell}$ . Hence

$$(1 - \chi(z)) \frac{h(\hat{z})}{|z|^n} k(\Delta_z \eta) \in K\mathcal{F}_{\text{tail}, \geq \ell}.$$

Applying Proposition 4.37 gives

$$\begin{aligned} & \int_{\mathbb{R}^d} (1 - \chi(z)) \frac{h(\hat{z})}{|z|^n} k(\Delta_z \eta) \omega_z dz \\ &= \text{Op}^{\text{BW}} \left( f_{\geq \ell}^{\text{tail}, \mathbb{R}}[\omega](\eta; x) \right) \eta + R_{[\ell, \geq \ell]}^{\text{lin}, \mathbb{R}}(\eta, \omega) \cdot \begin{bmatrix} \eta \\ \omega \end{bmatrix}, \end{aligned}$$

after absorbing the two smoothing remainders in the generic smoothing vector. Combining the local and tail contributions proves Eq. (B.8).  $\square$

**Lemma B.3** (Localized Bony remainder). *For every prescribed  $\rho \geq 0$ , the localized Bony remainder satisfies*

$$t_{h,k}^R(\eta)[\omega] = r_{[\rho, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad r_{[\rho, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^\rho. \quad (\text{B.11})$$

*In particular, the remainder is real-to-real.*

*Proof.* Since  $n < d$ , the localized homogeneous factor  $\chi(z) h(\hat{z}) |z|^{-n}$  has local order  $-n > -d$ . By Remark 4.19, after increasing the baseline index according to the prescribed  $\rho$ , the family

$$R^{\text{BW}}[k(\Delta_z \eta), \omega_z]$$

is a  $z$ -dependent smoothing operator with background vanishing order  $\ell$ . Multiplication by the localized homogeneous factor preserves a locally integrable kernel-smoothing class. Hence Lemma 4.34 gives Eq. (B.11). The real-to-real property follows from that of the Bony remainder and from the reality of the kernel.  $\square$

## B.2 Quasilinear contribution

**Lemma B.4** (Quasilinear contribution). *For every prescribed  $\rho \geq 0$ , the quasilinear branch in Eq. (B.9) satisfies*

$$t_{h,k}^{\text{QL}}(\eta)[\omega] = \text{Op}^{\text{BW}} \left( a_{-\beta}^{h,k;\omega}(\nabla \eta, \xi) + a_{[-\beta-2, \geq \ell]}(\eta; x, \xi) \right) \omega + r_{[\rho, \geq \ell]}^{\mathbb{R}}(\eta)\omega. \quad (\text{B.12})$$

*The objects in this equation have the following properties.*

- $a_{-\beta}^{h,k;\omega}$  is defined in Eq. (B.6) and belongs to  $\Gamma_{\geq \ell}^{-\beta}$ ;
- $a_{[-\beta-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-\beta-2}$  and  $r_{[\rho, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^\rho$  are generic terms in the sense of Notation 5.1. Both are real-to-real.

*Proof.*

**Part 1** (Proof of Eq. (B.13)). We first decompose, analogously as it was done in the critical case in Section 5.1

$$t_{h,k}^{\text{QL}} = t_{h,k}^{\text{QL}, -\beta} + t_{h,k}^{\text{QL}, -\beta-1} + t_{h,k}^{\text{QL}, \leq -\beta-2}, \quad (\text{B.13})$$

where

$$t_{h,k}^{\text{QL}, -\beta}(\eta)[\omega] := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} \text{Op}^{\text{BW}}(k(v[\eta])) \omega_z dz, \quad (\text{B.14a})$$

$$t_{h,k}^{\text{QL}, -\beta-1}(\eta)[\omega] := -\frac{1}{2} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n-1}} \text{Op}^{\text{BW}}(k'(v[\eta])) h[\eta] \omega_z dz, \quad (\text{B.14b})$$

$$t_{h,k}^{\text{QL}, \leq -\beta-2}(\eta)[\omega] := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} \text{Op}^{\text{BW}}(k_{[2, \geq \ell]}^{\text{loc}, \mathbb{R}}) \omega_z dz.$$

**Part 2** (Analysis of Eq. (B.14a)). We prove

$$t_{h,k}^{\text{QL},-\beta}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(a_{-\beta}^{h,k;\omega}(\nabla\eta, \xi) + \frac{i}{2}(\nabla_x \cdot \nabla_\xi) a_{-\beta}^{h,k;\omega}(\nabla\eta, \xi) + a_{[-\beta-2, \geq \ell]}\right)\omega + r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad (\text{B.15})$$

where

$$a_{[-\beta-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-\beta-2}, \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\ell}.$$

