Grand zigzag knight's paths

ALEA 2024

Laboratoire d'Informatique de Bourgogne & Universidad Nacional de Colombia

Jean-Luc Baril, Nathanaël Hassler, Sergey Kirgizov, José L. Ramìrez

Table of content

Knight's moves

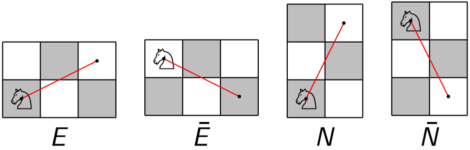

Figure – The four possible right-moves of a knight on a chessboard.

Knight's paths

Definition

A *grand knight's path* is

- a lattice path in $\mathbb{Z}^2$,
- starting at the origin,
- consisting of steps $N = (1,2)$, $\bar{N} = (1,-2)$, $E = (2,1)$, and $\bar{E} = (2,-1)$.

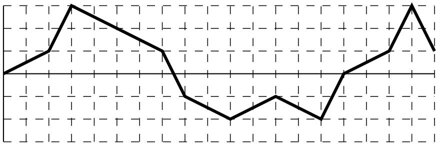

Figure – The grand knight's path $EN\bar{E}\bar{E}\bar{N}\bar{E}EE\bar{E}NEN\bar{N}$.

Grand zigzag knight's paths

Definition

A *grand zigzag knight's path* is

- a grand knight's path,
- such that two consecutive steps cannot be in the same direction.

Equivalently, two consecutive steps cannot be NN, NE, $\bar{N}\bar{N}$, $\bar{N}\bar{E}$, EE, EN, $\bar{E}\bar{E}$, $\bar{E}\bar{N}$.

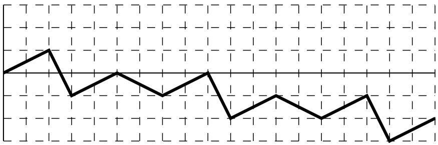

Figure – The grand zigzag knight's path $E\bar{N}E\bar{E}E\bar{N}E\bar{E}E\bar{N}E$.

The number of grand zigzag knight's paths ending at (n, k)

Let $\mathcal{Z}_{n,k}$ be the set of grand zigzag knight's paths of size n ending at height k, and $\mathcal{Z}^{+}_{n,k}$ (resp. $\mathcal{Z}^{-}_{n,k}$) be the subset of $\mathcal{Z}_{n,k}$ of paths starting with E or N (resp. $\bar{E}$ or $\bar{N}$).

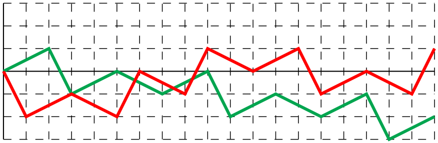

Figure – A path in $\mathcal{Z}^{+}_{19,-2}$, and a path in $\mathcal{Z}^{-}_{19,1}$.

Generating functions

Let

$$F(u,z) = \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} |\mathcal{Z}_{n,k}^{+}| u^k z^n,$$

$$G(u,z) = \sum_{k=0}^{+\infty} \sum_{n=0}^{+\infty} |\mathcal{Z}_{n,k}^{-}| u^k z^n,$$

$$H(u,z) = F(u,z) + G(u,z).$$

Functional equations

Theorem

F and G satisfy the following equations

$$F(u,z) = \left(1 + \frac{z^3 u}{1-z}\right) f_0(z) + zu(z+u)(G(u,z)+1),$$
$$G(u,z) = -\left(\frac{z}{u^2} + \frac{z^2}{u} + \frac{z^3}{u(1-z)}\right) f_0(z) + \left(\frac{z}{u^2} + \frac{z^2}{u}\right) F(u,z),$$

with $f_0(z) = \sum_{n=0}^{+\infty} |\mathcal{Z}^+_{n,0}| z^n$.

Kernel method

The previous equations can be restated as follows :

$$K(u,z)F(u,z) = a(u,z)f_0(z) + b(u,z),$$
$$K(u,z)G(u,z) = c(u,z)f_0(z) + d(u,z),$$

with $K(u,z)$ a polynomial that we call the **kernel** of the equation. The unknowns are in blue.

