Intervals in a family of Fibonacci lattices

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28 mai 2025

Overview

Lattices and intervals

Definition

A *lattice* is a poset in which each pair of elements admits a *meet* (greatest lower bound) and a *join* (lowest upper bound).

Definition

In a poset $(\mathcal{P}, \leq)$, an *interval* $[P, Q]$ is a set of the form

$$\{R \in \mathcal{P} \mid P \leq R \leq Q\}.$$

If $[P, Q] = \{P, Q\}$, then this interval is called a *covering*.

Hasse diagram

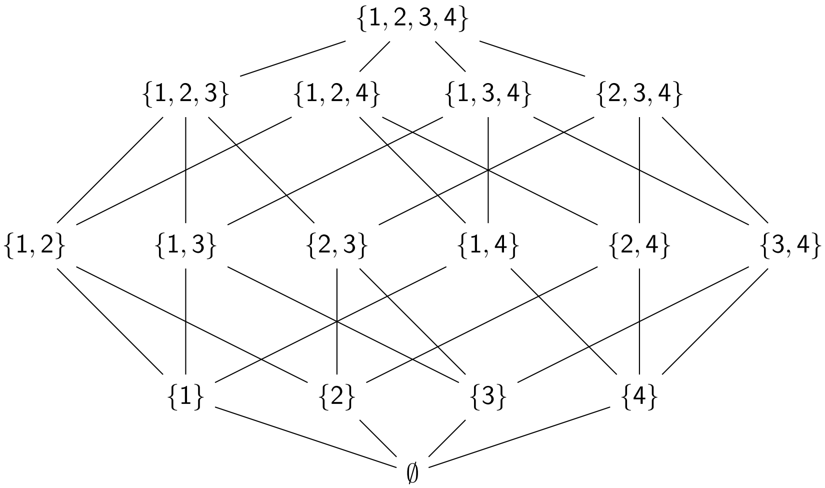

Figure – The Hasse diagram of the boolean lattice of size 4.

Dyck path

Definition

A *Dyck path* of semilength n is a lattice path starting at the origin $(0,0)$, ending at $(2n,0)$, and never going below the x-axis, consisting of up steps $U=(1,1)$ and down steps $D=(1,-1)$.

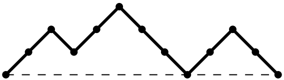

Figure – The Dyck path *UUDUUDDDUUDD*

Theorem

The Dyck paths of semilength n are enumerated by the *Catalan number*

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Examples of lattices

Examples of lattices enumerated by the Catalan numbers :

- the Stanley lattice [*Stanley*, 1975]
- the Tamari lattice [*Friedman, Tamari*, 1967]
- the Kreweras lattice [*Kreweras*, 1972]
- the Phagocyte lattice [*Baril, Pallo*, 2006]
- the Pruning-grafting lattice [*Baril,Pallo*, 2008]
- the Pyramid lattice [*Baril, Kirgizov, Naima*, 2023]
- the Ascent lattice [*Baril, Bousquet-Mélou, Kirgizov, Naima*, 2024]

Examples of intervals in the Stanley lattice

The Stanley lattice. The Stanley lattice Stan_n is the lattice on Dyck paths of semilength n where $P \leq Q$ if P is always under Q when we draw them together.

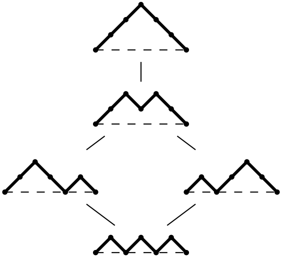

Figure – The Hasse diagram of Stan_3.

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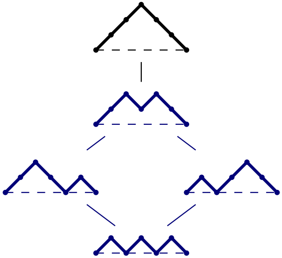

Figure – The Hasse diagram of Stan_3.

