
Sheet 5: Maximum principles and stability of steady states

Exercise 1. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a final time and $Q_T = (0,T) \times \Omega$. We consider the following differential operator

$$L = -\sum_{i,j=1}^{d} a^{i,j}(t,x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} b^i(t,x)\partial_{x_i} + c(t,x), \quad (t,x) \in Q_T,$$

the coefficients $a^{i,j}, b^i$ and c being bounded on Q_T, with moreover $a^{i,j} = a^{j,i}$. We assume that the operator $\partial_t + L$ is uniformly parabolic, that is,

$$\exists \theta > 0, \forall (t,x) \in Q_T, \forall \xi \in \mathbb{R}^d, \quad \sum_{i,j=1}^{d} a^{i,j}(t,x)\xi_i\xi_j \geq \theta|\xi|^2.$$

State as many maximum principles as you can for the parabolic operator $\partial_t + L$.

Exercise 2. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set, $T > 0$ be a positive time and $Q_T = (0,T) \times \Omega$. We also consider $f \in C^\infty(\mathbb{R})$ a smooth function. Let $u, v \in C^2(Q_T) \cap C^0(\bar{Q}_T)$ be two functions satisfying

$$\begin{cases} \partial_t v - \Delta v - f(v) \leq \partial_t u - \Delta u - f(u) & \text{in } Q_T, \\ v \leq u & \text{on } \partial Q_T. \end{cases}$$

Prove that $v \leq u$ on Q_T.

Application: Consider $u \in C^2(Q_T) \cap C^0(\bar{Q}_T)$ a solution of the equation

$$\begin{cases} \partial_t u - \Delta u = u(1-u)(u-a) & \text{in } Q_T, \\ u = 0 & \text{on } (0,T) \times \partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega, \end{cases}$$

where $0 < a < 1$ is a positive constant and u_0 is a smooth initial datum satisfying $0 \leq u_0 \leq 1$ in Ω. Prove that the function u is bounded as follows

$$\forall (t,x) \in Q_T, \quad 0 \leq u(t,x) \leq 1.$$

Can you be more precise when assuming $0 \leq u_0 < a$ in Ω ?

Exercise 3. Let $L > 0$. Prove that there exists a critical length $L_c > 0$ such that the equation

$$\begin{cases} q'' + q(1-q) = 0 & x \in (0,L), \\ q(0) = q(L) = 0, \end{cases} \tag{1}$$

has a non-trivial non-negative solution if and only if $L > L_c$. Why is this exercise in this sheet ?
Hint: The function $H(q_1, q_2) = q_1^2/2 + q_2^2/2 - q_1^3/3$ is a Lyapunov function for this equation.

Exercise 4. Let $L > 0$ be a length, $u_0 \in L^2(0,L)$ be an initial datum satisfying $u_0 > 0$ and u be the solution of the Fisher-KPP equation

$$\left\{\begin{array}{rcll} \partial_t u - \partial_{xx} u &=& u(1-u), & t>0,\ x\in(0,L), \\ u(t,0)=u(t,L) &=& 0, & t>0, \\ u(0,x) &=& u_0(x), & x\in(0,L), \end{array}\right. \tag{2}$$

We aim at proving that when $0 < L < \pi$, then

$$\forall x \in [0,L], \quad u(t,x) \underset{t\to+\infty}{\to} 0.$$

1. Find a subsolution $\underline{u}$ of the equation (2).
2. We consider $\overline{u}$ the solution of the equation

   $$\left\{\begin{array}{rcll} \partial_t \overline{u} - \partial_{xx} \overline{u} &=& \overline{u} & t>0,\ x\in(0,L), \\ \overline{u}(t,0)=\overline{u}(t,L) &=& 0 & t>0, \\ \overline{u}(0,x) &=& u_0(x) & x\in(0,L), \end{array}\right.$$

   Check that $\overline{u}$ is a supersolution of the equation (2).
3. Prove that

   $$\forall x \in [0,L], \quad \overline{u}(t,x) \underset{t\to+\infty}{\to} 0.$$

   Hint: Use Fourier series.
4. Conclude.

Exercise 5. We still consider the Fisher-KPP equation (2). Assuming this time that $L > \pi$, we aim at proving that there exists a supersolution $\overline{u}$ of the equation (2) such that $u(t,x) \le \overline{u}(t,x)$ for all $t \ge 0$ and $x \in (0,L)$, and satisfying

$$\forall x \in [0,L], \quad \overline{u}(t,x) \underset{t\to+\infty}{\to} q(x),$$

where q is the non-trivial non-negative steady state given by Exercice 3.

1. Let $\overline{u}$ be the solution of the equation

   $$\left\{\begin{array}{rcll} \partial_t \overline{u} - \partial_{xx} \overline{u} &=& \overline{u}(1-\overline{u}), & t>0,\ x\in(0,L), \\ \overline{u}(t,0)=\overline{u}(t,L) &=& 0, & t>0, \\ \overline{u}(0,x) &=& M, & x\in(0,L), \end{array}\right.$$

   with $M = \max(1, \sup_{(0,L)} u_0)$. Prove that $\overline{u}$ is a supersolution of the equation (2) which dominates the function u.
2. By comparing $\overline{u}(t+h,x)$ and $\overline{u}(t,x)$, prove that for all $x \in [0,L]$, the limit $w(x) = \lim_{t\to+\infty} \overline{u}(t,x)$ exists and satisfies the estimate $0 \le w(x) \le M$.
3. Admit that w is a solution of the equation (1). Deduce then that $w = q$ and conclude.

Remark: One can also prove that there exists a subsolution $\underline{u}$ converging pointwise to q and bounding the function u from below. As a consequence, $u(t,x) \to_{t\to+\infty} q(x)$ for all $x \in [0,L]$.