TD 10: Fourier transform and tempered distributions

Exercise 1. Let $A \in S_n^{++}(\mathbb{R})$ be a definite positive real matrix. Prove that the function u defined on $\mathbb{R}^n$ by $u(x) = e^{-\langle Ax,x\rangle}$ belongs to the Schwartz space $\mathscr{S}(\mathbb{R}^n)$ and that its Fourier transform is given by

$$\forall \xi \in \mathbb{R}^n, \quad \widehat{u}(\xi) = \sqrt{\frac{\pi^n}{\det A}}\, e^{-\frac{1}{4}\langle A^{-1}\xi,\xi\rangle}.$$

Application: Compute the Fourier transform of the following Gaussian function

$$f_\varepsilon(x) = e^{-\varepsilon|x|^2}, \quad \varepsilon > 0,\, x \in \mathbb{R}^d.$$

Hint: Begin by considering the case $n = 1$, and diagonalize the matrix A to treat the general case.

Exercise 2.

1. Let $A \subset \mathbb{R}^n$ be a measurable subset with finite measure. Prove that $\widehat{\mathbb{1}_A}$ belongs to $L^2(\mathbb{R}^n)$ but not to $L^1(\mathbb{R}^n)$.
2. Are there two functions $f, g \in \mathscr{S}(\mathbb{R}^n)$ not being identically equal to zero and satisfying the relation $f * g = 0$? Same question for some functions f et g with compact supports.
3. Prove that the equation $f * f = f$ has no non trivial solution in $L^1(\mathbb{R}^n)$, but has an infinite number of solutions in $L^2(\mathbb{R}^n)$.

Exercise 3. By computing the Fourier transform of the functions $f = \mathbb{1}_{[-1/2,1/2]}$ and $f * f$, show that

$$\int_{\mathbb{R}} \left(\frac{\sin t}{t}\right)^2 \mathrm{d}t = \pi.$$

Exercise 4. Let $I \subset \mathbb{R}$ be an interval and ρ be a weight function, meaning that ρ is measurable, positive, and satisfies

$$\forall n \in \mathbb{N}, \quad \int_I |x|^n \rho(x)\, \mathrm{d}x < +\infty.$$

Assume that there exists $a > 0$ such that

$$\int_I e^{a|x|}\rho(x)\, \mathrm{d}x < +\infty.$$

Let us denote by $L^2(I,\rho)$ the space of square integrable functions with respect to the measure $\rho\, \mathrm{d}x$.

1. Prove that there exists an orthonormal family of polynomials $(P_n)_{n\geq 0}$ such that $\deg P_n = n$ for all $n \geq 0$.

The aim is now to prove that $(P_n)_{n\geq 0}$ is a Hilbert basis of $L^2(I,\rho)$.

2. Let $f \in L^2(I,\rho)$. Check that the function φ defined by

$$\varphi(x) = \begin{cases} f(x)\rho(x) & \text{if } x \in I, \\ 0 & \text{if } x \notin I, \end{cases}$$

belongs to $L^1(\mathbb{R})$. Prove that its Fourier transform $\widehat{\varphi}$ can be extended to an holomorphic function F on the strip

$$B_a = \big\{z \in \mathbb{C} : |\operatorname{Im} z| < a/2\big\}.$$

3. Assume that the function $f \in L^2(I,\rho)$ is orthogonal to any monomial. By computing the derivatives of the function F at 0, prove that f is identically equal to zero and conclude.

Exercise 5 (Heisenberg's uncertainty principle). Prove that for all $f \in \mathscr{S}(\mathbb{R}^n)$ and $j \in \{1, \dots, n\}$,

$$\inf_{a\in\mathbb{R}} \big\|(x_j - a)f\big\|^2_{L^2(\mathbb{R}^n)} \inf_{b\in\mathbb{R}} \big\|(\xi_j - b)\widehat{f}\big\|^2_{L^2(\mathbb{R}^n)} \geq \frac{(2\pi)^n}{4}\|f\|^2_{L^2(\mathbb{R}^n)},$$

When is this inequality an equality ?

Exercise 6. Let us consider the interval $I = [-1,1]$ and the following subspace of $L^2(I)$

$$\mathrm{BL}^2(I) = \big\{u \in L^2(\mathbb{R}) : \widehat{u} = 0 \text{ almost everywhere on } \mathbb{R}\setminus I\big\}.$$

1. Prove that $\mathrm{BL}^2(I)$ is a Hilbert space.
2. Check that $\mathrm{BL}^2(I) \subset C^0_{\to 0}(\mathbb{R})$ and that the corresponding embedding is continuous.
3. Let us consider the continuous extension of $x \mapsto \sin x/x$, denoted sinc.
   (a) Prove that the family $(\pi^{-1/2}\tau_{2\pi k}\,\mathrm{sinc})_{k\in\mathbb{Z}}$ is a Hilbert basis of $\mathrm{BL}^2(I)$.
   (b) Prove (sampling theorem) that any element $u \in \mathrm{BL}^2(I)$ can be decomposed as follows

$$u(x) = \sum_{k\in\mathbb{Z}} u(2\pi k)\,\mathrm{sinc}(x - 2\pi k),$$

   the convergence being uniform in $\mathbb{R}$, and also holds in $L^2(\mathbb{R})$.

