TD 6: Existence and uniqueness of solutions for reaction-diffusion equations

Exercise 1. We consider the following reaction-diffusion equation:

$$\left\{\begin{aligned} \partial_t u - \Delta u &= u^2 \quad \text{in } (0,+\infty)\times\mathbb{R}, \\ u(0,\cdot) &= u_0 \quad \text{in } \mathbb{R}. \end{aligned}\right. \tag{1}$$

1. Establish *a priori* energy estimates for any smooth solution of the equation (1).
2. Assume that $u_0 \in H^1(\mathbb{R})$. We aim at proving, by using an iterative method, that there exist $T > 0$ and a solution $u \in C^0([0,T], H^1(\mathbb{R}))$ of the equation (1). We therefore consider the sequence $(u^n)_{n\geq 0}$ recursively defined by $u^0 = u_0$ and

$$\left\{\begin{aligned} \partial_t u^{n+1} - \Delta u^{n+1} &= (u^n)^2 \quad \text{in } (0,+\infty)\times\mathbb{R}, \\ u^{n+1}(0,\cdot) &= u_0 \qquad \text{in } \mathbb{R}. \end{aligned}\right. \tag{2}$$

   (a) Discuss the well-posedness of the sequence $(u^n)_{n\geq 0}$.

   (b) (Bound in H^1) Prove that there exists a positive time $T_1 > 0$ and a positive constant $c_1 > 0$ such that for all $n \geq 0$ and $0 \leq t \leq T_1$,
$$\|u^n(t,\cdot)\|_{H^1(\mathbb{R})} \leq c_1.$$

   (c) (Convergence in H^1) Prove that there exists another positive time $0 < T_2 < T_1$ and another positive constant $c_2 > 0$ satisfying that for all $n \geq 0$ and $0 \leq t \leq T_2$,
$$\|u^{n+1}(t,\cdot) - u^n(t,\cdot)\|_{H^1(\mathbb{R})} \leq \frac{c_2}{2^n}.$$

   (d) Conclude.
3. Is this solution unique ?

Exercise 2. Let $u_0 \in H^1(\mathbb{R})$ be a smooth initial datum. We consider $T > 0$ the positive time and $u \in C^0([0,T], H^1(\mathbb{R}))$ the solution of the equation (1), both given by the previous exercise. By using a bootstrap argument, prove that the function u is smooth, precisely $u \in C^\infty(]0,T[\times\mathbb{R})$.

Exercise 3. By adapting the strategy used in the first exercise, investigate the existence of solutions for the following reaction-diffusion equation:

$$\left\{\begin{aligned} \partial_t u - \Delta u &= \arctan(u) \quad \text{in } (0,+\infty)\times\mathbb{R}^d, \\ u(0,\cdot) &= u_0 \qquad\quad \text{in } \mathbb{R}^d, \end{aligned}\right. \tag{3}$$

with initial datum $u_0 \in L^2(\mathbb{R}^d)$. Assuming then that $d = 1$ and $\langle x\rangle u_0, \langle x\rangle \partial_x u_0 \in L^2(\mathbb{R})$, prove pointwise estimates for the function u.
Hint: The function arctan *is globally Lipschitz continuous, only* L^2 *estimates are required.*

Exercise 4. Let $T > 0$ and $u_0 \in L^2(\mathbb{R}^d)$. We consider the following initial value problem:

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& \sqrt{1+u^2} - 1, & \text{in } (0,T] \times \mathbb{R}^d, \\ u(0,\cdot) &=& u_0 & \text{in } \mathbb{R}^d, \end{array} \right. \tag{4}$$

We say that a continuous function $u \in C^0([0,T], L^2(\mathbb{R}^d))$ is a *mild* solution of the initial value problem (4) when it satisfies the following integral equation for all $0 \leq t \leq T$:

$$u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (\sqrt{1+u(s)^2} - 1)\,\mathrm{d}s,$$

where, for all $v \in L^2(\mathbb{R}^d)$, $e^{t\Delta}v$ denotes the solution of the heat equation posed on $\mathbb{R}^d$ with initial datum v.

1. We consider the function $F : C^0([0,T], L^2(\mathbb{R}^d)) \to C^0([0,T], L^2(\mathbb{R}^d))$ defined by

   $$(Fu)(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (\sqrt{1+u(s)^2} - 1)\,\mathrm{d}s.$$

   By using a fixed-point theorem on the function F, prove that the equation (4) admits a unique mild solution $u \in C^0([0,T], L^2(\mathbb{R}^d))$.
2. Check that the function $u_0 \in L^2(\mathbb{R}^d) \mapsto u \in C^0([0,T], L^2(\mathbb{R}^d))$ is Lipschitz continuous.

Exercise 5. Study the existence of mild solutions for the equation (3) and make the link with the solution constructed by iterative method in Exercise 3.

Exercise 6. Let $\Omega = (0,1)$, $t_0 > 0$ and $u_0 \in H_0^1(\Omega)$. We aim at proving that there exist a positive time $t^* > 0$ and a unique function $u \in C^0([t_0, t^*[, H_0^1(\Omega))$ solution of the following integral equation for all $t_0 \leq t < t^*$:

$$u(t) = e^{(t-t_0)\Delta} u_0 + \int_{t_0}^t e^{(t-s)\Delta} \sinh(u(s))\,\mathrm{d}s. \tag{5}$$

Let us recall that there exists a Hilbert basis $(e_n)_{n\geq 0}$ of the space $H_0^1(\Omega)$ composed of eigenvalues of the operator $-\Delta$. In the above integral equation, the operator $e^{t\Delta} \in \mathcal{L}(H_0^1(\Omega))$ is defined by

$$e^{t\Delta} = \sum_{n=0}^{+\infty} e^{-t\lambda_n} \langle \cdot, e_n \rangle_{H_0^1} e_n,$$

with $\lambda_n > 0$ the eigenvalue associated with the eigenfunction e_n.

1. By using a fixed-point theorem, prove that there exists a positive time $t_1 > t_0$ such that the equation (5) has a solution in the space $C^0([t_0, t_1], H_0^1(\Omega))$.
2. Explain how this solution can be extended to the interval $[t_0, t_1 + \delta]$ with $\delta > 0$. Deduce, proceeding by contradiction, that if $[t_0, t^*[$ stands for the maximal interval of existence of the solution u and if $t^* < +\infty$, then

   $$\lim_{t \nearrow t^*} \|u(t)\|_{H_0^1(\Omega)} = +\infty.$$

3. Investigate the uniqueness of such a solution.
4. Of which equation is the function u a mild solution ?