Banach-Nečas-Babuška theorem and proof

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I - Theorem

The scalar field is always $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Theorema 0.1: Banach–Nečas–Babuška

Let V, W be respectively Banach and reflexive Banach space.

$a : V \times W$ continuous bilinear form.
$f : W \to \mathbb{R}$ continuous linear form.

$$(\star\star) : \quad \text{find } u \in V, \quad a(u, w) = f(w) \quad \forall w \in W$$

$(\star\star)$ is well-posed if and only if

$$\textit{(BNB1)} \qquad \exists C_{\text{sta}} > 0, \forall v \in V, \quad \sup_{w \in W \setminus \{0\}} \frac{a(v, w)}{\|w\|_W} \geq C_{\text{sta}} \|v\|_V$$

$$\textit{(BNB2)} \qquad \forall w \in W, \quad (\forall v \in V, a(v, w) = 0) \implies w = 0$$

The following control holds true :

$$\|u\|_V \leq \frac{1}{C_{\text{sta}}} \|f\|_{W'}$$

II - Proof

1) Definitions

Definition 0.2: Well-posedness

$$(\star\star) : \quad \text{find } u \in V, \quad a(u, w) = \mathbf{f}(w) \quad \forall w \in W$$

is well-posed in the sense of Hadamard if and only if there exists a *unique* solution u and

$$\exists c > 0, \ \forall f \in W', \quad \|u\|_V \leq c \|f\|_{W'}$$

In the next section we always consider V, W real Banach spaces.
We also denote $A \in \mathcal{L}(V, W)$ a linear continuous operator.

Definition 0.3: Dual space

The dual space of V is $V' := \mathcal{L}(V, \mathbb{R})$ the space of continuous linear forms.
We denote $\langle A|v\rangle_{V',V} = Av$.

Remark :

Being given an operator $A : V \to W$, one can define a unique linear operator A^T by Riesz representation theorem that has the following property :

$$\forall v \in V, \forall w' \in W', \quad \left\langle A^T w' | v \right\rangle_{V',V} = \langle w' | Av \rangle_{W',W}$$

Definition 0.4: Dual operator

The dual operator $A^T : W' \to V'$ is defined by

$$\forall v \in V, \forall w' \in W', \quad \left\langle A^T w' | v \right\rangle_{V',V} = \langle w' | Av \rangle_{W',W}$$

Definition 0.5: Annihilator

For $M \subset V$, $N \subset V'$

$$M^\perp = \left\{ v' \in V' \;\middle/\; \forall m \in M, \langle v' | m \rangle_{V,V'} = 0 \right\} \subset V'$$

$$N^\perp = \left\{ v \in V \;\middle/\; \forall n' \in N, \langle n' | v \rangle_{V',V} = 0 \right\} \subset V$$

We note that $V^\perp = \{0_{V'}\}$ and $\{0_V\}^\perp = V'$.

2) Preliminary results

From these definitions, we can characterise the range and the kernel of an operator.

Lemma 0.6: Characterisation of the range and kernel

- $\text{Ker } A = (\text{Im } A^T)^\perp$
- $\text{Ker } A^T = (\text{Im } A)^\perp$
- $\overline{\text{Im } A} = (\text{Ker } A^T)^\perp$
- $\overline{\text{Im } A^T} \subset (\text{Ker } A)^\perp$

$\mathcal{D}$

Functional Analysis, Sobolev Spaces and Partial Differential Equations, [Bre10] p.45 □

Theorema 0.7: Closed Range

EQU :

- Im A is closed
- Im A^T is closed
- Im $A = (\text{Ker } A^T)^\perp$
- Im $A^T = (\text{Ker } A)^\perp$

$\mathcal{D}$

Functional Analysis, Sobolev Spaces and Partial Differential Equations, [Bre10] p.46 □

Theorema 0.8: Open Mapping

If A is surjective and U is an open set then

$$A(U) \text{ is open in } W$$

$\mathcal{D}$

Functional Analysis, Sobolev Spaces and Partial Differential Equations, [Bre10] p.35 □

3) Characterisation of a functionnal operator

Lemma 0.9

EQU

- Im A is closed
- There exists $\alpha > 0$ such that for all $w \in \text{Im } A$, there exists $v_w \in V$

$$Av_w = w \text{ and } \|w\|_W \geq \alpha \|v_w\|_V$$

$\mathcal{D}$

$\Rightarrow$: Suppose Im $A \subset W$ is closed.
Since W is Banach, Im A is also one.
Considering $B = B_V(0,1)$ the open unity sphere and using the Open Mapping theorem 0.8 with $A \simeq A|_{\text{Im } A} : V \to \text{Im } A$ surjective, $A(B)$ is open.
Since $0 \in A(B)$, there exists $\gamma > 0$ such that $B_W(0,\gamma) \subset A(B)$.
Let $w \in \text{Im } A$, then $\frac{\gamma}{2}\frac{w}{\|w\|_W} \in B_W(0,\gamma) \subset A(B)$.
Then there exists $z \in B$ such that

$$Az = \frac{\gamma}{2}\frac{w}{\|w\|_W} \iff A(\underbrace{\frac{2\|w\|_W}{\gamma}z}_{v_w}) = w$$

and

$$\|v_w\|_V = \underbrace{\|z\|_V}_{\leq 1}\frac{2}{\gamma}\|w\|_W \leq \frac{2}{\gamma}\|w\|_W \iff \underbrace{\frac{\gamma}{2}}_{:=\alpha}\|v_w\|_V \leq \|w\|_W$$