The point at which the subcritical assumption differs from the critical case is that  $n < d$ . Hence the kernel in Eq. (B.14a) is locally integrable at  $z = 0$ , and Proposition 4.35 applies directly to the leading branch. In particular, by Eq. (4.23),

$$t_{h,k}^{\text{QL},-\beta}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(b_{-\beta}^{h,k}(\eta; x, \xi)\right)\omega,$$

where

$$b_{-\beta}^{h,k}(\eta; x, \xi) := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} k\left(\nu[\eta]\left(x + \frac{z}{2}, z\right)\right) e^{-iz \cdot \xi} dz. \quad (\text{B.16})$$

For fixed  $z \neq 0$ , set

$$F_z(x) := k\left(\nu[\eta](x, z)\right).$$

Taylor's formula at  $x$  gives

$$F_z\left(x + \frac{z}{2}\right) = F_z(x) + \frac{1}{2}z \cdot \nabla_x F_z(x) + \frac{1}{4} \int_0^1 (1-t) (z \cdot \nabla_x)^2 F_z\left(x + \frac{t}{2}z\right) dt. \quad (\text{B.17})$$

Substituting the zeroth Taylor term into Eq. (B.16) gives exactly

$$a_{-\beta}^{h,k;\omega}(\nabla\eta(x), \xi)$$

by Eq. (B.6). For the first Taylor term, differentiating under the integral sign gives

$$\frac{1}{2} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} z \cdot \nabla_x [k(\nu[\eta](x, z))] e^{-iz \cdot \xi} dz = \frac{i}{2} (\nabla_x \cdot \nabla_\xi) a_{-\beta}^{h,k;\omega}(\nabla\eta(x), \xi).$$

It remains to consider the last term in Eq. (B.17). The two explicit factors of  $z$  improve the local order of the kernel from  $-n$  to  $-n+2$ , while differentiation in  $x$  preserves its background vanishing order  $\ell$ . Hence the corresponding kernel belongs to  $K\mathcal{F}_{\geq \ell}^{-n+2}$ . Since  $-n+2 > -d$ , another application of Proposition 4.35 gives a symbol in  $\Gamma_{\geq \ell}^{-((-n+2)+d)} = \Gamma_{\geq \ell}^{-\beta-2}$ . This yields the term  $a_{[-\beta-2, \geq \ell]}$  in Eq. (B.15) and proves the claim.

**Part 3** (Analysis of Eq. (B.14b)). We prove

$$t_{h,k}^{\text{QL},-\beta-1}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(-\frac{i}{2}(\nabla_x \cdot \nabla_\xi) a_{-\beta}^{h,k;\omega}(\nabla\eta, \xi) + a_{[-\beta-2, \geq \ell]}\right)\omega + r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad (\text{B.18})$$

with the classes in Eq. (B.15). Differentiating Eq. (B.6) under the integral sign gives

$$(\nabla_x \cdot \nabla_\xi) a_{-\beta}^{h,k;\omega}(\nabla\eta(x), \xi) = -i \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n-1}} k'(\nu[\eta](x, z)) h[\eta](x, z) e^{-iz \cdot \xi} dz.$$

Thus the uncentered Fourier integral of the kernel in Eq. (B.14b) is the explicit term in Eq. (B.18). The kernel has local order  $-n+1$ , so every nonconstant term in its expansion at  $x + z/2$  has local order at least  $-n+2$ . Applying Proposition 4.35 proves Eq. (B.18). Consequently, the two order- $(-\beta-1)$  terms in Eqs. (B.15) and (B.18) cancel:

$$\frac{i}{2}(\nabla_x \cdot \nabla_\xi) a_{-\beta}^{h,k;\omega} - \frac{i}{2}(\nabla_x \cdot \nabla_\xi) a_{-\beta}^{h,k;\omega} = 0. \quad (\text{B.19})$$

**Part 4** (Proof of Eq. (B.20)). We prove

$$t_{h,k}^{\text{QL}, \leq -\beta-2}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(a_{[-\beta-2, \geq \ell]}(\eta; x, \xi)\right)\omega + r_{[\ell, \geq \ell]}^{\mathbb{R}}(\eta)\omega, \quad (\text{B.20})$$

where

$$a_{[-\beta-2, \geq \ell]} \in \Gamma_{\geq \ell}^{-\beta-2}, \quad r_{[\ell, \geq \ell]}^{\mathbb{R}} \in \mathcal{R}_{\geq \ell}^{\ell}.$$

Since  $-n+2 > -d$ , this follows from Proposition 4.35. The real-to-real conclusions follow from Eq. (4.22) and the reality of the kernel.