Kernel method

The previous equations can be restated as follows :

$$K(u,z)F(u,z) = a(u,z)f_0(z) + b(u,z),$$
$$K(u,z)G(u,z) = c(u,z)f_0(z) + d(u,z),$$

with $K(u,z)$ a polynomial that we call the **kernel** of the equation. The unknowns are in blue. Let $r(z)$ be a root of the kernel : $K(r(z),z) = 0$. Thus,

$$f_0(z) = -\frac{a(r(z),z)}{b(r(z),z)}.$$

The generating function

Theorem

$$H(u,z) = -\frac{u^2 + u(r(z) + z + z^2) + z^2 s(z)(2f_0(z) - 1)}{z^2(u - s(z))},$$

with

$$s(z) = r(z)^{-1} = \frac{1 - z^4 - z^2 + \sqrt{z^8 - 2\,z^6 - z^4 - 2\,z^2 + 1}}{2z^3},$$

and

$$f_0(z) = \frac{r(z)(z-1)}{z^3(r(z)z^2 + z - 1)}.$$

Example

k \ n	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	2	0	4	2	10	6	22	16	52	44	126
1	0	0	1	2	2	4	4	10	11	26	28	64	71
2	0	1	0	1	0	3	2	7	6	16	18	40	52
3	0	0	0	0	1	0	2	0	6	2	16	8	41
4	0	0	0	0	0	0	0	1	0	3	0	10	2

Table – The number of grand zigzag knight's paths from $(0,0)$ to (n,k) for $(n,k) \in [\![0,15]\!] \times [\![0,4]\!]$.

Example

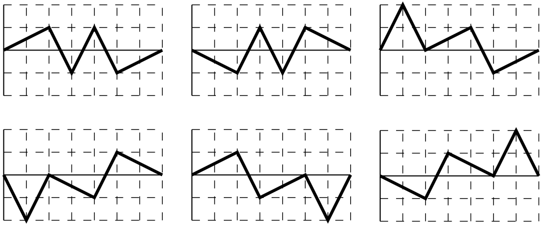

Figure – The 6 grand zigzag knight's paths of size 7 ending on the x-axis.

Average final height

Theorem

An asymptotic approximation for the expected final height of a grand zigzag knight's path of size n ending on or above the x-axis is

$$\frac{(1+\sqrt{5})\sqrt{7\sqrt{5}-15}}{2\sqrt{5}}\sqrt{\frac{n}{\pi}}.$$

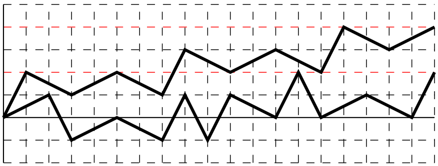

Figure – The final height of two grand zigzag knight's paths of size 19.

Pairs of compositions

Definition

Let $\mathcal{C}_{n,m}$ be the set of ordered pairs (X, Y) such that

- X is a composition of n,
- Y is a composition of m,
- all parts are equal to 1 or 2,
- X and Y have the same number of parts.

Example:

Let $X = (1, 2, 2, 1, 1)$ and $Y = (2, 2, 1, 2, 1)$. Then $(X, Y) \in \mathcal{C}_{7,8}$.

Cardinal of $C_{n,m}$

Lemma

If $n \leq m$, then

$$|\mathcal{C}_{m,n}| = |\mathcal{C}_{n,m}| = \sum_{i=0}^{n-\lceil m/2 \rceil} \binom{n-i}{i} \binom{n-i}{m-n+i}.$$

Bijection with pairs of compositions

Proposition

If $n, k \in \mathbb{N}$ with $n = k \mod 2$, then there is a bijection ϕ between $\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ and $\mathcal{Z}^+_{n,k}$, and a bijection ψ between $\mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$ and $\mathcal{Z}^-_{n,k}$.

Proof. Let $X = (x_1, \ldots, x_i)$ and $Y = (y_1, \ldots, y_i)$ such that $(X, Y) \in \mathcal{C}_{\frac{n-k}{2},\frac{n+k}{2}}$. We define $\phi(X, Y)$ as the path $\phi(x_1, y_1) \cdots \phi(x_k, y_k)$, where

$$\phi(x_j, y_j) = \begin{cases} E\bar{E}, & \text{if } x_j = y_j = 2, \\ N\bar{N}, & \text{if } x_j = y_j = 1, \\ N\bar{E}, & \text{if } x_j = 1 \text{ and } y_j = 2, \\ E\bar{N}, & \text{if } x_j = 2 \text{ and } y_j = 1. \end{cases}$$

Example

$$X = (2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 2,\ 1)$$
$$Y = (1,\ 2,\ 1,\ 2,\ 2,\ 1,\ 2,\ 1,\ 2)$$
$$\phi(X, Y) = E\bar{N}\,E\bar{E}\,E\bar{N}\,N\bar{E}\,N\bar{E}\,N\bar{N}\,N\bar{E}\,E\bar{N}\,N\bar{E}$$