$\Leftarrow$ Suppose there exists $\alpha > 0$ such that for all $w \in \text{Im } A$, there exists $v_w \in V$

$$Av_w = w \text{ and } \|w\|_W \geq \alpha \|v_w\|_V$$

Let $(w_n)_n \in (\text{Im } A)^{\mathbb{N}}$ be a converging sequence to $w \in W$.
There exists $(v_n)_n$ such that $\begin{cases} Av_n = w_n \\ \|w_n\|_W \geq \alpha \|v_n\|_V \end{cases} \forall n.$
$(w_n)_n$ convergence implies $(w_n)_n$ is a Cauchy sequence and by the inequality $(v_n)_n$ is a Cauchy sequence in V.
Yet V is a Banach thus $(v_n)_n$ converge to $v \in V$.
A being continuous we can then write

$$Av = w$$

Hence $w \in \text{Im } A$ i.e. Im A is closed.

$\square$

Lemma 0.10

EQU :

- (i) A^T is surjective
- (ii) A is injective and Im A is closed
- (iii) There exists $\alpha > 0$ such that $\forall v \in V, \|Av\|_W \geq \alpha \|v\|_V$.
- (iv) There exists $\alpha > 0$ such that $\forall v \in V, \inf_{W'} \sup_{V} \frac{\langle A^T w' | v \rangle_{V',V}}{\|w'\|_{W'} \|v\|_V} \geq \alpha.$

$\mathcal{D}$

(i) $\Rightarrow$ (ii) :
A^T is surjective hence Im $A^T = V'$ is closed because V' is a Banach.
Hence Im A is closed using Closed Range Theorem 0.7.
Then $(\text{Im } A^T)^\perp = \{0\} = \text{Ker } A$.

(ii) $\Rightarrow$ (i) :
By Closed Range Theorem 0.7, Im A closed $\Rightarrow$ Im $A^T = (\text{Ker } A)^\perp$.
Yet A is injective hence Ker $A = \{0\}$ thus Im $A^T = V'$ i.e. A^T is surjective.

(ii) $\Rightarrow$ (iii) :
Im A is closed and Im $A = \{Av \ / \ v \in V\}$.
Using Lemma 0.9 one can construct α as in the proof to have

$$\|Av\|_w \geq \alpha \|v\|_V$$

(iii) $\Rightarrow$ (ii) :
The injectivity of A comes directly from the inequality. Im A is closed using the same proof as 3) in the $\Leftarrow$ part.

(iii) $\Rightarrow$ (iv) :
Using the corollary of Hahn-Banach theorem :

$$\sup_{w' \in W'} \frac{\langle w' | Aw \rangle_{W',W}}{\|w'\|_W} = \|Av\|_W \geq \alpha \|v\|_V$$

Hence dividing by $\|v\|_V$ and taking the inf

$$\inf_{v \in V} \sup_{w' \in W'} \frac{\langle w' | Aw \rangle_{W',W}}{\|w'\|_W \|v\|_V} \geq \alpha$$

(iv) $\Rightarrow$ (iii) :
Take $v \in V$.

$$\begin{aligned} \|Av\|_W &= \sup_{w' \in W'} \frac{\langle w' | Aw \rangle_{W',W}}{\|w'\|_W} \\ &= \sup_{w' \in W'} \frac{\langle w' | Aw \rangle_{W',W}}{\|w'\|_W \|v\|_V} \|v\|_V \\ &\geq \inf_{v \in V} \sup_{w' \in W'} \frac{\langle w' | Aw \rangle_{W',W}}{\|w'\|_W \|v\|_V} \|v\|_V \geq \alpha \|v\|_V \end{aligned}$$

□

Theorema 0.11: Bijectivity characterisation

A is bijective if and only if

$$\begin{cases} A^T : W' \to V' \text{ injective} \\ \exists \alpha > 0,\ \forall v \in V,\ \|Av\|_W \geq \alpha \|v\|_V \end{cases}$$

$\mathcal{D}$

$\Rightarrow$:
Ker $A^T \overset{0.6}{=} (\text{Im } A)^\perp = \{0\}$ because A is surjective. i.e. A^T injective.
Yet Im $A = W$ is closed. Using Lemma 0.10 : $\|Av\|_W \geq \alpha \|v\|_V$.

