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TD 5: EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR REACTION-DIFFUSION EQUATIONS

**EXERCISE 1.** We consider the following reaction-diffusion equation:

$$(1) \quad \begin{cases} \partial_t u - \Delta u = u^2 & \text{in } (0, +\infty) \times \mathbb{R}, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}. \end{cases}$$

1. Establish *a priori* energy estimates for any smooth solution of the equation (1).
2. Assume that  $u_0 \in H^1(\mathbb{R})$ . We aim at proving, by using an iterative method, that there exist  $T > 0$  and a solution  $u \in C^0([0, T], H^1(\mathbb{R}))$  of the equation (1). We therefore consider the sequence  $(u^n)_{n \geq 0}$  recursively defined by  $u^0 = u_0$  and

$$(2) \quad \begin{cases} \partial_t u^{n+1} - \Delta u^{n+1} = (u^n)^2 & \text{in } (0, +\infty) \times \mathbb{R}, \\ u^{n+1}(0, \cdot) = u_0 & \text{in } \mathbb{R}. \end{cases}$$

- (a) Discuss the well-posedness of the sequence  $(u^n)_{n \geq 0}$ .
- (b) (Bound in  $H^1$ ) Prove that there exists a positive time  $T_1 > 0$  and a positive constant  $c_1 > 0$  such that for all  $n \geq 0$  and  $0 \leq t \leq T_1$ ,

$$\|u^n(t, \cdot)\|_{H^1(\mathbb{R})} \leq c_1.$$

- (c) (Convergence in  $H^1$ ) Prove that there exists another positive time  $0 < T_2 < T_1$  and another positive constant  $c_2 > 0$  satisfying that for all  $n \geq 0$  and  $0 \leq t \leq T_2$ ,

$$\|u^{n+1}(t, \cdot) - u^n(t, \cdot)\|_{H^1(\mathbb{R})} \leq \frac{c_2}{2^n}.$$

- (d) Conclude.

3. Is this solution unique ?

**EXERCISE 2.** Let  $u_0 \in H^1(\mathbb{R})$  be a smooth initial datum. We consider  $T > 0$  the positive time and  $u \in C^0([0, T], H^1(\mathbb{R}))$  the solution of the equation (1), both given by the previous exercise. By using a bootstrap argument, prove that the function  $u$  is smooth, precisely  $u \in C^\infty(]0, T[ \times \mathbb{R})$ .

**EXERCISE 3.** By adapting the strategy used in the first exercise, investigate the existence of solutions for the following reaction-diffusion equation:

$$(3) \quad \begin{cases} \partial_t u - \Delta u = \arctan(u) & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^d, \end{cases}$$

with initial datum  $u_0 \in L^2(\mathbb{R}^d)$ . Assuming then that  $d = 1$  and  $\langle x \rangle u_0, \langle x \rangle \partial_x u_0 \in L^2(\mathbb{R})$ , prove pointwise estimates for the function  $u$ .

*Hint: The function  $\arctan$  is globally Lipschitz continuous, only  $L^2$  estimates are required.*

**EXERCISE 4.** Let  $T > 0$  and  $u_0 \in L^2(\mathbb{R}^d)$ . We consider the following initial value problem:

$$(4) \quad \begin{cases} \partial_t u - \Delta u = \sqrt{1 + u^2} - 1, & \text{in } (0, T] \times \mathbb{R}^d, \\ u(0, \cdot) = u_0 & \text{in } \mathbb{R}^d, \end{cases}$$

We say that a continuous function  $u \in C^0([0, T], L^2(\mathbb{R}^d))$  is a *mild* solution of the initial value problem (4) when it satisfies the following integral equation for all  $0 \leq t \leq T$ :

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (\sqrt{1 + u(s)^2} - 1) ds,$$

where, for all  $v \in L^2(\mathbb{R}^d)$ ,  $e^{t\Delta} v$  denotes the solution of the heat equation posed on  $\mathbb{R}^d$  with initial datum  $v$ .

1. We consider the function  $F : C^0([0, T], L^2(\mathbb{R}^d)) \rightarrow C^0([0, T], L^2(\mathbb{R}^d))$  defined by

$$(Fu)(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (\sqrt{1 + u(s)^2} - 1) ds.$$

By using a fixed-point theorem on the function  $F$ , prove that the equation (4) admits a unique mild solution  $u \in C^0([0, T], L^2(\mathbb{R}^d))$ .

2. Check that the function  $u_0 \in L^2(\mathbb{R}^d) \mapsto u \in C^0([0, T], L^2(\mathbb{R}^d))$  is Lipschitz continuous.

**EXERCISE 5.** Study the existence of mild solutions for the equation (3) and make the link with the solution constructed by iterative method in Exercise 3.

**EXERCISE 6.** Let  $\Omega = (0, 1)$ ,  $t_0 > 0$  and  $u_0 \in H_0^1(\Omega)$ . We aim at proving that there exist a positive time  $t^* > 0$  and a unique function  $u \in C^0([t_0, t^*], H_0^1(\Omega))$  solution of the following integral equation for all  $t_0 \leq t < t^*$ :

$$(5) \quad u(t) = e^{(t-t_0)\Delta} u_0 + \int_{t_0}^t e^{(t-s)\Delta} \sinh(u(s)) ds.$$

Let us recall that there exists a Hilbert basis  $(e_n)_{n \geq 0}$  of the space  $H_0^1(\Omega)$  composed of eigenvalues of the operator  $-\Delta$ . In the above integral equation, the operator  $e^{t\Delta} \in \mathcal{L}(H_0^1(\Omega))$  is defined by

$$e^{t\Delta} = \sum_{n=0}^{+\infty} e^{-t\lambda_n} \langle \cdot, e_n \rangle_{H_0^1(\Omega)} e_n,$$

with  $\lambda_n > 0$  the eigenvalue associated with the eigenfunction  $e_n$ .

1. By using a fixed-point theorem, prove that there exists a positive time  $t_1 > t_0$  such that the equation (5) has a solution in the space  $C^0([t_0, t_1], H_0^1(\Omega))$ .
2. Explain how this solution can be extended to the interval  $[t_0, t_1 + \delta]$  with  $\delta > 0$ . Deduce, proceeding by contradiction, that if  $[t_0, t^*]$  stands for the maximal interval of existence of the solution  $u$  and if  $t^* < +\infty$ , then

$$\lim_{t \nearrow t^*} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

3. Investigate the uniqueness of such a solution.
4. Of which equation is the function  $u$  a mild solution ?