Example

$$X = (2,\ 2,\ 2,\ 1,\ 1,\ 1,\ 1,\ 2,\ 1)$$
$$Y = (1,\ 2,\ 1,\ 2,\ 2,\ 1,\ 2,\ 1,\ 2)$$
$$\phi(X,Y) = E\bar{N}\ E\bar{E}\ E\bar{N}\ N\bar{E}\ N\bar{E}\ N\bar{N}\ N\bar{E}\ E\bar{N}\ N\bar{E}$$

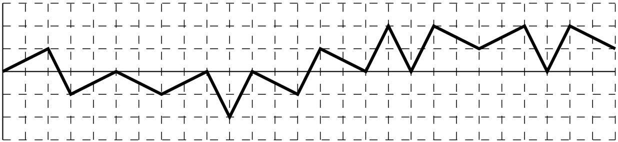

Figure – The path $\phi(X, Y)$ is a grand zigzag knight's path of size 27 and final height 1.

Grand zigzag knight's paths with i steps

For $n/2 \leq i \leq n$, let $\mathcal{Z}^{i}_{n,k}$ be the subset of $\mathcal{Z}_{n,k}$ consisting of paths with i steps. We define similarly $\mathcal{Z}^{i,+}_{n,k}$ and $\mathcal{Z}^{i,-}_{n,k}$.

Theorem

For $n = k \mod 2$ and i even,

$$|\mathcal{Z}^{i,+}_{n,k}| = |\mathcal{Z}^{i,-}_{n,k}| = \binom{i/2}{\frac{n-i-k}{2}}\binom{i/2}{\frac{n-i+k}{2}}.$$

For $n \neq k \mod 2$ and i odd,

$$|\mathcal{Z}^{i,+}_{n,k}| = \binom{\frac{i+1}{2}}{\frac{n-k-i}{2}+1}\binom{\frac{i-1}{2}}{\frac{n+k-i}{2}-1},$$

$$|\mathcal{Z}^{i,-}_{n,k}| = \binom{\frac{i-1}{2}}{\frac{n-k-i}{2}-1}\binom{\frac{i+1}{2}}{\frac{n+k-i}{2}+1}.$$

Closed form for the number of grand zigzag knight's paths

Corollary:
If $n = k \mod 2$ with $(n, k) \neq (0, 0)$,

$$|\mathcal{Z}_{n,k}| = 2 \sum_{\substack{i=0 \\ i \text{ even}}}^{n-k} \binom{i/2}{\frac{n-i-k}{2}} \binom{i/2}{\frac{n-i+k}{2}}.$$

If $n \neq k \mod 2$,

$$|\mathcal{Z}_{n,k}| = \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-k+1} \left[\binom{\frac{i+1}{2}}{\frac{n-k-i}{2}+1} \binom{\frac{i-1}{2}}{\frac{n+k-i}{2}-1} + \binom{\frac{i-1}{2}}{\frac{n-k-i}{2}-1} \binom{\frac{i+1}{2}}{\frac{n+k-i}{2}+1} \right].$$

Average number of steps

Theorem

An asymptotic approximation for the expected number of steps of a grand zigzag knight's path ending on the x-axis of size $2n$ is

$$\frac{1+\sqrt{5}}{2\sqrt{5}}(2n).$$

Grand zigzag knight's paths staying above a horizontal line

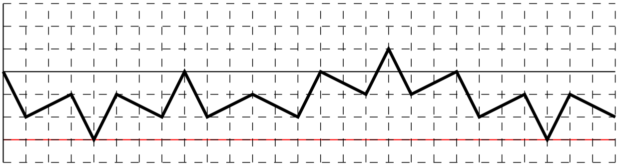

Figure – A grand zigzag knight's path staying above the line $y = -3$, with size 27, and ending at height -2.

Bijection with pairs of compositions

When $n = k \mod 2$, there is a bijection between those paths ending at (n, k) and pairs of compositions $(X, Y) \in \mathcal{C}_{\frac{n-k}{2}, \frac{n+k}{2}}$ such that for all j,

$$-m \leq \sum_{i=1}^{j} (y_i - x_i).$$

Generating functions

Theorem
The bivariate generating functions for the number of grand zigzag knight's paths staying above $y = -m$ and ending with respectively an up-step and a down-step with respect to the size and the final height are

$$F_m(u,z) = -\frac{u(u^m(1+z^2u+zu^2) - r^{m-1}z(u+z)(z+rz^2+r^2))}{z^3(u-r)(u-s)},$$

$$G_m(u,z) = \frac{r^{m-1}(z+r^2+z^2r) - zu^{m-1}(1+zu)(1+z^2u+zu^2)}{z^3(u-r)(u-s)}.$$

