# TD 2: Sobolev spaces

**Exercise 1.**

1. Show that $u(x) = |x|$ belongs to $W^{1,2}(-1,1)$ but not to $W^{2,2}(-1,1)$.
2. Check that $v(x) = \frac{\sin(x^2)}{\sqrt{1+x^2}}$ belongs to $L^2(\mathbb{R})$ but not to $W^{1,2}(\mathbb{R})$.
3. Show that $H^1(\mathbb{R}^2)$ is not included in $L^\infty(\mathbb{R}^2)$.
   *Hint: Consider a function of the form* $x \mapsto \chi(|x|)\,|\log|x||^{1/3}$.

**Exercise 2.** Let $\Omega = (0,1)$.

1. Prove that the following continuous embeddings hold
$$W^{1,1}(\Omega) \hookrightarrow C^0(\bar{\Omega}) \quad \text{and} \quad W^{1,p}(\Omega) \hookrightarrow C^{0,1-1/p}(\bar{\Omega}) \quad \text{when } p \in (1,\infty],$$
   with the convention $1/\infty = 0$.
2. Prove that for all $1 \le p < \infty$, the space $W_0^{1,p}(\Omega)$ is given by
$$W_0^{1,p}(\Omega) = \big\{u \in W^{1,p}(\Omega) : u(0) = u(1) = 0\big\}.$$

**Exercise 3.** The aim of this exercise is to give a characterization of the space $H^1(0,1)$.

1. (a) Prove that if $u \in C^1[0,1]$, then we have for any $\alpha \in (0,1/2)$, $x \in (\alpha, 1-\alpha)$ and $h \in \mathbb{R}$ such that $|h| < \alpha$,
$$|u(x+h) - u(x)|^2 \le h^2 \int_0^1 |u'(x+sh)|^2 \, \mathrm{d}s.$$
   (b) Deduce that for any $u \in H^1(0,1)$, any $\alpha \in (0,1/2)$ and $h \in \mathbb{R}$ such that $|h| < \alpha$, we have
$$\|\tau_h u - u\|_{L^2(\alpha,1-\alpha)} \le |h|\|u'\|_{L^2(0,1)}.$$
2. Conversely, we assume that $u \in L^2(0,1)$ is such that there exists a constant $C > 0$ such that for any $\alpha \in (0,1/2)$ and for any $h \in \mathbb{R}$ such that $|h| < \alpha$, we have
$$\|\tau_h u - u\|_{L^2(\alpha,1-\alpha)} \le C|h|.$$
   (a) Let $\phi \in C_0^1(0,1)$ and $\alpha > 0$ such that $\phi$ is supported in $(\alpha, 1-\alpha)$. Prove that for any $|h| < \alpha$, we have
$$\int_\alpha^{1-\alpha} (u(x+h) - u(x))\phi(x) \, \mathrm{d}x = \int_0^1 u(x)(\phi(x-h) - \phi(x)) \, \mathrm{d}x.$$
   Deduce that
$$\left| \int_0^1 u(x)\phi'(x) \, \mathrm{d}x \right| \le C\|\phi\|_{L^2(0,1)}.$$

(b) Conclude that $u \in H^1(0,1)$.

**Exercise** 4. Let $p \in [1,+\infty)$ and let $\Omega$ be an open subset of $\mathbb{R}^d$.

1. Assume that $\Omega$ is bounded in one direction, meaning that $\Omega$ is contained in the region between two parallel hyperplanes. Prove Poincaré's inequality: there exists $c > 0$ such that for every $f \in W_0^{1,p}(\Omega)$,
$$\|f\|_{L^p(\Omega)} \leq c\|\nabla f\|_{L^p(\Omega)}.$$
*Hint: Consider first the case* $\Omega \subset \mathbb{R}^{d-1} \times [-M, M]$.
2. Assume that $\Omega$ is bounded. Prove Poincaré-Wirtinger's inequality: there exists a constant $c > 0$ such that for any $f \in W^{1,p}(\Omega)$ satisfying $\int_\Omega f = 0$,
$$\|f\|_{L^p(\Omega)} \leq c\|\nabla f\|_{L^p(\Omega)}.$$

**Exercise** 5. Let $\Omega$ be an open subset of $\mathbb{R}^d$ and let $p \in (1,+\infty)$. Prove that for all $F \in W_0^{1,p}(\Omega)'$, there exist $f_0, f_1, \ldots, f_d \in L^q(\Omega)$ (with $\frac{1}{p} + \frac{1}{q} = 1$) such that for all $g \in W_0^{1,p}(\Omega)$,
$$\langle F, g\rangle_{W_0^{1,p}(\Omega)', W_0^{1,p}(\Omega)} = \int_\Omega f_0 g \,\mathrm{d}x + \sum_{i=1}^d \int_\Omega f_i \partial_i g \,\mathrm{d}x.$$
Assuming that $\Omega$ is bounded, prove that we may take $f_0 = 0$.

**Exercise** 6. Given some real number $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{R}^d)$ by
$$H^s(\mathbb{R}^d) = \big\{u \in \mathscr{S}'(\mathbb{R}^d) : \langle\xi\rangle^s \widehat{u} \in L^2(\mathbb{R}^d)\big\},$$
equipped with the following scalar product
$$\langle u, v\rangle_{H^s} = \int_{\mathbb{R}^d} \langle\xi\rangle^{2s} \widehat{u}(\xi)\overline{\widehat{v}(\xi)} \,\mathrm{d}\xi, \quad u, v \in \ H^s(\mathbb{R}^d).$$

1. Prove that there exists a positive constant $c > 0$ such that for all $u \in \mathcal{S}(\mathbb{R}^3)$,
$$\|u\|_{L^\infty(\mathbb{R}^3)} \leq c\,\|u\|_{H^1(\mathbb{R}^3)}^{1/2}\|u\|_{H^2(\mathbb{R}^3)}^{1/2}.$$
*Hint: Considering $R > 0$, use the following decomposition*
$$\|\widehat{u}\|_{L^1(\mathbb{R}^3)} = \int_{|\xi|\leq R} \langle\xi\rangle|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle} + \int_{|\xi|>R} \langle\xi\rangle^2|\widehat{u}(\xi)|\frac{\mathrm{d}\xi}{\langle\xi\rangle^2}.$$
2. (a) Prove that if $s > d/2$, the space $H^s(\mathbb{R}^d)$ embeds continuously to $C^0_{\to 0}(\mathbb{R}^d)$, the space of continuous functions $u$ on $\mathbb{R}^d$ satisfying $u(x) \to 0$ as $|x| \to +\infty$.
   (b) State an analogous result in the case where $s > d/2 + k$ for some $k \in \mathbb{N}$. Deduce that $\bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^d) \subset C^\infty(\mathbb{R}^d)$.

**Exercise** 7. Let us consider the function
$$\gamma_0 : \varphi(x', x_d) \in C_0^\infty(\mathbb{R}^d) \mapsto \varphi(x', x_d = 0) \in C_0^\infty(\mathbb{R}^{d-1}).$$
Prove that for all $s > 1/2$, the function $\gamma_0$ can be uniquely extended as an application mapping $H^s(\mathbb{R}^d)$ to $H^{s-1/2}(\mathbb{R}^{d-1})$.
*Hint: For all $\varphi \in \ C_0^\infty(\mathbb{R}^d)$, begin by computing the Fourier transform of the function $\gamma_0\phi$.*