Séminaire Landau

Non linear controllability of ODE and PDE with several controls

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under the supervision of Karine Beauchard and Frédéric Marbach

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Introduction :

Controllability : ajust a function parameter of an ODE/PDE so that the solution reachs a target at a fixed time.

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ODE : affine systems (finite dimension).

PDE : bilinear Schrödinger equation with two controls.

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ODE

Definition (Affine Systems)

An ODE is an affine system if it is given by :

$$x' = f_0(x) + \sum_{i=1}^m u_i f_i(x),$$
(1)

where $x(t) \in \mathbb{R}^d$ is the stat, and $u(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_m(t) \end{pmatrix} \in \mathbb{R}^m$, the control. The

function f_0 is the **drift**.

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 ODE
 Schrödinger equation

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 Lie Brackets
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One supposes that f_0, \dots, f_m are regular functions, and $f_0(0) = 0$ (equilibrium). One considers x(t; u, p) the solution of (1) at time t, with the initial condition x(0; u, p) = p.

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 L^{∞} -Small Time Locally Controllable : $\forall T > 0, \forall \eta > 0, \exists \delta > 0$, so that $\forall x^* \in \mathbb{R}^d, ||x^*|| \leq \delta$, then, $\exists u \in L^{\infty}(0, T)$ such that $||u||_{\infty} \leq \eta$ and $x(T, u, 0) = x^*$.

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No SNC A few NC/SC are known \longrightarrow in term of Lie Brackets.

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Definition (Lie Brackets)

Let Ω be a nonempty openset of \mathbb{R}^d , and $X, Y : \Omega \to \mathbb{R}^d$ vector fields, \mathcal{C}^1 . One defines the Lie Bracket of X, and Y, [X, Y] as :

$$[X,Y]: x\in \Omega\mapsto DY(x)(X(x))-DX(x)(Y(x))\in \mathbb{R}^d.$$

 $[X, Y] \in \mathcal{C}^0(\Omega, \mathbb{R}^d).$

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Definition (Iterated Lie Brackets)

Let Ω be a nonempty openset of \mathbb{R}^d , and $X, Y : \Omega \to \mathbb{R}^d$ vector fields, \mathcal{C}^{∞} . One defines, for $k \in \mathbb{N}$, $ad_X^k(Y)$ by induction as :

$$\operatorname{ad}_X^0(Y) = Y$$
 and $\operatorname{ad}_X^{k+1}(Y) = [X, \operatorname{ad}_X^k(Y)].$

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Theorem (Hermann and Nagano, 1963-66)

Let f_0, \dots, f_m be analytic vector fields. One supposes that (1) is STLC, then, the (following) LARC is true i.e.

 $\{g(0), g \in Lie(f_0, \cdots, f_m)\} = \mathbb{R}^d.$

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Theorem (Linear Test)

Let f_0, f_1, f_2 be analytic vector fields (and $f_0(0) = 0$). One supposes that :

$$Span(ad_{f_0}^k(f_1)(0), ad_{f_0}^k(f_2)(0) \quad k \in \mathbb{N}) = \mathbb{R}^d.$$

Then, the affine system $x' = f_0(x) + uf_1(x) + vf_2(x)$ is STLC.

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Purpose : establish, with a new proof, a positive result of controllability $(S(\theta) \text{ condition of Sussmann})$ (already known) for affine systems. We want to formulate the result **in terms of Lie Brackets**. Then, we want to adapt this proof in the framework of **PDE (infinite dimension)** - **bilinear Schrödinger equation**.

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Strategy : approximate the solution of an ODE by a Lie Brackets series (Magnus formula) :

- solve a moment problems to give the value of the partial sum
- estimate the rest of the sum.

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We construct a suitable basis \mathcal{B} of $\mathcal{L}(X)$ (Viennot theorem).

$$\mathcal{B} \cap \mathcal{S}_{2} = \left\{ \underbrace{(X_{1}0^{j}, X_{1}0^{j+1})0^{k}, (X_{2}0^{j}, X_{2}0^{j+1})0^{k}}_{bad}, \underbrace{(X_{1}0^{j}, X_{2})0^{k}}_{good}, j, k \ge 0 \right\}$$

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Theorem

Let $L \in \mathbb{N}$, $\delta > 0$, $f_0, f_1, f_2 \in \mathcal{C}^{\omega}(B_{2\delta}, \mathbb{R}^d)$. One supposed that $f_0(0) = 0$,

$$Span(f_b(0), \ b \in \mathcal{B} \cap (\mathcal{S}_1 \cup \mathcal{S}_{2,good}), \ |b| \leqslant L) = \mathbb{R}^d.$$

If, for all $b \in S_{2,bad}$ with $|b| \leq L+1$,

 $f_b(0)\in \mathcal{S}_1(f)(0),$

then, $x' = f_0(x) + uf_1(x) + vf_2(x)$ is $L^{\infty} - STLC$.

