Kai Fu
22.04

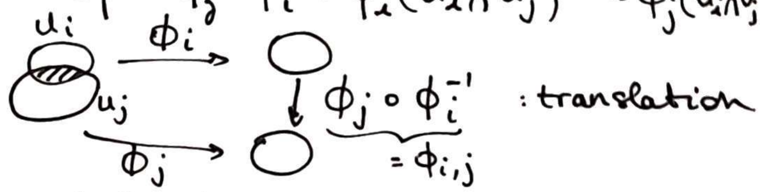
Recall: P = a finite set of polygons in $\mathbb{R}^2$
$f : E(P) \to E(P)$
$X = \bigsqcup_{P_i \in P} P_i \big/ f$ is a translation surface obtained by (P, f)

In this course today, we will see definitions which do not depend on P, f

1. Geometric definition

S: topological orientable connected surface of genus g
$\Sigma \subseteq S$: a finite subset of points

- A <u>translation atlas</u> $\mathcal{A} = \{(U_i, \phi_i)\}$ on (S, Σ) is a set of charts satisfying:
  1. (U_i) forms an open cover of $S \setminus \Sigma$
  2. Each ϕ_i is a homeomorphism from U_i to $\phi_i(U_i) \subseteq \mathbb{R}^2$
  3. For each i, j, the transition map $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a translation in $\mathbb{R}^2$.



<u>Remark</u>: In each U_i, we have $h_i = dx^2 + dy^2$
The transition maps are Euclidean isometries so they preserve (h_i) on (U_i), meaning that $\phi_{i,j}^*(h_j) = h_i$.
The family (h_i) forms a global <u>flat</u> metric on $S \setminus \Sigma$.
(If $\Sigma = \emptyset$, S must have genus 1.)

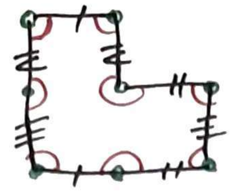
<u>Def</u> { A <u>geometric translation surface</u> X is a triple $(S, \Sigma, \mathcal{A})$ where $\mathcal{A}$ is a maximal translation atlas on (S, Σ) such that for any $p \in \Sigma$, there is a chart (U_p, ϕ_p) near p such that:

1. $\phi_p(U_p)$ is obtained by $2m_p$ half disks ($m_p \geq 1$) construction

$\phi_p(U_p) =$ 

2. $\forall q \in U_p$, there is a chart $(U, \phi) \in \mathcal{A}$ near q such that $\phi_p \circ \phi^{-1} : \phi(U \cap U_p) \to \phi_p(U \cap U_p)$ is a translation

- $(U, \phi) \in \mathcal{A}$ is called a <u>flat chart</u> on X (or flat coordinate)
- $p \in S \setminus \Sigma$ is called <u>regular</u>; $p \in \Sigma$ is called a <u>singularity</u>.
  (sometimes "fake sing." when $m_p = 1$)

Ex :  $S = P/\sim$, $\Sigma = \{\bullet\}$, $\mathcal{A}$

6 half disks

We saw last week that if X is obtained by (P, f), then X is a geometric translation surface: $S = \sqcup P_i / f$, $\Sigma = V(P)/f$

The converse is also true:

Lemma :

For any geometric translation surface $X = (S, \Sigma, \mathcal{A})$, there exist P and f such that:

1. $S = \bigsqcup_{P_i \in P} P_i / f$
2. $\Sigma = V(P)/f$
3. $\mathcal{A} = \mathcal{A}(P, f)$

2. Analytic definition

A Riemann surface X is a pair $(S, \mathcal{A})$ where $\mathcal{A}$ is a maximal atlas $\{(U_i, \phi_i)\}$ such that

1. $\phi_i : U_i \to \phi_i(U_i) \subseteq \mathbb{C}$ is a homeomorphism
2. The transition maps $\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ are biholomorphic

A holomorphic 1-form ω (or abelian differential) on a Riemann surface X assigns to each $(U, \phi) \in \mathcal{A}$ a holomorphic function $f : \phi(U) \subseteq \mathbb{C} \to \mathbb{C}$ such that in the overlapping chart:

$$\phi_i(U_i \cap U_j) \ni z_i \xrightarrow{f_i} \mathbb{C}$$

$$\phi_j(U_i \cap U_j) \ni z_j \xrightarrow{f_j} \mathbb{C}$$

$$\boxed{\phi_{ij}^*(f_j \, dz_j) = f_i \, dz_i}$$

equivalently:

$$f_j(\phi_{ij}(z_i)) \frac{d\phi_{ij}}{dz_i}(z_i) = f_i(z_i)$$

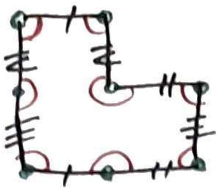
With $\phi_{ij}^* f_j = f_i$, (f_i) forms a holomorphic function on X

-> this function is a constant (holomorphic on a compact set)

Instead, $(f_i(z_i) \, dz_i)$ forms a holomorphic 1-form.

