Translation surfaces : definitions

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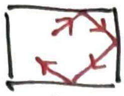
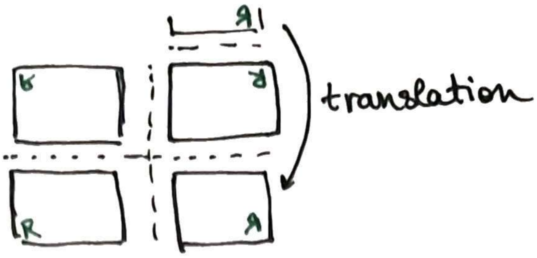
15.04

Recall last time :



billiards

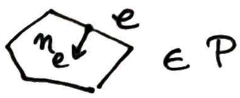


1. Constructive definition

P : a finite set of polygons in $\mathbb{R}^2$ (maybe non convex)

$E(P)$: the edges of the polygons in P

For every $e \in E(P)$, we define n_e the unit vector which is orthogonal to e and points toward the interior of the polygon

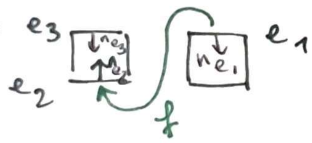


A map $f : E(P) \to E(P)$ is a pairing if

1. $f \circ f = id$, f has no fixed point
2. If $e_2 = f(e_1)$, then there exist $w_1 \in \mathbb{R}^2$ s.t. $e_2 = e_1 + w_1$
   we denote f_{e_1} the translation by w_1 in $\mathbb{R}^2$

3'. then $f_{e_1}(n_{e_1}) = -n_{e_2}$

Ex :



We could not have $f(e_1)$ because $f_{e_1}(n_{e_1}) = n_{e_3}$

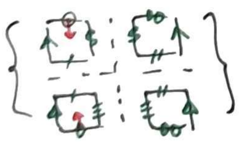
We define $X = \bigsqcup_{P_i \in P} P_i \big/ \sim_f$

where

$\sim_f$: $x_1 \sim x_2 \iff x_1 \in e_1,\ x_2 \in e_2,\ f(e_1) = f(e_2)$ and $f_{e_1}(x_1) = x_2$

Call X a translation surface obtained by P and f if X is connected.

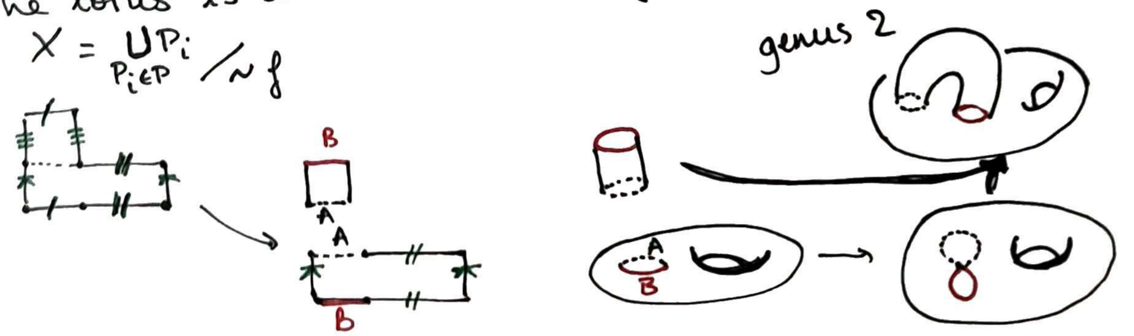
Ex :

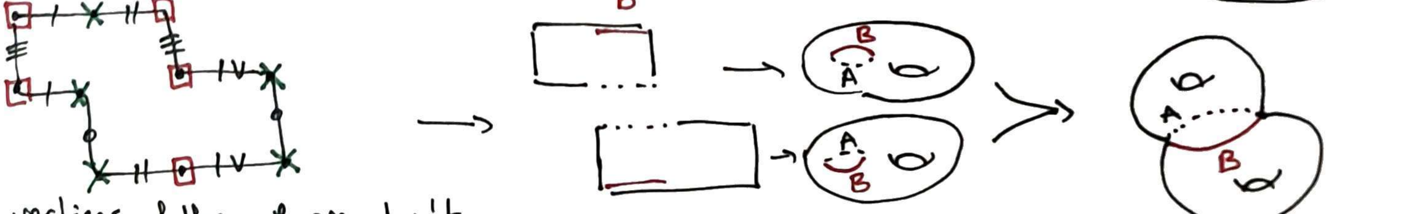
-  $= P$, f given by the marking o, /, s, ^, ...

  The torus is a translation surface obtained by P and f

  $X = \bigcup_{P_i \in P} P_i \big/ \sim_f$

- $P = \{P_0\}$ P_0



- 

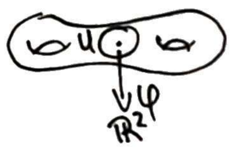
The vertices of the polyon don't identify to one point, but two.

genus 2 (again)

Rmk: If $f : A \to A$ with $A \subseteq E(P)$ satisfies 1. 2. and 2' then we can define a translation surface <u>with boundary</u>.

