TD 8: Pseudo-differential operators

Exercise 1.

1. Let $L = \sum_{|\alpha|\le m} a_\alpha(x)\partial_x^\alpha$ be a differential operator of order $m \ge 0$ with smooth and fast decaying coefficients $a_\alpha \in C^\infty(\mathbb{R}^d)$. Prove that for all $u \in \mathscr{S}(\mathbb{R}^d)$,
$$(Lu)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} a(x,\xi)\widehat{u}(\xi)\,\mathrm{d}\xi, \quad x \in \mathbb{R}^d,$$
where the symbol a is defined by
$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)(i\xi)^\alpha, \quad (x,\xi)\in\mathbb{R}^{2d}.$$

2. For all $u_0 \in L^2(\mathbb{R}^d)$ and $t \ge 0$, we set $e^{t\Delta}u_0$ as the mild solution at time t of the heat equation
$$\begin{cases} \partial_t u - \Delta u = 0 & \text{on } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R}^d. \end{cases}$$
Prove that for all $t \ge 0$, the evolution operator $e^{t\Delta}$ is a pseudo-differential operator and give the expression of its symbol.

3. Let $m \in \mathbb{R}$ and $A \in \mathrm{Op}(S^m)$. Prove that there exists a unique $a \in S^m$ such that $\mathrm{Op}(a) = A$.

Exercise 2. Let $a \in S^m$ be a symbol of order $m \in \mathbb{R}$.

1. We denote by $[\mathrm{Op}(a), \partial_{x_j}]$ the commutator between the operator $\mathrm{Op}(a)$ and the partial derivative ∂_{x_j} with respect to x_j. Prove that $[\mathrm{Op}(a), \partial_{x_j}]$ is also a pseudo-differential operator and compute its symbol as a function of a.
2. Same question with $[\mathrm{Op}(a), x_j]$, where x_j stands for the multiplication by x_j.

Exercise 3.

1. Let $m \in \mathbb{R}$ and $a \in S^m$. Prove that for all $s \in \mathbb{R}$, there exists a positive constant $c_s > 0$ such that
$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \|\mathrm{Op}(a)u\|_{H^s} \le c_s\|u\|_{H^{s+m}}.$$
Hint: Any operator in $\mathrm{Op}(S^0)$ *is bounded in* $L^2(\mathbb{R}^d)$.

2. Let $m_1, m_2 \in \mathbb{R}$ and $a_1 \in S^{m_1}$, $a_2 \in S^{m_2}$. Check that
$$[\mathrm{Op}(a_1), \mathrm{Op}(a_2)] - \mathrm{Op}\left(\frac{1}{i}\{a_1,a_2\}\right) \in \mathrm{Op}(S^{m_1+m_2-2}),$$
where $\{a_1,a_2\}$ stands for the following Poisson bracket
$$\{a_1,a_2\} = \nabla_\xi a_1\cdot\nabla_x a_2 - \nabla_x a_1\cdot\nabla_\xi a_2.$$

Exercise 4. Let $m \in \mathbb{R}$ and $a \in S^m$.

1. Assume that there exists $b \in S^{-m}$ such that $\operatorname{Op}(a)\operatorname{Op}(b) - I \in \operatorname{Op}(S^{-\infty})$. Prove that there exist $R > 0$ and $c > 0$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \quad |\xi| \geq R \Rightarrow |a(x,\xi)| \geq c\langle\xi\rangle^m. \tag{1}$$

   Hint: Begin by checking that $ab - 1 \in S^{-1}$.

2. Let us now assume that the symbol a satisfies the condition (1). We aim at proving that there exists a symbol $b \in S^{-m}$ such that $\operatorname{Op}(a)\operatorname{Op}(b) - I \in \operatorname{Op}(S^{-\infty})$. The operator $\operatorname{Op}(b)$ is called a *parametrix* of the operator $\operatorname{Op}(a)$. To that end, we will construct a sequence of symbols $(b_j)_j$ such that $b_j \in S^{-m-j}$ and

$$\forall n \geq 0, \quad a \sharp (b_0 + \cdots + b_n) - 1 \in S^{-n-1}.$$

   (a) Let $F \in C^\infty(\mathbb{C})$ such that $F(z) = 1/z$ when $|z| \geq c$. We set

$$b_0(x,\xi) = \frac{1}{\langle\xi\rangle^m} F(a(x,\xi)\langle\xi\rangle^{-m}), \quad (x,\xi) \in \mathbb{R}^{2d}.$$

   Prove that $b_0 \in S^{-m}$ and that $a \sharp b_0 - 1 \in S^{-1}$.

   (b) Construct then the other symbols b_j and conclude by using Borel's summation lemma.

   (c) Check that we also have $\operatorname{Op}(b)\operatorname{Op}(a) - I \in \operatorname{Op}(S^{-\infty})$.

   (d) *Application:* Prove that for all $s, t \in \mathbb{R}$, there exist some positive constants $a_s, b_{s,t} > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \|u\|_{H^{s+m}} \leq a_s \|\operatorname{Op}(a)u\|_{H^s} + b_{s,t}\|u\|_{H^t}. \tag{2}$$

Exercise 5. Let $m \in \mathbb{R}$ and $a \in S^m$ be a symbol satisfying that there exist $c, R > 0$ such that

$$\forall (x,\xi) \in \mathbb{R}^{2d}, \quad |\xi| \geq R \Rightarrow \operatorname{Re} a(x,\xi) \geq c\langle\xi\rangle^m.$$

1. Prove that there exists $r \in S^{m-1}$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \operatorname{Re}\langle \operatorname{Op}(a)u, u\rangle_{L^2} = \langle \operatorname{Op}(\operatorname{Re} a)u, u\rangle_{L^2} + \langle \operatorname{Op}(r)u, u\rangle.$$

2. Prove that for all $\tilde{r} \in S^{m-1}$, there exists a positive constant $c > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad |\langle \operatorname{Op}(\tilde{r})u, u\rangle_{L^2}| \leq c\|u\|^2_{H^{(m-1)/2}}.$$

3. Prove that there exists $b \in S^{m/2}$ which is elliptic in the sense that (1) holds with $m/2$, and such that $\operatorname{Op}(\operatorname{Re} a) - \operatorname{Op}(b)^*\operatorname{Op}(b) \in \operatorname{Op}(S^{m-1})$.

4. Check that there exist $c_0, c_1 > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \operatorname{Re}\langle \operatorname{Op}(a)u, u\rangle_{L^2} + c_1\|u\|^2_{H^{(m-1)/2}} \geq c_0\|u\|^2_{H^{m/2}}.$$

   Hint: Use the estimate (2) with the operator $\operatorname{Op}(b)$.

5. Prove finally that for all $s \in \mathbb{R}$, there exist some positive constants $a_s, b_s > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^d), \quad \operatorname{Re}\langle \operatorname{Op}(a)u, u\rangle_{L^2} + a_s\|u\|^2_{H^s} \geq b_s\|u\|^2_{H^{m/2}}.$$

   Hint: When $s < (m-1)/2$, use Young's inequality with the exponents $p = 2(m-2s)/(m-2s-1)$ and $q = 2(m-2s)$.