We say $\omega = f_i(z_i) \, dz_i$ under the chart (U_i, ϕ_i)

$(\omega = \phi_i^*(f(z_i) \, dz_i))$

Ex : 

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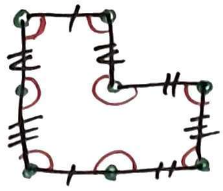
Ex :  $S = P/_\sim$, $\Sigma = \{\bullet\}$, $\mathcal{A}$

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A holomorphic 1-form ω on a Riemann surface X assigns to
(or abelian differential)
each $(U, \phi) \in \mathcal{A}$ a holomorphic function $f : \phi(U) \to \mathbb{C}$ (with $\phi(U) \subseteq \mathbb{C}$) such that in the overlapping chart :

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$\Rightarrow$ this function is a constant (holomorphic on a compact set)

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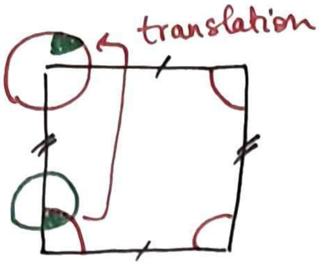


$\mathcal{A} = \mathcal{A}(P, f)$ is an atlas of a Riemann surface

Define ω :

$\omega = dz$ on $\mathring{P}$ and edges

$\omega = dz$ in the neighborhood of the vertices

Since translations preserves dz ($\phi_{ij}^*(dz) = d(z+c) = dz$), they define a holomorphic 1-form ω on X.

An <u>analytic translation surface</u> is a pair (X, ω) consisting of a Riemann surface X and a non-zero holomorphic 1-form ω.

<u>Rmk</u> (zeros of ω) In each chart $(U, \phi) \in \mathcal{A}$, $\omega = f(z)\,dz$ on $\phi(U)$

The zeros of ω on U correspond to the zeros of f.

In the overlapping charts , $\omega = f_i(z_i)\,dz_i = f_j(z_j)\,dz_j$.

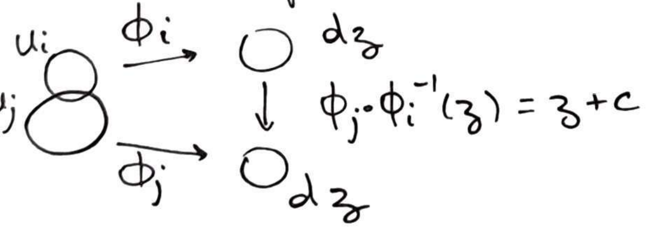
Since $f_j(\phi_{ij}(z_i)) \underbrace{\frac{d\phi_{ij}(z_i)}{dz_i}(z_i)}_{\neq 0} = f_i(z_i)$, the zeros of ω are well-defined.

<u>Lemma</u> : Geometric translation surfaces are analytic TSs.

<u>Proof</u> : Let $(S, \Sigma, \mathcal{A})$ be a geometric TS.
We want to build an atlas of a Riemann surf on S
+ a holomorphic 1. form (non-zero)

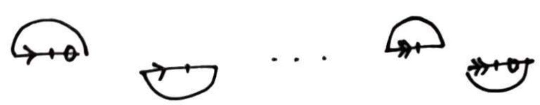
- $\mathcal{A} = \{(U_i, \phi_i)\}$ is a maximal atlas with transition maps $\in \{\text{translations}\} \subset \{\text{biholo.}\}$
then $S \setminus \Sigma$ is a Riemann surf. under $\mathcal{A}$.