Examples of lattices on Dyck paths

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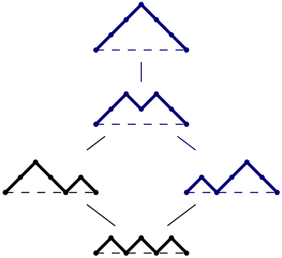

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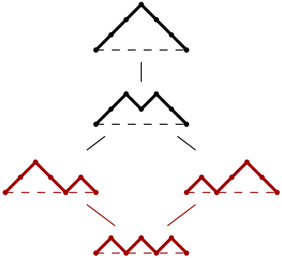


Figure – The Hasse diagram of Stan_3.

Enumeration of intervals

Intervals in Stan_n [*De Sainte-Catherine, Viennot, 1986*] :

$$\frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}.$$

Intervals in Tam_n [*Chapoton, 2006*] :

$$\frac{2(4n+1)!}{(n+1)!(3n+2)!}.$$

Linear intervals in both Stan_n and Tam_n [*Chenevrière, 2022*] :

$$\frac{1}{n+1}\binom{2n}{n} + \binom{2n-1}{n-2} + 2\binom{2n-1}{n+2}.$$

Generalized Fibonacci numbers

Definition

The p-generalized Fibonacci sequences are defined for every $p \geq 2$ by

$$F_n^p = F_{n-1}^p + F_{n-2}^p + \cdots + F_{n-p}^p$$

with initial conditions $F_i^p = 0$ for $i < 0$, and $F_0^p = 1$.

Dyck paths enumerated by the Fibonacci numbers

Definition

For $p \geq 2$, let $\mathcal{F}^p$ (resp. $\mathcal{F}^p_n$) be the set of Dyck paths (resp. of semilength n) avoiding the patterns DUU and D^{p+1}.

Definition

Let $\mathcal{F}^\infty$ (resp. $\mathcal{F}^\infty_n$) be the set of Dyck paths (resp. of semilength n) avoiding the pattern DUU.

Remark : For any $n \in \mathbb{N}$,

$$\mathcal{F}^2_n \subseteq \mathcal{F}^3_n \subseteq \cdots \subseteq \mathcal{F}^p_n \subseteq \mathcal{F}^{p+1}_n \subseteq \cdots \subseteq \mathcal{F}^\infty_n,$$

$$|\mathcal{F}^p_n| = F^p_n, \quad \text{and} \quad |\mathcal{F}^\infty_n| = 2^{n-1}.$$

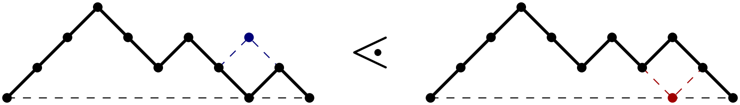
Lattice on $\mathcal{F}_n^p$

Let $\leq$ be the Stanley order.

Definition-Proposition

$\mathbb{F}_n^p = (\mathcal{F}_n^p, \leq)$ and $\mathbb{F}_n^\infty = (\mathcal{F}_n^\infty, \leq)$ are sublattices of the Stanley lattice.

Remark : The cover relation corresponds to transformations $DU \rightarrow UD$.



Lattice on $\mathcal{F}_n^p$

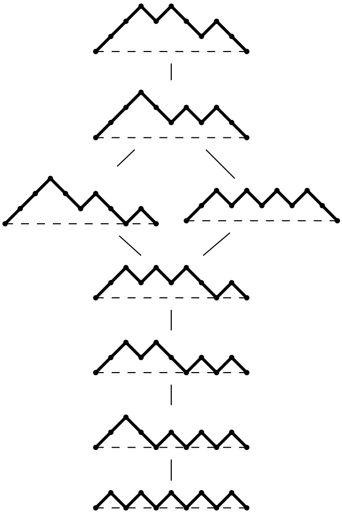


Figure – The Hasse diagram of $\mathbb{F}_5^2$.

Upper covers

Let $F_p(x,y)$ be the generating function where the coefficient of $x^n y^k$ is the number of elements in $\mathbb{F}_n^p$ that have exactly k upper covers.

