Interval Exchange Transformations

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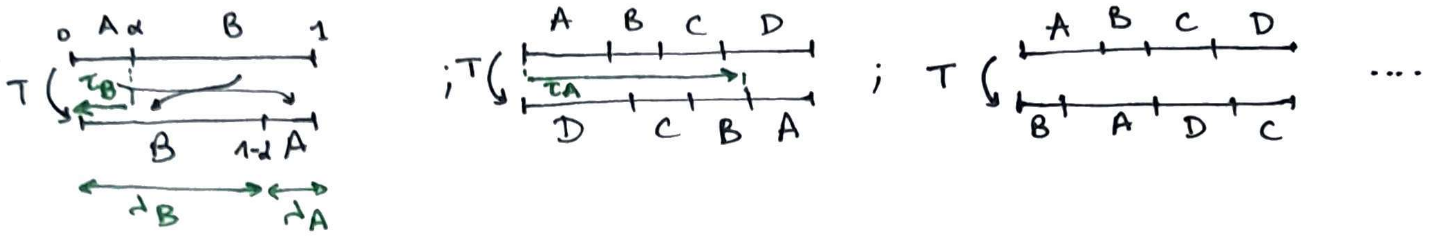
I. Definition

- Let $I \subset \mathbb{R}$ be an interval. An interval exchange transformation (IET) on I is a bijection $T : I \setminus D_1 \to I \setminus D_2$ where:
  - D_1 and D_2 are finite (of the same cardinality)
  - and on each connected component of $I \setminus D_1$, T is a translation

Ex:



There are two classical ways to describe an IET:

1. Give the tuples $\{\beta_i\} = D_1$ (including the endpoints of I) and $\{\tau_i\}$ of translation lengths (= length of the translation on each C.C. of $I \setminus D_1$)

or

2. Give the tuple $\{\lambda_i\}_{i \in [\![1,d]\!]}$ of lengths: the length of each C.C. of $I \setminus D_1$ and the bijection $\pi : [\![1,d]\!] \to [\![1,d]\!]$ of how T permutes the C.C. of $I \setminus D_1$

Note that the order in $\{\tau_i\}$ and in $\{\lambda_i\}$ is important!

Ex:
- $\{\beta_i\} = \{0, \alpha, 1\}$ corresponds to the first example above
  $\{\tau_i\} = \{1-\alpha, -\alpha\}$
- $\{\lambda_i\} = \{\alpha, 1-\alpha\}$
  $\pi = (1\ 2)$

We will mostly use (if not only) the second way, with a slight change: We consider an alphabet $\mathcal{A}$ of d symbols with which we label the C.C's. We give a pair of bijections (π_0, π_1), $\pi_i : \mathcal{A} \to [\![1,d]\!]$ s.t. $\pi = \pi_1 \circ \pi_0^{-1}$

Notation: $\pi = \begin{pmatrix} \pi_0^{-1}(1) & \dots & \pi_0^{-1}(d) \\ \pi_1^{-1}(1) & \dots & \pi_1^{-1}(d) \end{pmatrix}$

Giving (π_0, π_1) (and $\{\lambda_i\}$) is redundant: the product $\pi = \pi_1 \circ \pi_0^{-1}$ (and $\{\lambda_i\}$) is enough; but this is convenient (for the notation and for other tools that we will see later).

Ex: $\pi_0 : \begin{matrix} A \to 1 \\ B \to 2 \end{matrix}$ $\quad \pi = \begin{pmatrix} A & B \\ B & A \end{pmatrix}$; $\pi = \begin{pmatrix} A & B & C & D \\ D & C & B & A \end{pmatrix}$; $\pi = \begin{pmatrix} A & B & C & D \\ B & A & D & C \end{pmatrix}$

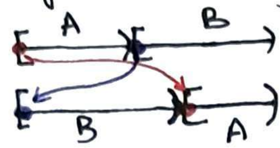
$\pi_1 : \begin{matrix} A \to 2 \\ B \to 1 \end{matrix}$ $\quad$ permutations corresponding to the above examples

Rmk: To get a dynamical system, we need to be able to iterate the transformation T, which requires that the codomain of T is the same as (or at least included in) the domain of T.

