Sheet 2: Elliptic regularity and maximum principle

Exercise 1 (Control of the L^∞ norm). Let Ω be an open bounded subset of $\mathbb{R}^d$ of class C^2. Let $A \in C^1(\overline{\Omega}, S_d(\mathbb{R}))$ satisfying the following ellipticity condition

$$\exists \alpha > 0, \forall (x,\xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2. \tag{1}$$

Let $f \in L^2(\Omega)$ and $u \in H^1_0(\Omega)$ be the weak solution of the following Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1. In this question, we assume that $d \leq 3$. Show that there exists a constant $C \geq 0$ depending only on Ω and d such that

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}). \tag{2}$$

2. We assume that $\Omega = B(0,R)$ where $R > 0$.
   (a) Compute Δv when $v(x) = \psi(|x|)$ is a radial function.
   (b) By considering the function $u(x) = \ln|\ln|x||$ and the case $A(x) = \mathrm{I}_d$, discuss the validity of the estimate (2) when $d \geq 4$.

Note: One can prove (this is a bit technical) that when $d \geq 4$ and $f \in L^p(\Omega)$, where $p > d/2$, there exists a positive constant $C > 0$ only depending on d, Ω and p such that the following estimate, somehow analogous to (2), holds

$$\|u\|_{L^\infty(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Exercise 2 (Hölder regularity). The purpose is to show a gains of derivatives in Hölder spaces for the solution u to the Laplace equation $-\Delta u = f$, where $f \in C(\mathbb{R}^3)$ is a function with compact support. Let $G(x) = \frac{1}{4\pi}\frac{1}{|x|}$ be the Green function of the Laplacian in dimension 3. Let us recall that the function $u = G * f$ is a weak solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^3$. We assume that $f \in C^\alpha(\mathbb{R}^3)$ for a given $\alpha \in (0,1)$, and we set

$$[f]_{\dot{C}^\alpha(\mathbb{R}^3)} = \sup_{x \neq z \in \mathbb{R}^3} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < +\infty.$$

Let K be a compact of $\mathbb{R}^3$. We first want to prove that $u, \nabla u \in C^\alpha(K)$ and that there exists a positive constant $c_1 > 0$ only depending on K, d, α and on the support of f such that

$$[u]_{\dot{C}^\alpha(K)} + [\nabla u]_{\dot{C}^\alpha(K)} \leq c_1 [f]_{\dot{C}^\alpha(\mathbb{R}^3)}. \tag{3}$$

1. Show that $u \in C^\alpha(K)$ and that the estimate (3) holds for u.

2. By introducing a cut-off function ω_ε of the forme $\omega_\varepsilon(x) = \theta(\varepsilon^{-1}|x|)$ and considering the approximation $u_\varepsilon = (G\omega_\varepsilon) * f$, prove that $\nabla u \in C^\alpha(K)$ and that the estimate (3) holds for the function ∇u.

Note: By using similar techniques, one can prove that for all $\delta \in (0, \alpha)$, we have $\nabla^2 u \in C^\delta(K)$ and also that there exists a positive constant $c_2 > 0$ depending only on K, d, α, δ and the support of the function f such that

$$[\nabla^2 u]_{\dot{C}^\delta(K)} \leq c_2 [f]_{\dot{C}^\alpha(\mathbb{R}^3)}.$$

Exercise 3 (A non-linear equation). Let Ω be a bounded subset of $\mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}$ be a 1-Lipschitz function. Prove that the equation

$$\begin{cases} -\Delta u + u = b(\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in H_0^1(\Omega)$. Assuming moreover that $b \in C^\infty(\mathbb{R}^d)$, check that this solution u belongs to $C^\infty(\Omega)$.

Exercise 4 (Strong maximum principle). Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary and $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying $\Delta u \leq 0$ on Ω. Proof by hand that

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Hint: Assume first that $\Delta u < 0$.

Exercise 5 (Weak maximum principle for weak solutions). Let $\Omega \subset \mathbb{R}^d$ be a bounded open set.