Theorem

The generating function $F_p(x,y)$ is given by

$$F_p(x,y) = \frac{(1-x)(1+(y-1)x^p)}{1-2x+x^{p+1}-(y-1)(x^2-x^p+x^{p+1}-x^{p+2})}.$$

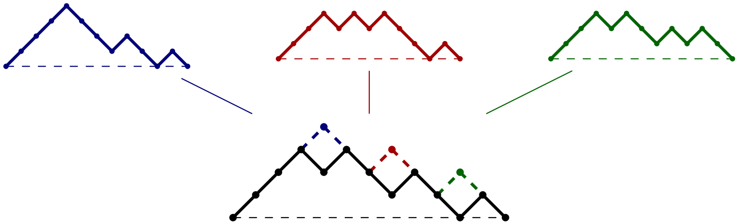


Figure – A path having 3 upper covers in $\mathbb{F}_6^3$.

Coverings

Corollary

The generating function for the number of coverings in $\mathbb{F}_n^p$, $n \geq 0$, is

$$\partial_y F_p(x,y)|_{y=1} = \frac{(1-x)\left(x^2 - x^{p+1}\right)(1-x^p)}{\left(1-2x+x^{p+1}\right)^2}.$$

Corollary

For any $p \geq 2$, the number of meet-irreducible elements in $\mathbb{F}_n^p$, is given by

$$b_p(n) = \left\lfloor \frac{n^2(p-1)}{2p} \right\rfloor,$$

which also counts the number of edges in the (n,p)-Turán graph, *i.e.* the maximum number of edges in a graph on n vertices and avoiding $(p+1)$-cliques.

Number of meet-irreducible elements

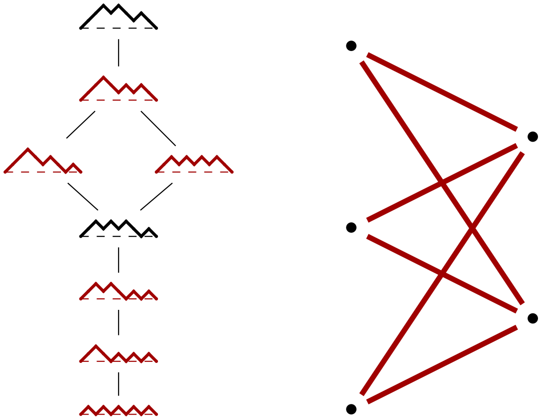


Figure – There are 6 meet-irreducible elements in $\mathbb{F}_5^2$, and 6 edges in the (5, 2)-Turán graph.

Boolean intervals

Definition

An interval is said *boolean* if it is isomorphic to the poset of subsets of $[n]$ ordered by inclusion.

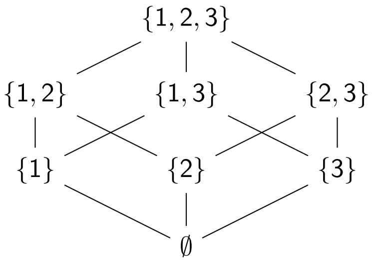


Figure – The boolean lattice of size 3.

Boolean intervals

Theorem

The generating function $B_p(x,y)$ for the number of boolean intervals in $\mathbb{F}_n^p$, with respect to the semilength $n \geq 0$, and the interval height is given by

$$B_p(x,y) = \frac{(1-x)(1+yx^p)}{1-2x+x^{p+1}-y(x^2-x^p+x^{p+1}-x^{p+2})}.$$

Proof. Since $\mathbb{F}_n^p$ is a distributive lattice, we have that

$$B_p(x,y) = F_p(x,1+y).$$

Structure of the linear intervals I

Definition

An interval is said *linear* when all its elements are pairwise comparable.

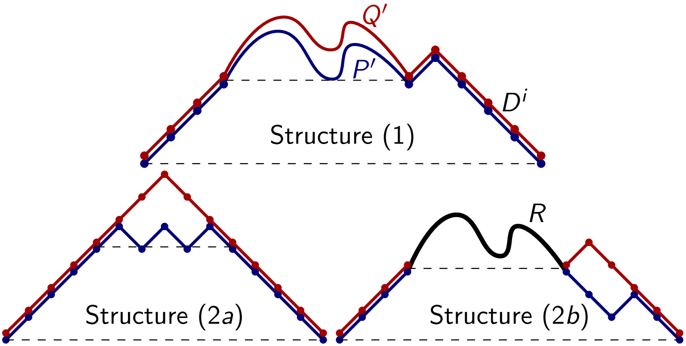


Figure – The structures of linear intervals $[P, Q]$ in $\mathbb{F}_n^p$.