$\Leftarrow$:
Using Lemma 0.10 (iii) $\Rightarrow$ (ii) , we get Im A closed and A injective.
Since Im A is closed by Closed Range theorem 0.7, Im $A = (\text{Ker } A^T)^\perp = W$ hence A bijective. □

Corollary 0.12

$A \in \mathcal{L}(V, W)$ is associated with $a \in \mathcal{L}(Z_1 \times Z_2, \mathbb{R})$ such that

$$a(z_1, z_2) = \langle Az_1 | z_2 \rangle_{Z_2', Z_2}$$

i.e. $V = Z_1$ and $W = Z_2'$.
If Z_2 is reflexive the following equivalence holds :

- For all $f \in Z_2'$ there is a unique $u \in Z_1$ such that

$$a(u, z_2) = \langle f | z_2 \rangle_{Z_2', Z_2} \, \forall z_2 \in Z_2$$

- There exists $\alpha > 0$ such that :

$$\inf_{z_1 \in Z_1} \sup_{z_2 \in Z_2} \frac{a(z_1, z_2)}{\|z_1\|_{Z_1} \|z_2\|_{Z_2}} \geq \alpha$$

  and

$$\forall z_2 \in Z_2,\ (\forall z_1 \in Z_1, a(z_1, z_2) = 0) \Longrightarrow (z_2 = 0)$$

$\mathcal{D}$

$\forall f \in Z_2', \exists!\ u \in Z_1,\ a(u, z_2) = \langle f | z_2 \rangle_{Z_2', Z_2} \quad \forall z_2 \in Z_2$
$\Longleftrightarrow \ \forall f \in Z_2', \exists!\ u \in Z_1, \langle Au | z_2 \rangle_{Z_2', Z_2} = \langle f | z_2 \rangle_{Z_2', Z_2} \quad \forall z_2 \in Z_2$
$\Longleftrightarrow \ \forall f \in Z_2', \exists!\ u \in Z_1, \langle Au - f | z_2 \rangle_{Z_2', Z_2} = 0 \quad \forall z_2 \in Z_2$
$\Longleftrightarrow \ \forall f \in Z_2', \exists!\ u \in Z_1, \quad Au - f \in (Z_2)^\perp = \{0\}$
$\Longleftrightarrow \ \forall f \in Z_2', \exists!\ u \in Z_1, \quad Au = f$

$\iff$ A is bijective.
$\iff$ Theorem 0.11

When the proceed to prove that the two results of this theorem are equivalent to those of the corollary :
$\exists\alpha > 0,\ \forall z_1 \in Z_1,\ \|Az_1\|_{Z_2'} \geq \alpha \|z_1\|_{Z_1}$.
Yet $\|Az_1\|_{Z_2'} = \sup\limits_{z_2 \in Z_2} \dfrac{\langle Az_1 | z_2 \rangle_{Z_2', Z_2}}{\|z_2\|_{Z_2}} = \sup\limits_{z_2 \in Z_2} \dfrac{a(z_1, z_2)}{\|z_2\|_{Z_2}}$.
Then dividing by $\|z_1\|_{2_1}$ and taking the infinimum gives the result.

Then with the second claim it follows :
$A^T : Z_2 \to Z_1'$ injective
$\iff \forall z_2 \in Z_2,\quad A^T z_2 = 0 \Rightarrow z_2 = 0$
$\iff \forall z_2 \in Z_2,\quad (\forall z_1 \in Z_1, \langle A^T z_2 | z_1 \rangle_{Z_1', Z_1} = 0) \Rightarrow (z_2 = 0)$
$\iff \forall z_2 \in Z_2,\quad (\forall z_1 \in Z_1, \langle z_2 | Az_1 \rangle_{Z_2, Z_2'} = 0) \Rightarrow (z_2 = 0)$
$\iff \forall z_2 \in Z_2,\quad (\forall z_1 \in Z_1, a(z_1, z_2) = 0) \Rightarrow (z_2 = 0)$

□

$\mathcal{D}$: Proof Banach-Nečas-Babuška theorem

The corollary 0.12 is a rewriting of Banach-Nečas-Babuška Theorem 0.1.
The a priori estimate results from :

$$\|f\|_{V'} = \sup_{v \in V} \frac{|f(v)|}{\|v\|_V} = \sup_{v \in V} \frac{a(u,v)}{\|v\|_V} \geq \alpha \|u\|_W$$

□

Refer to *Theory and Practice of Finite Elements* [EG10], Ern and Guermond, for most of the results and further informations.
See also Norikazu Saito's *Notes on the Banach-Necas-Babuska theorem and Kato's minimum modulus of operators* for historical context and more results.

References

[Bre10] Haïm Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York, NY, 2010. DOI: https://doi.org/10.1007/978-0-387-70914-7.

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