Sheet 8: Systems of reaction-diffusion equations

Exercise 1. Rabies may infect all warm-blooded animals, also birds, and also humans, and affects the central nervous system. Vaccines are available (but expensive); but no further cure is known. The spread seems to occur in waves, e.g. one coming from the Polish-Russian border; the spread velocity is approx. 30-60 km/year.

Let us consider two groups of foxes:

- Susceptible foxes (S), with no diffusion (as they are territorial),
- Infective foxes (I), with diffusion (loss of sense of territory), constant death rate.

The infection rate is assumed to be proportional to their densities, no reproduction or further spread:

$$\begin{cases} \partial_t S = -rIS, & t > 0,\ x \in \mathbb{R}, \\ \partial_t I = rIS - aI + \nu \partial_{xx}^2 I, & t > 0,\ x \in \mathbb{R}. \end{cases}$$

The non-dimensionalised version of the above system is the following:

$$\begin{cases} \partial_t S = -IS, & t > 0,\ x \in \mathbb{R}, \\ \partial_t T = IS - mI + \partial_{xx}^2 I, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

with $m = a/(rS_0)$, S_0 being the initial (maximum) susceptible density. We look for a travelling wave solution of this system of the form

$$S(t,x) = S(x-ct) = S(z) \quad \text{and} \quad I(t,x) = I(x-ct) = I(z),$$

where $z = x - ct$, the wave fronts S and I satisfying $0 \le S \le 1$ and $0 \le I \le 1$.

1. Write the system of ODEs satisfied by the functions S and I.
2. Justify the following boundary conditions: $S(+\infty) = 1$, $I(+\infty) = 0$, $S'(-\infty) = 0$, $I(-\infty) = 0$.
3. Check that
$$S(-\infty) - m \ln S(-\infty) = 1.$$
Deduce the fraction of susceptibles which survives the "rabies wave" (draw a picture).
4. Draw the phase plane associated with the system satisfied by S and I.
5. Explain why $c = 2\sqrt{1-m}$ is the minimal wave speed.
6. Draw the shapes of the waves fronts S and I.

Exercise 2. We consider a simple predator-prey model with logistic growth of the prey

$$\begin{cases} \partial_t u = u(1-u-v) + \nu \partial_{xx}^2 u, & t > 0,\ x \in \mathbb{R}, \\ \partial_t v = av(u-b) + \partial_{xx}^2 v, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

with $a > 0$, $0 < b < 1$ and $\nu \ge 0$ some constants.

1. Check that the spatially independent system admits three stationary points, namely $(0,0)$, $(1,0)$ and $(b, 1-b)$. Study their stability.

We look for constant shape travelling wavefront solutions moving to the left:

$$u(t,x) = U(z) \quad \text{and} \quad v(t,x) = V(z),$$

where z denotes the wave variable $z = x + ct$ and $c > 0$ is positive.

2. Write the system of ODEs satisfied by the wave fronts U and V.

As a simpler case, we assume that the prey is diffusing much slower than the predators (e.g. consider a system where animals eat some plants), thus $\nu = 0$ is assumed.

3. Transform the system obtained in Question 2 in a new system of three ODEs of order one. Check that its stationary points are $(0,0,0)$, $(1,0,0)$ and $(b, 1-b, 0)$.
4. Compute the Jacobian matrix $J(U,V,W)$ of this system.
5. By studying the stability of the point $(1,0,0)$, explain why the wave speed c should necessarily satisfy $c \geq \sqrt{4a(1-b)}$ for keeping the possibility of a travelling wavefront.
6. Check that the point $(0,0,0)$ is unstable.
7. Let p be the characteristic polynomial of the matrix $J(b, 1-b, 0)$. What can we say about the local maxima of p ? Draw the typical graph of p for various values of a.
8. Justify that we could find some solutions with the following boundary conditions:

$$U(-\infty) = 1, \quad V(-\infty) = 0, \quad U(+\infty) = b, \quad V(+\infty) = 1-b, \tag{1}$$

and / or:

$$U(-\infty) = 0, \quad V(-\infty) = 0, \quad U(+\infty) = b, \quad V(+\infty) = 1-b.$$

9. By considering the boundary conditions (1), draw the possible shapes for the fronts U and V.

Exercise 3. We consider the Belousov-Zhabotinskii chemical reaction modeled by the following system:

$$\begin{cases} \partial_t u \;=\; Lrv + u(1-u-rv) + \partial^2_{ss} u, & t > 0,\ x \in \mathbb{R}, \\ \partial_t v \;=\; -Mv - buv + \partial^2_{ss} v, & t > 0,\ x \in \mathbb{R}, \end{cases}$$

where L and M are of order 10^{-4}, b is of order 1, r is something between 5 and 50.

1. Check that the spatially homogeneous stationary states are $(0,0)$ and $(1,0)$.

Due to $L \ll 1$ and $M \ll 1$, we neglect the corresponding terms, which yields a model for the leading edge of travelling waves in the Belousov-Zhabotinskii reaction:

$$\begin{cases} \partial_t u \;=\; u(1-u-rv) + \partial^2_{ss} u, & t > 0,\ x \in \mathbb{R}, \\ \partial_t v \;=\; -buv + \partial^2_{ss} v, & t > 0,\ x \in \mathbb{R}. \end{cases}$$

We search for travelling wavefront solutions $u(t,x) = U(x+ct)$ and $v(t,x) = V(x+ct)$ for this new system, moving to the left and satisfying the boundary conditions

$$U(-\infty) = 0, \quad V(-\infty) = 1, \quad U(+\infty) = 1, \quad V(+\infty) = 0.$$

2. * By using what was stated previously for the Fisher-KPP equation and the comparison theorem, show that necessarily, the wave speed satisfies $c \leq 2$.

Remark: The best known result is *a priori* $((r^2 + \frac{2b}{2})^{1/2} - r)(2(b+2r))^{-1/2} \leq c \leq 2$.