Structure of linear intervals II

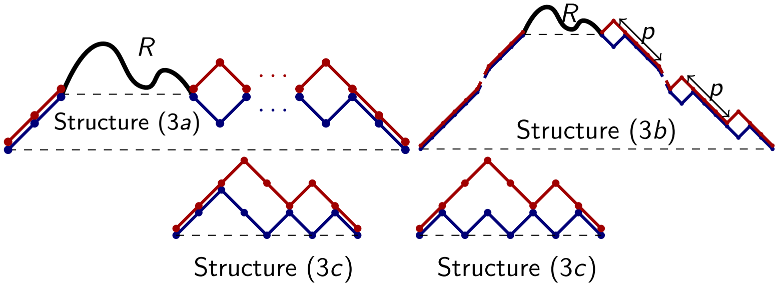


Figure – The structures of linear intervals $[P, Q]$ in $\mathbb{F}_n^p$.

Enumeration of the linear intervals

Corollary

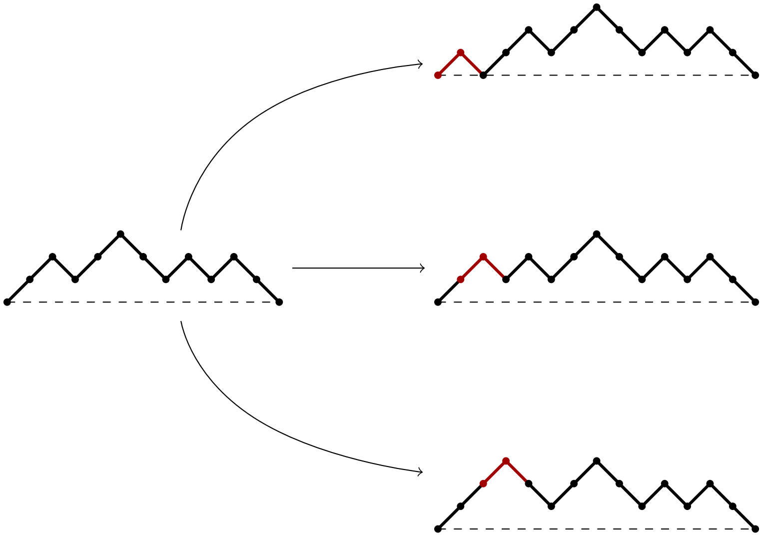
The generating function $L_2(x, y)$ of the number of linear intervals in $\mathbb{F}_n^2$ with respect to n and the interval height is given by

$$L_2(x, y) = \frac{x^4y^4 + y^3x^4 + 1}{1 - x - x^2} + \frac{x^2y\left(x^2 - 1\right)\left(x^3y^2 - 1\right)}{(xy - 1)\left(x^2 + x - 1\right)^2\left(x^2y - 1\right)}.$$

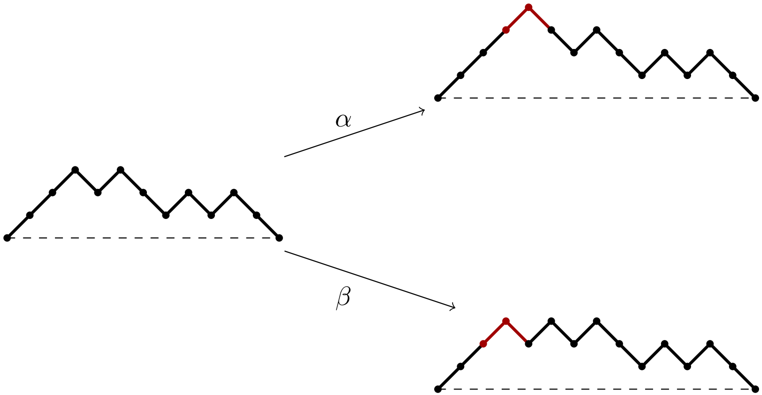
Theorem

Asymptotically, the number of linear intervals in $\mathbb{F}_n^p$ is proportional to the number of coverings.