**Part 5** (Proof of Eq. (B.12)). Combining Eqs. (B.13), (B.15), (B.18) and (B.20) and using Eq. (B.19) proves Eq. (B.12). The symbol estimates in Proposition 4.35, applied to the Fourier integral in Eq. (B.6), give

$$a_{-\beta}^{h,k;\omega}(\nabla\eta, \xi) \in \Gamma_{\geq \ell}^{-\beta}.$$

All kernels are real-valued; hence Eq. (4.22) gives the real-to-real symbols and remainders in Eq. (B.12).  $\square$

### B.3 Transported contribution

**Lemma B.5** (Transported contribution). *For every prescribed  $\varrho \geq 0$ , the transported branch in Eq. (B.10) satisfies*

$$t_{h,k}^{\text{tr}}(\eta)[\omega] = \text{Op}^{\text{BW}}\left(\omega(x) a_1^{h,k;\eta}(\nabla\eta(x), \xi) + a_{[0,\geq \ell]}[\omega](\eta; x, \xi)\right)\eta + r_{[\varrho,\geq \ell]}^{\mathbb{R}}[\omega](\eta)\eta. \quad (\text{B.21})$$

The objects in this equation have the following properties.

- $a_1^{h,k;\eta}$  is defined in Eq. (B.5) and belongs to  $\Gamma_{\geq \ell-1}^1$ ;
- $a_{[0,\geq \ell]}[\cdot] \in \mathcal{L}(H^{s_0}; \Gamma^0)$  and  $r_{[\varrho,\geq \ell]}^{\mathbb{R}}[\cdot] \in \mathcal{L}(H^{s_0}; \mathcal{R}^\varrho)$ . If  $\ell \geq 2$ , both ranges improve to vanishing order  $\ell - 1$ . The symbol and the remainder are real-to-real.

*Proof.*

**Part 1** (Proof of Eq. (B.22)). Applying Lemma 4.20 to  $k(\Delta_z\eta)$  gives

$$k(\Delta_z\eta) = \text{Op}^{\text{BW}}(k'(\Delta_z\eta))\Delta_z\eta + q_k(\Delta_z\eta)[\Delta_z\eta], \quad (\text{B.22})$$

where  $q_k := R_k$ . Substitution into Eq. (B.10) gives

$$t_{h,k}^{\text{tr}} = t_{h,k}^{\text{tr,I}} + t_{h,k}^{\text{tr,II}},$$

with

$$\begin{aligned} t_{h,k}^{\text{tr,I}}(\eta)[\omega] &:= \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} \text{Op}^{\text{BW}}(\omega_z) \text{Op}^{\text{BW}}(k'(\Delta_z\eta)) \Delta_z\eta dz, \\ t_{h,k}^{\text{tr,II}}(\eta)[\omega] &:= \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} \text{Op}^{\text{BW}}(\omega_z) q_k(\Delta_z\eta) [\Delta_z\eta] dz. \end{aligned}$$

**Part 2** (Proof of Eq. (B.24)). We prove

$$t_{h,k}^{\text{tr,I}}(\eta)[\omega] = t_{h,k}^{\text{tr,I,loc}}(\eta)[\omega] + \text{Op}^{\text{BW}}(a_{[0,\geq \ell]}[\omega])\eta + r_{[\varrho,\geq \ell]}^{\mathbb{R}}[\omega](\eta)\eta, \quad (\text{B.24})$$

where the first term is

$$t_{h,k}^{\text{tr,I,loc}}(\eta)[\omega] := \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n+1}} \text{Op}^{\text{BW}}(\omega(x) k'(\nu[\eta])) \delta_z\eta dz, \quad (\text{B.25})$$

and the generic terms have the classes stated in Lemma B.5. Recall  $\nu[\eta]$  from Eq. (5.35). The identities

$$\omega_z - \omega(x) = -|z| \int_0^1 \nabla\omega(x - tz) \cdot \hat{z} dt, \quad k'(\Delta_z\eta) - k'(\nu[\eta]) = (\Delta_z\eta - \nu[\eta]) \int_0^1 k''(\nu[\eta] + t(\Delta_z\eta - \nu[\eta])) dt,$$

together with Eq. (5.38), give

$$\omega_z k'(\Delta_z\eta) = \omega(x) k'(\nu[\eta]) + e_{[1]}^{\text{tr}}[\omega], \quad e_{[1]}^{\text{tr}}[\cdot] \in \mathcal{L}(H^{s_0}; K\mathcal{F}_{\text{loc}}^1). \quad (\text{B.26})$$