Probability that a path stays above a horizontal line

Proposition

For $m \geq 0$, the probability that a grand zigzag knight's path chosen uniformly at random among all grand zigzag knight's paths of size n stays above the line $y = -m$ is asymptotically $c_m/\sqrt{n}$, with

$$c_m = \begin{cases} \frac{2+\sqrt{5}}{2}\sqrt{\frac{7\sqrt{5}-15}{\pi}}, & \text{if } m = 0, \\ \frac{4m+3-\sqrt{5}}{4(\sqrt{5}-2)}\sqrt{\frac{7\sqrt{5}-15}{\pi}}, & \text{if } m \geq 1. \end{cases}$$

Grand zigzag knight's paths staying in a tube

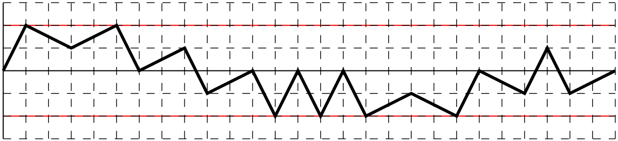

Figure – A grand zigzag knight's path staying between the lines $y = -2$ and $y = +2$, with size 27 and ending at height 0.

Bijection with pairs of compositions

When $n = k \mod 2$, there is a bijection between those paths ending at (n, k) and pairs of compositions $(X, Y) \in \mathcal{C}_{\frac{n-k}{2}, \frac{n+k}{2}}$ such that for all j,

$$\left| \sum_{i=1}^{j} (y_i - x_i) \right| \leq m.$$

Generating function

Theorem

The generating functions $F_{m,m}(u,z)$ and $G_{m,m}(u,z)$ counting the number of grand zigzag knight's paths staying between $y=-m$ and $y=+m$ with respect to the size and the final height are given by :

$$F_{m,m}(u,z) = -\frac{(z(\mathbb{1}_{[m=1]}-f_{-m+1})u^{2m+1} - z^2(u+z)f_{-m+1} + u^m(1+z^2u+zu^2))u}{z^3(u-r)(u-s)},$$

$$G_{m,m}(u,z) = -\frac{z(1+zu)(\mathbb{1}_{[m=1]}-f_{-m+1})u^{2m+1} - uf_{-m+1} + u^m(1+zu)(1+z^2u+zu^2)}{z^2(u-r)(u-s)}.$$

A bijection in a small case

Proposition
There is an explicit bijection Φ between grand zigzag knight's paths from $(0,0)$ to $(2n+4,0)$, starting with E and staying between $y=-1$ and $y=1$ and compositions of n with parts in $\{2,1,3,5,7,9,11\ldots\}$.

Proof. Indeed, such a path can be uniquely decomposed as $E\mathbf{P}\bar{E}$ with $\mathbf{P}$ of size $2n$ and having its steps in $\{\bar{E}E\}\cup\bigcup_{k\geq 0}\{\bar{N}(E\bar{E})^kN\}$. If $\mathbf{P}=S_1\cdots S_j$, then we set $\Phi(E\mathbf{P}\bar{E})=(\Phi(S_1),\ldots,\Phi(S_j))$ with $\Phi(\bar{E}E)=2$ and $\Phi(\bar{N}(E\bar{E})^kN)=2k+1$.

Example

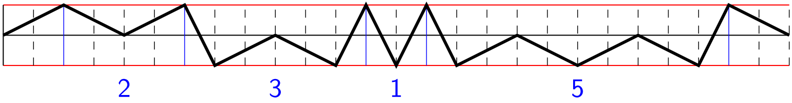

Figure – The path $E\bar{E}E\bar{N}E\bar{E}N\bar{N}N\bar{N}E\bar{E}E\bar{E}N\bar{E}$ of size $26 = 2 \times 11 + 4$ is mapped to the composition $(2, 3, 1, 5)$ of 11.

Example

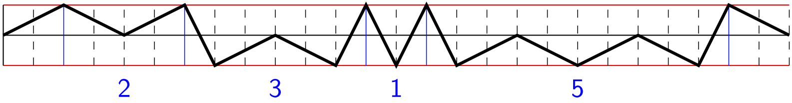

Figure – The path $E\bar{E}E\bar{N}E\bar{E}N\bar{N}N\bar{N}E\bar{E}E\bar{E}N\bar{E}$ of size $26 = 2 \times 11 + 4$ is mapped to the composition $(2, 3, 1, 5)$ of 11.

Thank you for your attention !