Extending a Dyck path



Extending a Dyck path in $\mathbb{F}_n^\infty$



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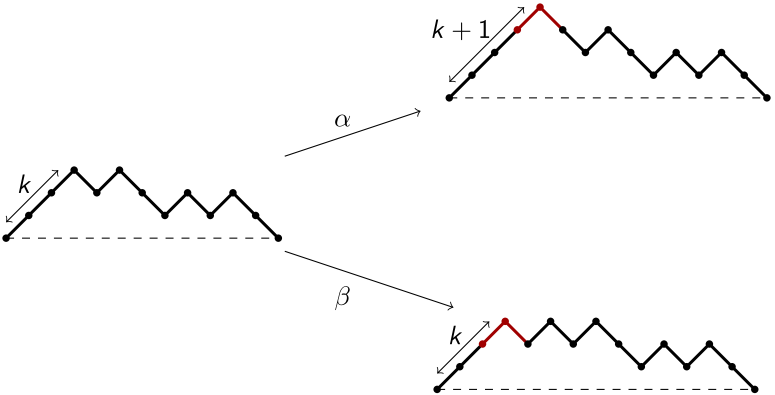
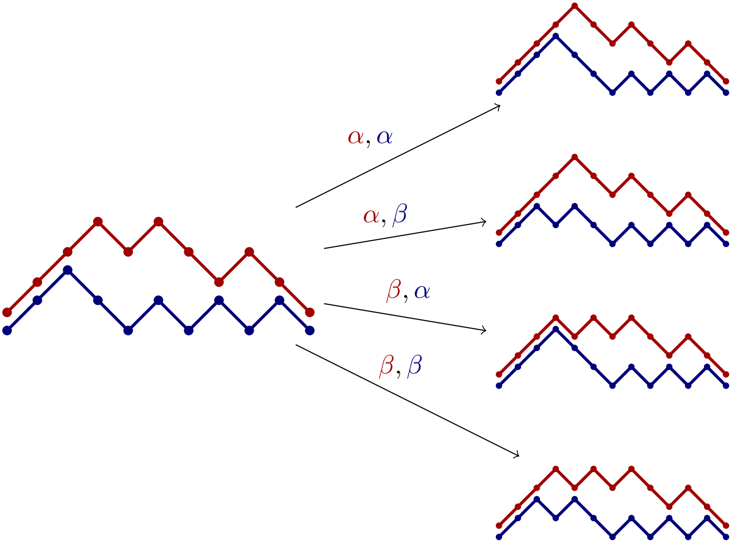


Generating intervals in $\mathbb{F}_n^\infty$



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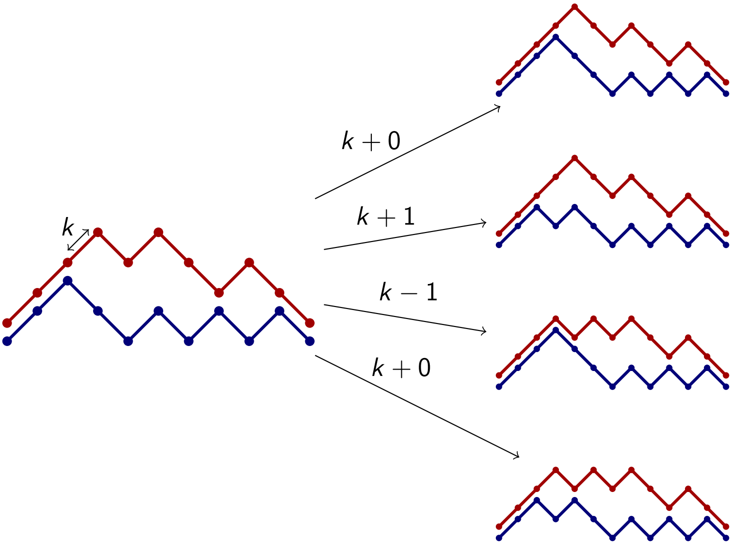

