08.04.26

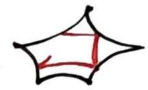
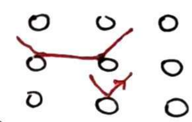
Introduction : the billiards, a motivation to study translation surfaces and interval exchange transformations

I. General setting

A mathematical billiards is a <u>dynamical system</u> (= evolving system, formal definition later) with a ball moving on a billiard table. It is idealised :

- → no effect on the ball : the trajectory is a broken line
- → elastic collisions : the incident angle = the reflected angle
- → no loss of energy : the trajectory goes for ever

There are lots of billiards, studied with different techniques.

- on smooth convex shapes $\subset \mathbb{R}^2$ → 
- on convex hyperbolic polygons $\subset \mathbb{H}^2$ → 
- dispersing billiards, as Sinai billiards → 
- polygons, <u>rational polygons</u> $\subset \mathbb{R}^2$ → 
- even more : outer billiards, tiling billiards ...

Here we will focus on rational polygons.
We take the convention that a trajectory stops if it goes into a corner.

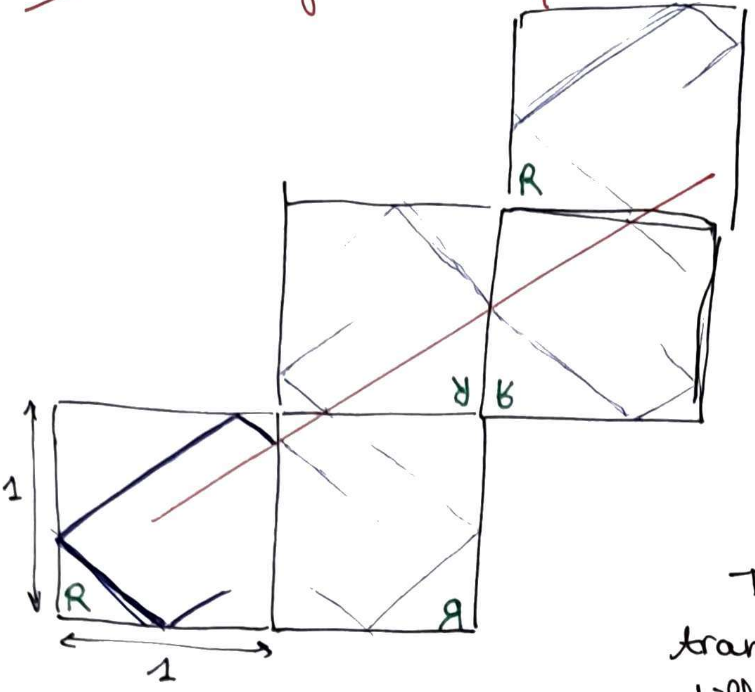
<u>Questions</u>: What can we say about the trajectories and their long-term behavior ?

- → Are they <u>periodic</u> ? = comes back at the starting point with the same direction
- → Are they <u>dense</u> ? = $\forall\, U$ open set of the table, $\forall T \in \mathbb{R}$, $\exists\, t \geq T$ such that the trajectory is in U at time t.
- → If so, do they <u>uniformely distribute</u> on the table ?

$\forall\, U$ measurable open set of the table,
$$\frac{\mathrm{Leb}_{\mathbb{R}}\left(\{t \in [0,T] \mid \text{the trajectory is in } U \text{ at time } t\}\right)}{T} \xrightarrow[T \to +\infty]{} \frac{\mathrm{Leb}_{\mathbb{R}^2}(U)}{\mathrm{Leb}_{\mathbb{R}^2}(\text{table})}$$

We will begin to investigate these questions in the case of a square table.

II. Case of the square billiards



We unfold the billiards: each time the trajectory bounces against a side, we consider the symmetric table with respect to this side.

We get a line in $\mathbb{R}^2$ that corresponds to the trajory and a subset of the square tiling of $\mathbb{R}^2$.

There are 4 different copies (up to translation) corresponding to 4 different directions of the trajectory.

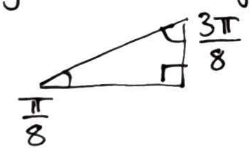
R : ↗ , Я : ↖ , ꓤ : ↙ and ᴚ : ↘

Q° : What does it mean for the line that the trajectory is periodic?

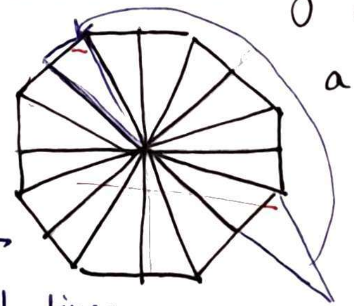
A : ~~It is equivalent to~~ starts at $x \in \mathbb{R}^2$ and goes through $x + 2\binom{p}{q}$ for some $p, q \in \mathbb{Z}$, which is equivalent to have a rational slope.

- What other polygons can be unfold?

e.g. A rectangle gives another tiling of $\mathbb{R}^2$



gives



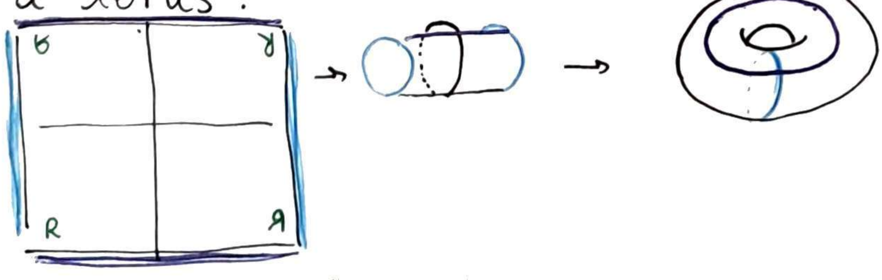
a regular octogon, which does not tile $\mathbb{R}^2$.

