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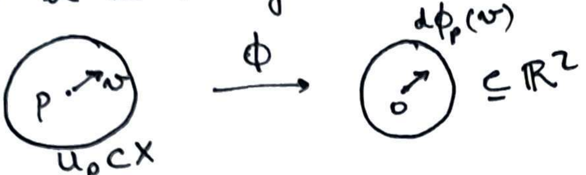
29.04

I. Straight lines flows on translation surfaces
(or directional or translation flow)

· Direction of a tangent vector

Let X be a translation surf. (translation atlas on $X \setminus \Sigma$)

Let v be a tangent vector at $p \in X \setminus \Sigma$



The <u>direction</u> of v is defined as the direction of $d\phi_p(v) = re^{i\theta}$, $\theta \in [0, 2\pi)$ which is θ.

<u>Rmk</u>: Transition maps between flat charts are translations on $\mathbb{R}^2$.
In particular, they preserve directions of vectors in $\mathbb{R}^2$ ∴ θ is the same under different choices of flat charts.

· Straight line flow in direction θ on X

Let $\theta \in [0, 2\pi)$.

Consider the flow $F_t^\theta : \mathbb{C} \to \mathbb{C}$
$z \mapsto z + te^{i\theta}$

Let v_θ be the vector field on $\mathbb{C}$ (tangent to F_t^θ) : $(v_\theta)_z = e^{i\theta}$

Let X be a translation surface.
Define a vector field V_θ on $X \setminus \Sigma$ as follows :

$$V_\theta |_{U_p} := (d\phi_p)^{-1} v_\theta$$

<u>Rmk</u>: Again, transition maps are translations.

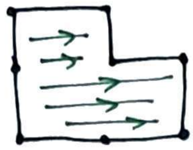
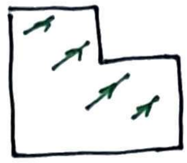
Let (U_i, φ_i), (U_j, φ_j) be two flow charts.

$$\varphi_j \circ \varphi_i^{-1}(z) = z + \alpha \quad , \quad \alpha \in \mathbb{C}$$

$$(d\varphi_j)\, d\varphi_i^{-1}(v_\theta) = v_\theta$$

Then, V_θ is well defined on $X \setminus \Sigma$.

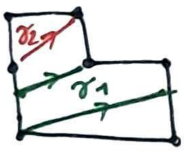
Ex : •  for $\theta \in [0, 2\pi)$

•  $\theta = 0$;  $\theta \in (0, 2\pi)$

Rmk : We have a flat metric h on $X \setminus \Sigma$.
Each orbit of ϕ_t^θ is a geodesic of h.

. Special orbits / geodesic of h

A path $\gamma : [a,b] \to X$ is called a <u>saddle connection</u> if $\gamma(a), \gamma(b) \in \Sigma$ and $\gamma|(a,b)$ is a geodesic of the flat metric on $X \setminus \Sigma$.

Ex : 

γ_1, γ_2 and each edge of the polygon is a saddle connection.

. Closed geodesic

A path $\gamma : [a,b] \to X$ is a <u>closed geodesic</u>
if $\gamma \subseteq X \setminus \Sigma$ is a $\underbrace{\text{closed geodesic}}$ of the flat metric.

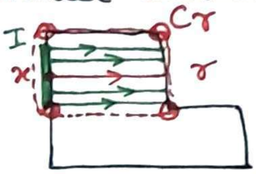
$$\begin{cases} \gamma(a) = \gamma(b) \\ \text{and} \\ \dot\gamma(a) = \dot\gamma(b) \end{cases}$$

Rmk : On a translation surface, $\gamma(a) = \gamma(b) \Rightarrow \dot\gamma(a) = \dot\gamma(b)$
Since γ is a geodesic of the flat metric h, in each local chart (U_i, φ_i), γ is a straight line

$\subseteq \mathbb{C}$

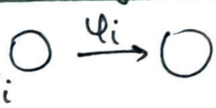
$d\varphi_i(\dot\gamma(t))$ are the same vector on $\mathbb{C}$ along the straight line.
Therefore, if $\gamma(a) = \gamma(b)$, the tangent vector coincide.

Rmk : If γ is a closed geodesic on X, then there is a family of parallel closed geodesic.



This family is a cylinder.
The maximal family of closed geodesic parallel to γ forms a cylinder C_γ and ∂C_γ consists of saddle connections.
(if there is no singularity in ∂C_γ, we can extend C_γ)

Rmk : $(X, \phi_t^\theta, \text{Leb})$
Leb is the <u>volume form</u> of the flat metric h

In local chart (U_i, φ_i) 

$h = dx^2 + dy^2$
$\text{Leb} = dx \wedge dy$

Claim : Φ_t^θ preserves the volume form Leb .

Proof : It suffices to show that Φ_t^θ preserves the flat metric h

$$U_i \xrightarrow{\Phi_t} U_j$$

$$z \xmapsto{F_t^\theta} z + te^{i\theta}$$

F_t^θ is a translation and hence a Euclidean isometry .