If  $\ell \geq 2$ , the range improves to  $K\mathcal{F}_{\text{loc}, \geq \ell-1}^1$ . Using the composition defect Eq. (5.72) and  $\Delta_z\eta = \delta_z\eta/|z|$ , we obtain the exact splitting

$$\begin{aligned} t_{h,k}^{\text{tr,I}}(\eta)[\omega] &= t_{h,k}^{\text{tr,I,loc}}(\eta)[\omega] \\ &\quad + \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n+1}} \text{Op}^{\text{BW}}(e_{[1]}^{\text{tr}}[\omega]) \delta_z\eta dz \\ &\quad + \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^n} c_z[\omega](\eta) [\Delta_z\eta] dz. \end{aligned} \quad (\text{B.27})$$

We analyze the three terms in Eq. (B.27) in their order of appearance.

**Subpart 2.1** (Proof of Eq. (B.28)). We first prove

$$\begin{aligned} t_{h,k}^{\text{tr},1,\text{loc}}(\eta)[\omega] &= \text{Op}^{\text{BW}}\left(\omega(x) a_1^{h,k;\eta}(\nabla\eta(x), \xi)\right)\eta \\ &\quad + \text{Op}^{\text{BW}}\left(a_{[0,\geq\ell]}[\omega]\right)\eta + r_{[\varrho,\geq\ell]}^{\mathbb{R}}[\omega](\eta)\eta. \end{aligned} \quad (\text{B.28})$$

The first difference in Eq. (B.25) cannot in general be split as  $\delta_z\eta = \eta - \eta_z$ : the two terms would be separately locally integrable only for  $\beta > 1$ . We therefore retain the first difference. In the Bony–Weyl Fourier representation Eqs. (4.8) and (4.24), let  $\varsigma$  be the Fourier variable of the coefficient and  $\xi$  the midpoint frequency. We use

$$\begin{aligned} 1 - e^{-i(\xi - \varsigma/2) \cdot z} &= 1 - e^{-i\xi \cdot z} \\ &\quad + e^{-i\xi \cdot z} (1 - e^{i\varsigma \cdot z/2}). \end{aligned} \quad (\text{B.29})$$

By Eq. (B.5), the first term in Eq. (B.29) produces exactly the principal symbol in Eq. (B.28). For the second term,

$$1 - e^{i\varsigma \cdot z/2} = -\frac{i}{2}(\varsigma \cdot z) \int_0^1 e^{it\varsigma \cdot z/2} dt, \quad |1 - e^{i\varsigma \cdot z/2}| \lesssim |\varsigma| |z|.$$

The factor  $\varsigma$  differentiates the coefficient, while  $|z|$  lowers the local singularity from  $-(n+1)$  to  $-n > -d$ . Thus Proposition 4.35 gives an order-zero symbol and a smoothing remainder, as in Eq. (B.28).

To record the order of  $a_1^{h,k;\eta}$ , decompose the localized kernel in Eq. (B.5) dyadically. On an annulus  $|z| \sim 2^j$ , its  $L_z^1$ -size is  $O(2^{j(\beta-1)})$ , while  $1 - e^{-iz \cdot \xi}$  gives  $\min\{2^j |\xi|, 1\}$ . The resulting bounds, and their  $p$ - and  $\xi$ -derivatives, imply

$$\left| \partial_p^\alpha \partial_\xi^\gamma a_1^{h,k;\eta}(p, \xi) \right| \lesssim_{\alpha, \gamma} \langle \xi \rangle^{1-|\gamma|}, \quad |p| \leq r_*. \quad (\text{B.30})$$

Together with Eq. (5.14), this proves  $a_1^{h,k;\eta}(\nabla\eta, \xi) \in \Gamma_{\geq \ell-1}^1$ .

**Subpart 2.2** (Proof of Eq. (B.31)). For the second term in Eq. (B.27), we prove

$$\begin{aligned} \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n+1}} \text{Op}^{\text{BW}}(e_{[1]}^{\text{tr}}[\omega]) \delta_z \eta dz &= \text{Op}^{\text{BW}}(a_{[0,\geq\ell]}[\omega](\eta; x, \xi))\eta \\ &\quad + r_{[\varrho,\geq\ell]}^{\mathbb{R}}[\omega](\eta)\eta. \end{aligned} \quad (\text{B.31})$$

By Eq. (B.26), the kernel after multiplication by  $\chi h |z|^{-n-1}$  has local order  $-n > -d$ . We may therefore split  $\delta_z\eta = \eta - \eta_z$ . Then Remark 4.28 applies to the unshifted term and Proposition 4.35 to the translated one, with  $\omega$  as a separately estimated linear parameter and  $\eta$  as the argument. The latter symbol has order  $-\beta < 0$  and is included in the generic order-zero term. This proves Eq. (B.31), with the classes stated in Lemma B.5.