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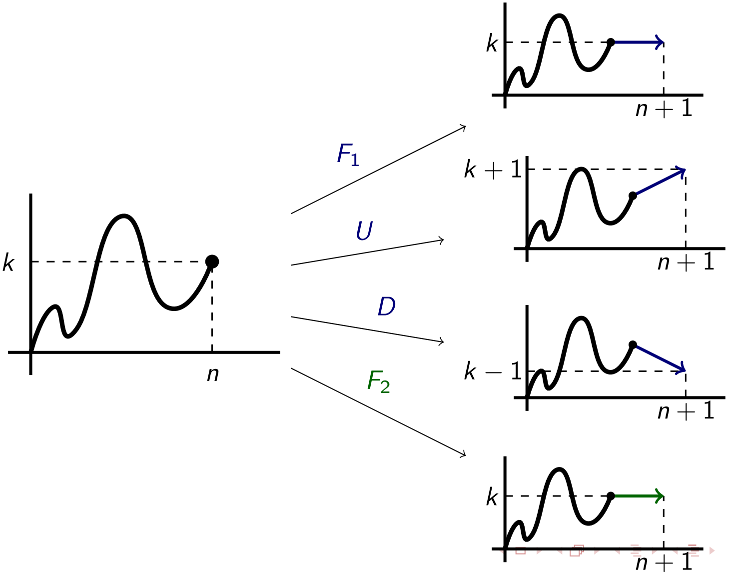

Generating intervals in $\mathbb{F}_n^\infty$

Theorem

There is a bijection between intervals in $\mathbb{F}_n^\infty$ and bicolored Motzkin paths of length $n-1$ in the quarter plane.

Corollary

There are $\binom{2n-1}{n}$ intervals in $\mathbb{F}_n^\infty$.

Generating intervals in $\mathbb{F}_n^\infty$

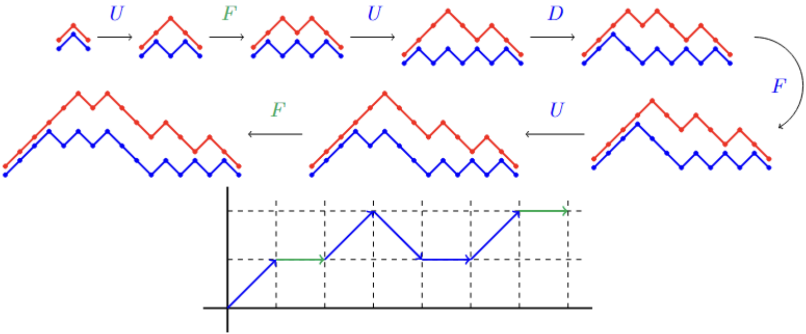

Figure – The generation of the interval $[U^2(UD)^3D^2(UD)^3, U^4(UD)^2D^2(UD^2)^2]$. This interval is thus associated with the bicolored Motzkin path $UF_2UDF_1UF_2$.

Intervals in $\mathbb{F}_n^p$

Theorem

There is a bijection between intervals in $\mathbb{F}_n^p$ and bicolored Motzkin paths of length $n-1$ and avoiding the $2^{p+1}-1$ consecutive patterns of the set $\{F_2, U\}^p \cup \{F_2, D\}^p$.

Corollary

The generating function $J(x)$ for the number of intervals in $\mathbb{F}_n^2$ is

$$J(x) = \frac{-x^2+3x-1+\sqrt{x^4-2x^3-x^2-2x+1}}{2x(x^2-3x+1)(x+1)}.$$

The coefficient of x^n in the series expansion is asymptotically

$$\frac{11+5\sqrt{5}}{20}\sqrt{\frac{14\sqrt{5}-30}{\pi}} \cdot n^{-1/2}\left(\frac{3+\sqrt{5}}{2}\right)^n.$$

Compositions of n

Proposition

There is a bijection between the elements of $\mathcal{F}_n^p$ and the compositions of n with parts in $[1, p]$.