But these 16 copies are enough, all the others would translated copies of them.

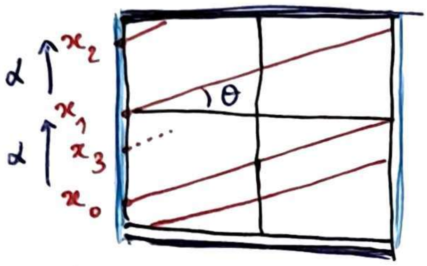
This is an example of a translation surface that Kai will define next time.

We can stop unfolding after finitely many copies if and only if the angles of the polygon are all in $\pi\mathbb{Q}$.

Going back to the square: its 4 copies and their identifications give a torus.



We can again symplify the dynamical system we are working with by looking at the sequence of crossing points of the trajectory and a segment (here, a circle)



The transformation $x_i \mapsto x_{i+1}$ is a rotation (EXERCISE) of angle α (EX: relate to θ)

In general, we will get an <u>interval exchange transformation</u> (defined later in the course)

This gives a more precise answer to our question:

<u>1st case</u>: $\alpha \in \mathbb{Q}$. Every orbit is periodic

<u>2nd case</u>: $\alpha \notin \mathbb{Q}$. Every orbit is dense (= the system is <u>minimal</u>)
More precisely, the system is <u>uniquely ergodic</u>.

III. Some ergodic theory

- A <u>dynamical system</u> is a (X, φ_t) with X a set (here: topological space)
  $\varphi_t : X \to X$ (here: continuous)
  $\varphi_{t+s} = \varphi_t \circ \varphi_s$

<u>Eg.</u>: (1) $X = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, φ_α = rotation of angle $\alpha \in \mathbb{R}$ (discrete time)

(2) X = unit tangent bundle of the square with identification

$$\varphi_t(x,v) = \begin{cases} (x+tv, v) & \text{until } x+tv \text{ lies on the boundary} \\ (x+tv, v) \sim (x+tv, r(v)) & \text{when it lies on the boundary} \end{cases}$$

(continuous time)

- A measure μ on X is <u>φ-invariant</u> if $\mu(\varphi^{-1}(A)) = \mu(A)$ for every measurable set A

We can transport the measure with the flow, it stays the same: $\varphi_* \mu = \mu$.

<u>E.g.</u>: (1) the Lebesgue mes. on $\mathbb{S}^1$ is invariant for φ_α

(2) Leb $\times\, d\theta$ on X (2^{nd} example) is φ_t - invariant

- A measure μ is <u>ergodic</u> for φ if $\forall$ measurable set A, $\underbrace{\varphi^{-1}(A) = A}_{A \text{ is } \varphi\text{-invariant}} \Rightarrow \begin{cases} \mu(A) = 0 \\ \text{or} \\ \mu(A^c) = 0 \end{cases}$

W.r.t. μ, we cannot decompose the system.

<u>E.g.</u>: (1) Leb is ergodic iff $\alpha \notin \mathbb{Q}$. (EXERCISE)

(2) Leb $\times\, d\theta$ is not ergodic (We can decompose the system with the direction of the trajectory)

- (X, φ) is <u>uniquely ergodic</u> if there exists a unique ergodic measure for φ.

<u>E.g.</u>: (1) If $\alpha \notin \mathbb{Q}$, $(\mathbb{S}^1, \varphi_\alpha)$ is uniquely ergodic

IV. A final word : Veech dichotomy

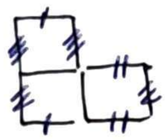
We have seen that for the square billiards, each direction is <u>either</u> <u>completely periodic</u>
<u>or</u> <u>uniquely ergodic</u>.

This is called the <u>Veech dichotomy</u>.

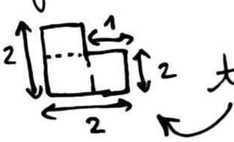
It is an open question to know which surfaces satisfy this and which ones do not, although we have sufficent condition.

For example, a <u>square tiled surface</u> satisfies it.

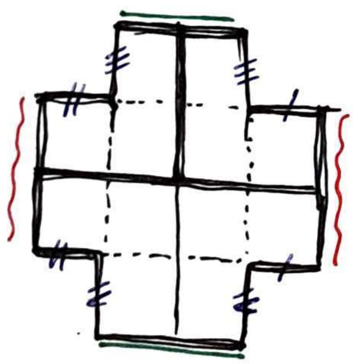
= a surface that can be decomposed with a finite collection of squares with each top side identified to a bottom side
— left — — right —

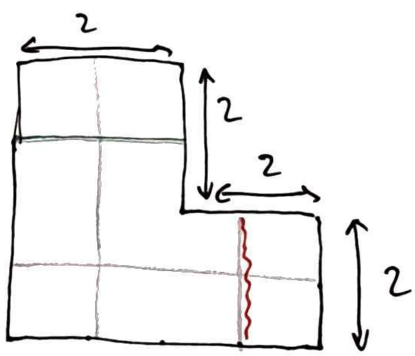
e.g. 

<u>Remark</u>: A priori, a square tiled surface does not necessarily comes from the unfolding of a billiards.

But here, if we unfold 

this billiards, we get



which can be rearranged into 

a bigger L-shape!