We can:

1. Take segments that are open on the right, closed on the left so that we define T everywhere on $I = [a, b)$

e.g.



or

2. Remove the backward orbit of each point in D_1 / forward orbit of each point in D_2 $\Big\} \to$ get $I \setminus \Delta$

Δ is countable (in particular $\mathrm{Leb}_{\mathbb{R}}(\Delta) = 0$)

Explanation: T is not defined on D_1, if x is such that $T(x) \in D_1$ then we can define $T(x)$ but not $T^2(x)$ so we remove x.
But then $y = T^{-1}(x)$ (assume it is defined) has again a finite well defined orbit $T(y)\ (= x)$, $T^2(y)\ (= T(x))$ and $T^3(y)$ is not def.
This is why we remove y, and, inductively, all backward orbits of D_1. Denote $D_1^\infty = \bigcup_{n \in \mathbb{N}} T^{-n}(D_1)$.

Now we have $T : I \setminus D_1^\infty \to I \setminus D_2 \setminus D_1^\infty$ and we can iterate

For the sake of symmetry we take $T : I \setminus \Delta \to I \setminus \Delta$

with $\Delta = \bigcup_{n \in \mathbb{N}} T^{-n}(D_1) \cup \bigcup_{n \in \mathbb{N}} T^{n}(D_2)$

II. From a translation surface to an IET

Proposition: Let S be a translation surface, and φ_t its vertical flow. Let $I \subset S$ be an interval transverse to φ_t.
Then, the first return of φ_t to I defines an IET on I.

$$T : I \to I, \quad x \mapsto \varphi_{t_x}(x) \qquad \text{where } t_x = \min\{t > 0 \mid \varphi_t(x) \in I\}$$

T might be not defined (if t_x is not finite). We will show that it is almost everywhere defined using the

Poincaré recurrence theorem

Let (X, μ) be a measured space and $T : X \to X$ be a μ-preserving map.
Let $A \subset X$ be a measurable set. (= $\forall A$ measurable set, $\mu(T^{-1}(A)) = \mu(A)$)
For almost every $x \in A$: $\forall N \in \mathbb{N}, \exists n \geq N$ s.t. $T^n(x) \in A$.

Proof of Poincaré recurrence theorem:

We have

$$\{x \in X \mid \forall N \in \mathbb{N}, \exists n \geq N, T^n(x) \in A\} = \bigcap_{N \in \mathbb{N}} \underbrace{\bigcup_{n \geq N} T^{-n}(A)}_{=: B_N}$$

We want to show that $\mu\left(A \setminus \bigcap_{N \in \mathbb{N}} B_N\right) = 0$

Note that (1) $B_{N+1} \subset B_N \subset B_0$

(2) $B_{N+1} = T^{-1}(B_N)$ so $\mu(B_{N+1}) \underset{\uparrow}{=} \mu(B_N) \underset{\uparrow}{=} \mu(B_0)$

$\boxed{\text{T preserves } \mu}$ induction

hypothesis of the thm

We have:

$$\mu\left(A \setminus \bigcap_{N \in \mathbb{N}} B_N\right) \underset{\substack{\uparrow \\ \text{same sets}}}{=} \mu\left(\bigcup_{N \in \mathbb{N}} A \setminus B_N\right) \underset{\substack{\uparrow \\ A \subset B_0}}{\leq} \mu\left(\bigcup_{N \in \mathbb{N}} B_0 \setminus B_N\right)$$

$$\leq \sum_{N \in \mathbb{N}} \underbrace{\mu(B_0 \setminus B_N)}$$

$= \mu(B_0) - \mu(B_N)$ because (1)

$= 0$ because (2)

$\square$

Rmk: It works as well for continuous dynamical system.

(e.g. translation flow φ_t)

for almost every $x \in A$, $\forall T_0 \geq 0$, $\exists t \geq T_0$ s.t. $\varphi_t(x) \in A$