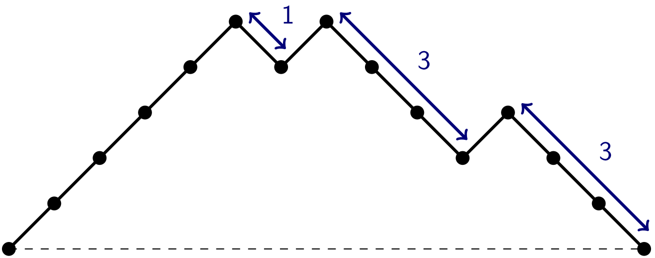

Figure – The path $P = U^5 D (UD^3)^2 \in \mathcal{F}_7^\infty$ is associated with the composition $\lambda(P) = (3, 3, 1)$.

Compositions of n

Proposition

The order induced by $\mathbb{F}_n^p$ on the compositions of n with parts in $[1, p]$ is known as the *dominance* order, defined by $\lambda \leq \mu$ if and only if for all k

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i.$$

$$\begin{tikzcd}[column sep=small]
& (3,1) \arrow[d, dash] & \\
& (2,2) \arrow[dl, dash] \arrow[dr, dash] & \\
(1,3) \arrow[dr, dash] & & (2,1,1) \arrow[dl, dash] \\
& (1,2,1) \arrow[d, dash] & \\
& (1,1,2) \arrow[d, dash] & \\
& (1,1,1,1) &
\end{tikzcd}$$

Figure – The lattice $\mathbb{F}_4^3$ on the compositions of 4 with parts in $[1, 3]$.

Compositions of n

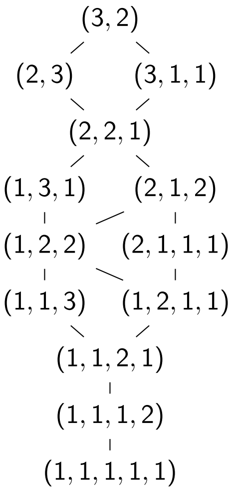

Figure – The lattice $\mathbb{F}_5^3$ on the compositions of 5 with parts in $[1, 3]$.

Powerset of $[1, n-1]$

Proposition

There is a bijection between $\mathcal{F}_n^p$ and the subsets of $[1, n-1]$ having no p consecutive elements.

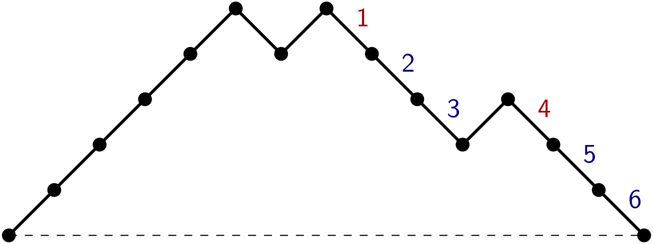

Figure – The path $P = U^5D(UD^3)^2 \in \mathcal{F}_7^\infty$ is associated with the subset $A(P) = \{2, 3, 5, 6\} \subseteq [1, 6]$.

Powerset of $[1, n-1]$

Proposition

The order induced by $\mathbb{F}_n^p$ on those subsets is the following :

$$A \lessdot B \iff \begin{cases} 1 \notin A \text{ and } B = A \cup \{1\}, \text{ or} \\ \text{there exists a unique } x \in A \text{ such that } B = \{x+1\} \cup A \backslash \{x\}. \end{cases}$$

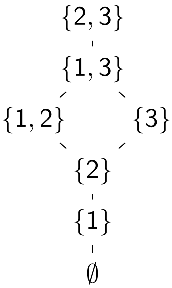

Figure – The lattice $\mathbb{F}_4^3$ on the subsets of $[1, 3]$ having no 3 consecutive elements.

Powerset of $[1, n-1]$

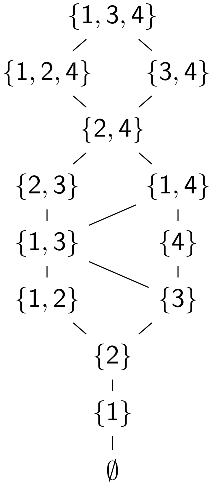

Figure – The lattice $\mathbb{F}_5^3$ on the subsets of $[1,4]$ having no 3 consecutive elements.