1. Let $G \in C^1(\mathbb{R})$ a function with bounded derivative satisfying $G(0) = 0$.
   a) Check that for all $u \in H^1(\Omega)$, we have $G \circ u \in L^2(\Omega)$.
   b) Prove that $G \circ u \in H^1(\Omega)$ and that for all $1 \leq j \leq n$,

$$\partial_{x_j}(G \circ u) = (G' \circ u)\partial_{x_j} u.$$

2. We consider the following operator $L = -\operatorname{div}(A(x)\nabla u)$, where $A \in L^\infty(\Omega, M_d(\mathbb{R}))$ satisfies the following ellipticity assumption

$$\exists \alpha > 0, \forall (x, \xi) \in \Omega \times \mathbb{R}^d, \quad A(x)\xi \cdot \xi \geq \alpha |\xi|^2.$$

   We want to prove that if $u \in H_0^1(\Omega)$ is a weak solution of the equation $Lu \leq 0$, then $u \leq 0$ a.e. in Ω.
   (a) Prove that there exists a non-negative function $G \in C^1(\mathbb{R})$ with bounded derivative such that $G' > 0$ on $(0, +\infty)$ and $G' = 0$ on $(-\infty, 0]$.
   (b) By considering $\langle Lu, G \circ u \rangle_{L^2(\Omega)}$, prove that

$$\int_\Omega |\nabla u(x)|^2 (G' \circ u)(x) \, \mathrm{d}x \leq 0.$$

   (c) Conclude.

Exercise 6 (Estimates on the gradient). Let Ω be an open bounded subset of $\mathbb{R}^d$. Let A be a symmetric definite positive $d\times d$ matrix and $f \in \mathrm{Lip}(\bar{\Omega})$. We will establish gradient estimates for solutions u to the equation $Lu = f$ with Dirichlet homogeneous boundary condition, where L is the elliptic operator $Lu = -\operatorname{div}(A\nabla u)$, under the assumption that there exists a function $\psi \in \mathrm{Lip}(\Omega)\cap C^2(\Omega)$ such that $L\psi \geq f$ in Ω and $\psi = 0$ on $\partial\Omega$. For simplicity, we will consider the case where the function f is constant.

1. Let $\omega \subset \Omega$ and $u, v \in C^2(\omega)\cap C(\bar{\omega})$ satisfying $Lu \leq Lv$ in ω. Show that

$$\sup_{\overline{\omega}}(u-v) \leq \sup_{\partial\omega}(u-v).$$

2. Let $u \in C^2(\Omega)\cap C(\bar{\Omega})$ satisfying $Lu = f$ in Ω.
   (a) Prove that

$$\sup\left\{\frac{|u(x)-u(y)|}{|x-y|} : x, y \in \Omega, x \neq y\right\} = \sup\left\{\frac{|u(x)-u(y)|}{|x-y|} : x \in \Omega, y \in \partial\Omega\right\}.$$

   Hint: given $x_1, x_2 \in \Omega$ with $\tau = x_2 - x_1 \neq 0$, compare u and $u_\tau \colon x \mapsto u(x+\tau)$ in $\omega = \Omega \cap (-\tau + \Omega)$.

   (b) We assume furthermore that $u = 0$ on $\partial\Omega$. Show that $\mathrm{Lip}(u) \leq \mathrm{Lip}(\psi)$.

3. A ψ as above is called a *barrier function*. Construct a barrier function in the case $\Omega = B(0,1)$. *Hint*: consider $\psi(x) = -\gamma|x|^2/2 + C$ for some given constants $\gamma > 0$ and $C \in \mathbb{R}$.

Exercise 7. (Localization) Let Ω be an open subset of $\mathbb{R}^d$. We consider $A \in L^\infty_{\mathrm{loc}}(\Omega, M_d(\mathbb{R}))$, $b \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^d)$ and $c \in L^\infty_{\mathrm{loc}}(\Omega)$ and L the operator defined by

$$Lu = -\operatorname{div}(A(x)\nabla u) + b\cdot\nabla u + cu.$$

Assume that A satisfies the following ellipticity assumption

$$\exists \alpha > 0, \forall (x,\xi) \in \Omega\times\mathbb{R}^d, \quad A(x)\xi\cdot\xi \geq \alpha|\xi|^2.$$

Let Ω' and Ω'' be open subsets of Ω satisfying $\overline{\Omega'} \subset \Omega''$ and $\overline{\Omega''} \subset \Omega$. Prove that there exists a positive constant $C > 0$ such that for all $u \in H^1_{\mathrm{loc}}(\Omega)$,

$$\|\nabla u\|_{L^2(\Omega')} \leq C\big(\|Lu\|_{H^{-1}(\Omega'')} + \|u\|_{L^2(\Omega'')}\big).$$