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 $\ensuremath{\textbf{PDE}}$: generalise the result of affine systems in the framework of infinite dimension.

Bilinear Schrödinger equation with two controls : for T > 0,

$$\begin{cases} i\partial_{t}\psi(t,x) = -\partial_{xx}^{2}\psi(t,x) - u(t)\mu_{1}(x)\psi - v(t)\mu_{2}(x)\psi(t,x), \\ x \in (0,1), t \in (0,T) \\ \psi(0,x) = \varphi_{1}(x) \end{cases}$$
(2)

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We consider $(\varphi_j)_{j \ge 1}$, the orthonormal basis of $L^2(0,1)$, given by the engeinvectors of the Laplacian with Dirichlet boundary conditions, with $\lambda_j = (j\pi)^2$.

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We linearise this equation (in terms of (ψ, u)) around the trajectory $(\psi_1, u, v \equiv 0)$ (where ψ_1 is the ground state) $\longrightarrow \Psi(\mathcal{T})$.

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K. Beauchard and C. Laurent proved an equivalent of the linear test for (2). We want to generalise this result, with the **quadratic term** $\longrightarrow \xi(T)$.

We obtain :

$$\psi(T) \simeq \psi_1(T) + \Psi(T) + \xi(T).$$

One supposes $(\Psi(T), \varphi_{\mathcal{K}})_{L^2(0,1)} = 0$ (lost direction). Moreover,

$$\begin{split} (\xi(T),\varphi_{\mathcal{K}}e^{-i\lambda_{1}T})_{L^{2}(0,1)} &= \int_{0}^{T}\left(\int_{0}^{t}h_{1}(t,s)u(s)\mathrm{d}s\right)u(t)\mathrm{d}t + \\ \int_{0}^{T}\left(\int_{0}^{t}h_{2}(t,s)v(s)\mathrm{d}s\right)u(t)\mathrm{d}t + \int_{0}^{T}\left(\int_{0}^{t}h_{3}(t,s)u(s)\mathrm{d}s\right)v(t)\mathrm{d}t + \\ \int_{0}^{T}\left(\int_{0}^{t}h_{4}(t,s)v(s)\mathrm{d}s\right)v(t)\mathrm{d}t, \end{split}$$

where

$$h_1 = h_{\mu_1,\mu_1}, \qquad h_2 = h_{\mu_2,\mu_1}, \qquad h_3 = h_{\mu_1,\mu_2}, \qquad h_4 = h_{\mu_2,\mu_2},$$

and, for $1\leqslant i,j\leqslant 2$,

$$h_{\mu_i,\mu_j}(t,s) = -\sum_{k=1}^{+\infty} (\mu_i \varphi_1, \varphi_k)_{L^2} (\varphi_k, \mu_j \varphi_K)_{L^2} e^{i(\lambda_k(s-t)+\lambda_K(t-T)+\lambda_1(T-s))}.$$

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Under good hypotheses, we obtain :

$$\left(\xi(T),\varphi_{K}e^{-i\lambda_{1}T}\right)_{L^{2}(0,1)}=(-i)^{n}T^{n+2}\gamma_{K}^{n}\int_{0}^{1}\bar{u}_{n+2}(t)\bar{v}(t)\mathrm{d}t+\mathcal{O}(T^{n+3}),$$

with

$$\gamma_{K}^{n} = (ad_{\Delta}^{n}(\mu_{1}), \mu_{2})\varphi_{1}, \varphi_{K})_{L^{2}(0,1)}.$$

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The proof is divised in two steps :

1. In the first step, on (0, T): we use moment problems to fix the value of $(\xi(T), \varphi_K)_{L^2(0,1)}$. The linear can evolve freely.

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PDE

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- 2. We use the result about small-time exact controllability in projection to force the value under :

 $\overline{\mathsf{Span}(\varphi_j, \ j \in \mathbb{N}^* \setminus \{K\})}.$

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We must do **carfully** in order not to destroy the first step. Then, we conclude by the Brouwer fixed-point theorem

This work is in progress...

Thank you for your attention !

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