Claim: Given any rational polygon P (the angle of P are rational multiple of π), there exist finitely many reflections of P along its sides (produced by the unfolding process), and there exists a pairing f s.t. $X = \bigcup_{P_i \text{ reflection of } P} P_i \,/\!\sim_f$ is a translation surf.
(maybe proved later)

<u>Why is X (obtained by P and f) a surface?</u>



<u>Local geometry of X</u>

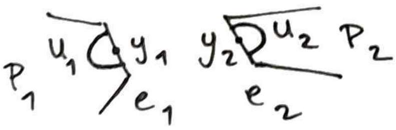
The map $\bigsqcup_{P_i \in P} P_i \xrightarrow{\pi} X = \bigsqcup P_i /\!\sim_f$ is such that:

1. $\pi|_{\sqcup \mathring{P}_i}$ is injective
2. $\pi|_{\sqcup_{e \in E(P)} \mathring{e}}$ is 2 to 1
3. $\pi|_{\sqcup_{v \in V(P)} v}$ is many to 1 (where $V(P)$ is the set of vertices)

In case 1., for $x \in X$, $y \in \mathring{P}_i$ s.t. $\pi(y) = x$, we have $U_y \subset \mathring{P}_i$ a Euclidean ball, $U_x = \pi(U_y)$

$(U_x, \pi^{-1}|_{U_x})$ is a chart around x.

In case 2., $x = \pi(y_1) = \pi(y_2)$, $y_i \in \mathring{e}_i$



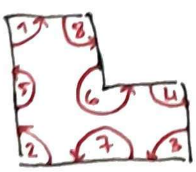
We have half-disks U_1 and U_2 centered at y_1 and y_2 within P_1 and P_2.

$U_x = \pi(U_1) \cup \pi(U_2)$

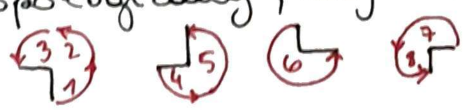
$U_x \xrightarrow{\varphi} B_{y_1}$ gives a chart around x.

In case 3., $x = \pi(y_1) = \dots = \pi(y_k)$

Ex:

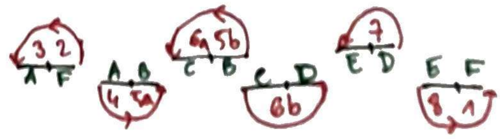


Topologically, they are glued to a disk



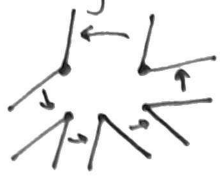
This shows that X is a surface.

We can also use the <u>half-disks construction</u>:



Claim Let X be obtained by P and if $x \in X$, $x = \pi(y_1) = \dots = \pi(y_k)$ for $y_i \in V(P)$.
Then there is a neighborhood U_x of x obtained by the half-disks construction for $2m$ half-disks, $m \in \mathbb{N}_{>0}$.

Proof: Every identification is by translation



so we must do complete "turns" before "closing" the neighborhood.
The total angle has to be $m \times 2\pi$. $\square$

Rmk: The cone angle at x is the sum of cone angles of y_i's in their polygon P_i. The cone angle is $m . 2\pi$ with $m \in \mathbb{N}_{>0}$.

Computation of the genus of X

$$2 - 2g = \chi(X) = \#V - \#E + \#F = \#V - \frac{\#E(P)}{2} + \#P$$

⚠ $\#V \neq \# V(P)$

Let $x_1, \dots, x_s$ be points in X from vertices, and $m_1 . 2\pi, \dots, m_s . 2\pi$ be their cone angles.

Define $n_i = m_i - 1$ the degree.

Lemma: $\boxed{\sum_{i=1}^{s} n_i = 2g - 2}$

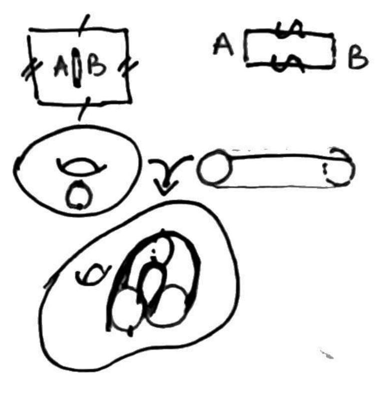
Pf: If P is a polygon with l edges,
The the total angle (sum of all inner angles) of P is $(l-2)\pi$.
We compute:

$$2\pi \sum_{i=1}^{s} n_i + 2\pi \#V = \sum_{i=1}^{s} (n_i + 1) 2\pi = \sum_{i=1}^{s} m_i 2\pi = \sum_{P_j \in P} \text{total angle of } P_j$$
$$= \sum_{P_j \in P} (l_{P_j} - 2)\pi = \pi \# E(P) - 2\pi \# P$$

So: $\sum_{i=1}^{s} n_i = -\#V + \frac{\#E(P)}{2} - \#P = -\chi(S) = 2g - 2$ $\square$

Rmk 1: $\forall i$, $m_i \geq 1$ so $n_i \geq 0$ and there is no translation surface of genus 0.

Rmk 2: $\forall g \geq 1$, there is a translation surface of genus g.
We know  of genus 1.
We can add genus with a slit construction:



With k slits (and k rectangles) on a square, we get a surface of genus $k+1$.