**Subpart 2.3** (Proof of Eq. (B.32)). For the last term in Eq. (B.27), we prove the equivalent reduction

$$\begin{aligned} t_{h,k}^{\text{tr},\text{I}}(\eta)[\omega] &= \int_{\mathbb{R}^d} \chi(z) \frac{h(\hat{z})}{|z|^{n+1}} \text{Op}^{\text{BW}}(\omega_z k'(\Delta_z \eta)) \delta_z \eta dz \\ &\quad + r_{[\varrho,\geq\ell]}^{\mathbb{R}}[\omega](\eta)\eta. \end{aligned} \quad (\text{B.32})$$

The two symbols in Eq. (5.72) are independent of  $\xi$ . Hence every non-principal term in their Bony–Weyl product vanishes, and the composition defect is arbitrarily smoothing by Proposition 4.24 and Lemma 4.33. Since its homogeneous factor has local order  $-n > -d$ , Lemma 4.34 applies directly, without a frozen subtraction. This proves Eq. (B.32); the composition defect contributes only to the smoothing remainder in Eq. (B.24).

Combining Eqs. (B.26), (B.31) and (B.32) proves Eq. (B.24).

**Part 3** (Proof of Eq. (B.33)). We prove

$$t_{h,k}^{\text{tr},\text{II}}(\eta)[\omega] = r_{[\varrho,\geq\ell]}^{\mathbb{R}}[\omega](\eta)\eta. \quad (\text{B.33})$$

By Lemma 4.20, choosing a smoothing gain larger than the prescribed  $\varrho$ , we have

$$q_k \in \mathcal{R}_{\geq 1}^\varrho.$$

Moreover,  $q_k(u)[u]$  vanishes to order at least  $\ell$ , while its differential vanishes to order at least  $\ell - 1$ . Since the homogeneous factor has local order  $-n > -d$ , Remark 4.19 and Lemma 4.34 apply directly to the second line of Eq. (B.23) and give Eq. (B.33). If  $\ell \geq 2$ , the range vanishes to order at least  $\ell - 1$ .

**Part 4** (Proof of Eq. (B.21)). Combining Eqs. (B.23), (B.24), (B.28) and (B.33) proves Eq. (B.21). The symbol classes and their vanishing orders follow from Eqs. (B.26) and (B.30) and the reductions above. Conjugating Eq. (B.5) and replacing  $\xi$  with  $-\xi$  gives the principal real-to-real symmetry. For the other terms, the same property follows from Eqs. (4.14) and (4.22) and Lemma 4.34. □

## B.4 Proof of Theorem B.1

*Proof.* For smooth inputs, combining Eqs. (B.8), (B.11), (B.12) and (B.21) gives

$$\begin{aligned} t_{h,k}^{(n)}(\eta)[\omega] &= \text{Op}^{\text{BW}} \left( \omega(x) a_1^{h,k;\eta}(\nabla\eta, \xi) + a_{[0,\geq\ell]}[\omega] + f_{\geq\ell}^{\text{tail},\mathbb{R}}[\omega](\eta) \right) \eta \\ &\quad + \text{Op}^{\text{BW}} \left( a_{-\beta}^{h,k;\omega}(\nabla\eta, \xi) + a_{[-\beta-2,\geq\ell]} \right) \omega \\ &\quad + \left( r_{[\ell,\geq\ell]}^{\mathbb{R}}[\omega](\eta) + r_{[\ell,\geq\ell]}^{\text{tail},\mathbb{R}}[\omega](\eta) \right) \eta + r_{[\ell,\geq\ell]}^{\mathbb{R}}(\eta) \omega. \end{aligned}$$

The first principal term is identified in Eq. (B.28). The second follows from Eq. (B.15), while Eq. (B.19) removes the intermediate order  $-\beta - 1$ . The symbol and remainder classes, their vanishing orders, and their real-to-real properties follow from Lemmas B.2 to B.5.

Absorbing the tail function in the generic order-zero coefficient and the tail remainders in the corresponding smoothing components, and using Remark 4.31, the bracketed terms have total vanishing order at least  $\ell$ . Collecting the terms according to Eqs. (B.4) and (B.7) then gives Eq. (B.3). Choose a common baseline  $s_0$ , finite loss  $\mu$ , and radius  $r_0$  for the preceding reductions. Smooth approximation and their tame estimates extend the identity to the Sobolev inputs of the theorem. □

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