Sheet 9: Reviews

Exercise 1 (Faber-Krahn inequality). Let Ω be an open bounded subset of $\mathbb{R}^d$, with $d \geq 3$, and $V \in L^\infty(\Omega)$ such that $V \geq 0$. We consider the problem

$$\begin{cases} -\Delta u = Vu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

1. Give the definition of a weak solution to the equation (1).
2. Can you apply the Lax-Milgram theorem here?
3. Let $r > \frac{d}{2}$. Show that there is a constant $c_d > 0$ depending on d only such that, if (1) has a non-trivial weak solution, then
$$|\Omega|^{\frac{2}{d}-\frac{1}{r}}\|V\|_{L^r(\Omega)} \geq c_d.$$
Hint: Use the following Sobolev inequality
$$\|u\|_{L^{2^*}(\Omega)} \leq M_d\|\nabla u\|_{L^2(\Omega)}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d},$$
which holds for all $u \in H_0^1(\Omega)$, where M_d depends on d only.
4. What do you obtain in the particular case $V = \lambda = \text{cst}$?

Exercise 2 (Estimates on the gradient). Let Ω be an open bounded subset of $\mathbb{R}^d$. Let A be a symmetric definite positive $d \times d$ matrix and $f \in \mathrm{Lip}(\bar{\Omega})$. We will establish gradient estimates for solutions u to the equation $Lu = f$ with Dirichlet homogeneous boundary condition, where L is the elliptic operator $Lu = -\operatorname{div}(A\nabla u)$, under the assumption that there exists a function $\psi \in \mathrm{Lip}(\Omega) \cap C^2(\Omega)$ such that $L\psi \geq f$ in Ω and $\psi = 0$ on $\partial\Omega$. For simplicity, we will consider the case where the function f is constant.

1. Let $\omega \subset \Omega$ and $u, v \in C^2(\omega) \cap C(\bar{\omega})$ satisfying $Lu \leq Lv$ in ω. Show that
$$\sup_{\overline{\omega}}(u - v) \leq \sup_{\partial\omega}(u - v).$$
2. Let $u \in C^2(\Omega) \cap C(\bar{\Omega})$ satisfying $Lu = f$ in Ω.
   (a) Prove that
   $$\sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : x, y \in \Omega, x \neq y\right\} \leq \sup\left\{\frac{|u(x) - u(y)|}{|x - y|} : x \in \Omega, y \in \partial\Omega\right\}.$$
   Hint: given $x_1, x_2 \in \Omega$ with $\tau = x_2 - x_1 \neq 0$, compare u and $u_\tau \colon x \mapsto u(x + \tau)$ in $\omega = \Omega \cap (-\tau + \Omega)$.
   (b) We assume furthermore that $u = 0$ on $\partial\Omega$. Show that $\mathrm{Lip}(u) \leq \mathrm{Lip}(\psi)$.
3. A ψ as above is called a *barrier function*. Construct a barrier function in the case $\Omega = B(0, 1)$. *Hint*: consider $\psi(x) = -\gamma|x|^2/2 + C$ for some given constants $\gamma > 0$ and $C \in \mathbb{R}$.

Exercise 3 (The method of continuity).

1. Solve the equation $u - \Delta u = f$ on $\mathbb{T}^d$ and show that it defines a map $L^2(\mathbb{T}^d) \to H^2(\mathbb{T}^d)$.
2. Let X, Y be some Banach spaces. Let $(T_t)_{t\in[0,1]}$ be a *continuous* path of linear operators from X to Y satisfying

$$\exists C \geq 0, \forall u \in X, \forall t \in [0,1], \quad \|u\|_X \leq C\|T_t u\|_Y. \tag{2}$$

   Prove that T_0 is surjective if and only if T_1 is surjective as well.
3. Let $(a_{i,j})_{1\leq i,j\leq d}$ be a family of maps of class C^1 on $\mathbb{T}^d$. We assume that the following ellipticity condition holds

$$\exists \alpha > 0, \forall x \in \mathbb{T}^d, \forall \xi \in \mathbb{R}^d, \quad a_{i,j}(x)\xi_i\xi_j \geq \alpha|\xi|^2.$$

   We define the path $(T_t)_{t\in[0,1]}$ of operators $H^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$ by the formula

$$T_t u = u - \partial_i(a_{ij}^{(t)}(x)\partial_j u), \quad a_{ij}^{(t)} = ta_{ij} + (1-t)\delta_{ij}.$$

   (a) Show that $t \mapsto T_t$ is continuous.
   (b) Check that (2) is satisfied.
   (c) What to conclude ?

Exercise 4 (The heat equation). Let $u_0 : \mathbb{R} \to \mathbb{R}$ be a continuous, piecewise C^1 and 2π-periodic function. Prove that there exists a unique function $u \in C^0([0,+\infty) \times \mathbb{R}) \cap C^\infty((0,+\infty) \times \mathbb{R})$ satisfying

$$\begin{cases} \partial_t u(t,x) = \partial_x^2 u(t,x), & (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

the function $u(t,\cdot)$ being moreover 2π-periodic for all $t \geq 0$.

Exercise 5 (A reaction-diffusion equation). We consider the following reaction-diffusion equation:

$$\begin{cases} \partial_t u - \Delta u = u^3 & \text{in } (0,+\infty) \times \mathbb{R}, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}, \end{cases} \tag{3}$$

with initial datum $u_0 \in H^1(\mathbb{R})$.

1. Establish *a priori* energy estimates for the equation (3).
2. By using an iterative method, prove that there exists a positive time $T > 0$ and a unique solution $u \in C^0([0,T], H^1(\mathbb{R}))$ of the equation (3). Check that $u \in C^\infty((0,T) \times \mathbb{R})$.
3. Assuming moreover that the initial datum u_0 is fast decaying, establish pointwise estimates for the solution u.

Exercise 6 (The Fisher-KPP equation with Allee effect). We consider the one-dimensional Fisher-KPP equation with Allee effect posed on the whole space

$$\partial_t u - \partial_{xx} u = u(1-u)(u-a), \quad t > 0,\ x \in \mathbb{R}, \tag{4}$$

where $0 < a < 1/2$ is a parameter. Study the existence of traveling wave solutions for this equation, that is, solutions of the form

$$u(t,x) = \phi(x - ct), \quad t > 0,\ x \in \mathbb{R},$$

with $c > 0$.