+ Define a holo. 1-form ω on $S\setminus\Sigma$ by $\omega|_{(U_i, \varphi_i)} = dz$



$(d(z+c) = dz)$

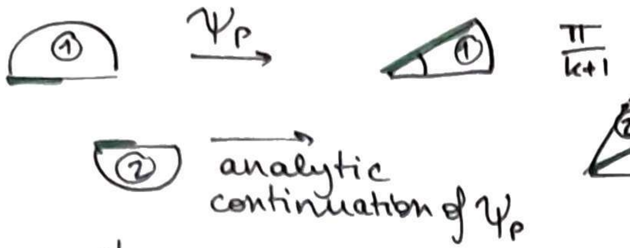
- For $p \in \Sigma$, we have a chart (U_p, ϕ_p) where

$\phi_p(U_p) =$  $2n$ half disks

We want to find $\psi_p : \phi_p(U_p) \to W_p \subseteq \mathbb{C}$

Consider the map $\psi_p(z) = \left[(k+1)z\right]^{\frac{1}{k+1}}$ (pick one branch) defined on $\frown$

where $k+1 = m$.



etc.

Finally, we build a map $\phi_p(U_p) \xrightarrow{\psi_p}$  $2\pi = 2\overbrace{(k+1)}^{=m} \cdot \frac{\pi}{k+1}$

each angle: $\frac{\pi}{k+1}$

Define a new chart near p by $(U_p, \psi_p \circ \phi_p)$.

For any $(U_i, \phi_i) \in A$, the transition map is $\underset{\text{holo}}{\psi_p} \circ \underbrace{(\phi_p \circ \phi_i)^{-1}}_{\text{translation}}$ which is biholomorphic.

+ Define a holo 1-form

Claim $\psi_p^*(z^k dz) = dz$ on each half disk.

Proof:

$$\psi_p^*(z^k dz) = \psi_p(z)^k \frac{d\psi_p(z)}{dz} dz$$

$$= ((k+1)z)^{\frac{k}{k+1}} \cdot \frac{k+1}{k+1} ((k+1)z)^{\frac{1}{k+1}-1} dz$$

$$= dz$$

So $\omega = z^k dz$ on W_p is compatible with dz on $S \setminus \Sigma$. $\square$

Rmk: (S, Σ, A)

We build A' on S and ω s.t. $\omega|_{U_i} = dz$, $\omega|_{U_p} = z^k dz$ $(k = m-1)$

Lemma: Analytic TSs are geometric TSs.

Proof: Let (X, ω) be an analytic translation surface.

Set $\Sigma = \{\text{zeros of } \omega\}$

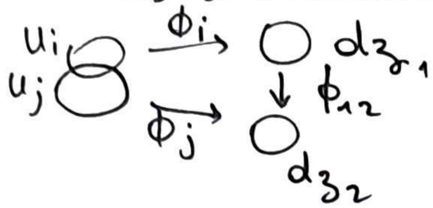
We need two statements:

1. For $p \notin \Sigma$, we can find a chart (U, ϕ) near p s.t. $\omega|_U = dz$ in the chart.
2. For $p \in \Sigma$, we can find a chart (U, ϕ) near p s.t. $\omega|_U = z^k dz$ where k is the order of p as a zero of ω.

Assume ① and ② hold.

- Define $\mathcal{A} = \{(U, \phi) \text{ near } p \notin \Sigma \text{ given by } ①\}$ on $S \setminus \Sigma$.

  Then $\mathcal{A}$ is a translation atlas because:



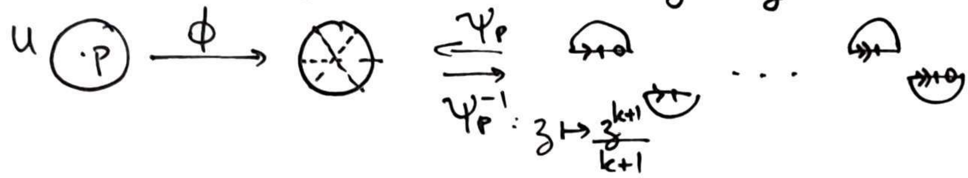
$$\phi_{12}^*(dz_2) = dz_1 \iff \frac{d\phi_{12}}{dz_1} = 1$$

compatibility condition for the holo. 1-form

$$\iff \phi_{12}(z_1) = z_1 + c$$

- It remains to build (U_p, ϕ_p) for $p \in \Sigma$

  By ②, we have (U, ϕ) and $\omega = z^k dz$



$(U, \psi_p^{-1} \circ \phi)$ gives a neighborhood of p obtained by $2(k+1)$ half disks

$\square$