Exercise 7. Prove that the following distributions are tempered and compute their Fourier transform:

1. δ_0
2. 1,
3. H (Heaviside),
4. $\mathrm{p.v.}(1/x)$,
5. $|x|$ in $\mathbb{R}$.

Indication : $\mathrm{p.v.}(1/x)$ *is an odd distribution, so its Fourier transform is also odd.*

Exercise 8. The aim of this exercice is to compute the Fourier transform of the following tempered distribution on $\mathbb{R}^2$

$$\langle T, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int_{\mathbb{R}} \varphi(x,x)\,\mathrm{d}x, \quad \varphi \in \mathscr{S}(\mathbb{R}^2).$$

1. Let $\psi \in \mathscr{S}(\mathbb{R}^2)$. Prove that

$$\langle \widehat{T}, \psi\rangle_{\mathscr{S}',\mathscr{S}} = \lim_{\varepsilon\to 0^+} I_\varepsilon \quad \text{où} \quad I_\varepsilon = \int_{\mathbb{R}} e^{-\varepsilon x^2}\widehat{\psi}(x,x)\,\mathrm{d}x.$$

2. By using the expression of $\widehat{\psi}(x, x)$, show that

$$I_\varepsilon = 2\sqrt{\pi} \int_{\mathbb{R}^2} e^{-\zeta^2} \psi(\xi, 2\sqrt{\varepsilon}\zeta - \xi)\,\mathrm{d}\xi\,\mathrm{d}\zeta.$$

3. Deduce the expression of $\widehat{T}$.

Exercise 9. Given some real number $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^d)$ by

$$H^s(\mathbb{R}^d) = \big\{u \in \mathscr{S}'(\mathbb{R}^d) : \langle \xi \rangle^s \widehat{u} \in L^2(\mathbb{R}^d)\big\},$$

equipped with the following scalar product

$$\langle u, v \rangle_{H^s} = \int_{\mathbb{R}^d} \langle \xi \rangle^{2s} \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\,\mathrm{d}\xi, \quad u, v \in \ H^s(\mathbb{R}^d).$$

1. Show that $H^{s_1}(\mathbb{R}^d)$ embeds continuously into $H^{s_2}(\mathbb{R}^d)$ for $s_1 \geq s_2$.
2. Check that $\delta_0 \in H^s(\mathbb{R}^d)$ for $s < -d/2$.
3. When $s \in \mathbb{N}^*$ is a nonnegative integer, the Sobolev space is also given by

$$H^s(\mathbb{R}^d) = \big\{u \in L^2(\mathbb{R}^n) : \forall |\alpha| \leq s,\ \partial^\alpha u \in L^2(\mathbb{R}^d)\big\}.$$

4. Prove that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^3)$,

$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c\,\|u\|^{1/2}_{H^1(\mathbb{R}^3)}\|u\|^{1/2}_{H^2(\mathbb{R}^3)}.$$

   Hint: Considering $R > 0$, use the following decomposition

$$\|\widehat{u}\|_{L^1(\mathbb{R}^3)} = \int_{|\xi| \leq R} \langle \xi \rangle |\widehat{u}(\xi)| \frac{\mathrm{d}\xi}{\langle \xi \rangle} + \int_{|\xi| > R} \langle \xi \rangle^2 |\widehat{u}(\xi)| \frac{\mathrm{d}\xi}{\langle \xi \rangle^2}.$$

5. (a) Prove that if $s > d/2$, the space $H^s(\mathbb{R}^d)$ embeds continuously to $C^0_{\to 0}(\mathbb{R}^d)$, the space of continuous functions u on $\mathbb{R}^d$ satisfying $u(x) \to 0$ as $|x| \to +\infty$.
   (b) State an analogous result in the case where $s > d/2 + k$ for some $k \in \mathbb{N}$. Deduce that $\bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.

Exercise 10. Let us consider the function

$$\gamma_0 : \varphi(x', x_d) \in C_0^\infty(\mathbb{R}^d) \mapsto \varphi(x', x_d = 0) \in C_0^\infty(\mathbb{R}^{d-1}).$$

Prove that for all $s > 1/2$, the function γ_0 can be uniquely extended as an application mapping $H^s(\mathbb{R}^d)$ to $H^{s-1/2}(\mathbb{R}^{d-1})$.
Hint: For all $\varphi \in\ C_0^\infty(\mathbb{R}^d)$, begin by computing the Fourier transform of the function $\gamma_0 \phi$.