Small Time Local Controllability of the multi-input bilinear Schrödinger equation thanks to a quadratic term

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$(*) : x' = f_0(x) + u f_1(x) + v f_2(x)$, with $f_0, f_1, f_2 \in \mathcal{C}^\omega(\mathbb{R}^d)$.

We assume that $f_0(0) = 0$, *i.e.* $(0, (0, 0))$ is an **equilibrium** trajectory of the system $(*)$.

We focus on small time and small controls : the solution is well-defined, and we note it $x(\cdot; (u, v), 0)$.

Definition (E-STLC)

$(*)$ is $\mathbf{E-STLC}$ around the equilibrium if : for all $T > 0$, $\varepsilon > 0$, there exists $\delta > 0$ such that, for all target $x_f \in \mathbb{R}^d$ such that $\|x_f\| \leqslant \delta$, there exists $u, v \in E$ with $\|(u, v)\|_E \leqslant \varepsilon$ such that $x(T; (u, v), 0) = x_f$.

Historical definition : $E = L^\infty$.

Definition (Lie Brackets)

For f, g, regular vectors fields on $\mathbb{R}^d$, we define the vector field $[f, g]$ as :

$$[f, g] : x \in \mathbb{R}^d \mapsto g'(x)f(x) - f'(x)g(x).$$

By induction, one defines :

$$ad_f^0 g = g \qquad \forall k \in \mathbb{N}, ad_f^{k+1}(g) = [f, ad_f^k(g)].$$

We want to prove **sufficient conditions** of controllability in terms of the evaluation at $x = 0$ of **Lie brackets** of f_0, f_1 and f_2

Theorem (K. Beauchard, F. Marbach)

The solution of $(*)$ is given by

$$x(T;(u,v),0) = \sum_{b\in\mathcal{B}_{[\![1,2]\!]},|b|\leqslant L} \underbrace{\xi_b(T,(u,v))}_{\text{explicit functional in (u,v)}} \times \underbrace{f_b}_{\in Lie(f_0,f_1,f_2)}(0) + \textit{remainders},$$

where $\mathcal{B}_{[\![1,2]\!]}$ is a set of brackets.

The set $\mathcal{B}_{[\![1,2]\!]}$ is defined as :

$$\mathcal{B} := \underbrace{\mathcal{B}_1}_{\substack{\text{linear terms : brackets}\\ \text{with } f_1 \text{ or } f_2 \text{ one time}}} \cup \underbrace{\mathcal{B}_{2,good} \cup \mathcal{B}_{2,bad}}_{\substack{\text{quadratic terms : brackets}\\ \text{with } f_1 \text{ or } f_2 \text{ two times}}} .$$

For $\mathcal{B}_{2,bad}$,

$$\xi_b(t,(u,v)) \geqslant 0, \qquad \text{for example } [f_1,[f_1,f_0]] \to \int_0^t u_1(s)^2 \mathrm{d}s.$$

For $\mathcal{B}_{2,good}$,

$$\xi_b(t,(-u,v)) = -\xi_b(t,(u,v)).$$

Theorem (Nagano (1966))

If the system $(*)$ is $L^\infty - STLC$, then LARC holds, i.e.

$$Lie(f_0, f_1, f_2)(0) = \mathbb{R}^d.$$

Theorem (Linear Test)

If $\{f_b(0),\ b \in \mathcal{B}_1\} = \mathbb{R}^d$, then system $(*)$ is $L^\infty - STLC$.

$\mathcal{B}_1$ is good. With **mono-control system**, $\mathcal{B}_2 = \mathcal{B}_{2,bad}$, and $\mathcal{B}_{2,good} = \emptyset$.

Theorem

Let $L > 0$. One supposes that :

$$\text{Span}\left(f_b(0),\ b \in \mathcal{B}_1 \cup \mathcal{B}_{2,good}, \quad |b| \leqslant L\right) = \mathbb{R}^d.$$

$$\text{For all } b \in \mathcal{B}_{2,bad},\ |b| \leqslant L \Rightarrow f_b(0) \in \mathcal{B}_1(f)(0).$$

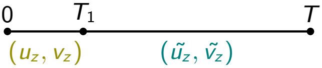
Then, the system $(*)$ is $L^\infty - STLC$.

Included in the Sussmann's $\mathcal{S}(\theta)$ condition, with $\theta = 0$.

Idea of proof (in the case codim$(\mathcal{S}_1(f)(0)) = 1$) One considers a basis of $\mathbb{R}^d$ given by the LARC :

$$\mathbb{R}^d = \text{Span}(f_{b_1}(0), \cdots, f_{b_{d-1}}(0), f_{\tilde{b}}(0)),$$

with $b_1, \cdots, b_{d-1} \in \mathcal{B}_1$ and $\tilde{b} \in \mathcal{B}_{2,good}$. One considers $\mathbb{P}$ such that $\mathbb{P}(f_{b_i}(0)) = 0$ for $i \in [\![1, d-1]\!]$ and $\mathbb{P}(f_{\tilde{b}}(0)) = 1$.



The proof is divised in two steps :

1. We construct (u_z, v_z) such that :

$$\mathbb{P}(x(T_1; (u_z, v_z), 0)) = z + O\left(|z|^{1+\delta}\right), \text{ with } \delta > 0,$$

$$\text{and } x(T_1; (u_z, v_z), 0) = O(|z|^s), \text{ with } s > \frac{1}{2}.$$

2. STLC in Span$(f_{b_1}(0), \cdots, f_{b_{d-1}}(0))$ + Brouwer fixed-point theorem.

Step 1 : Let $\bar{u}, \bar{v} \in L^2((0,1),\mathbb{R})$ *with enough vanishing moments*. Let $T_1(z) > 0$, $\varepsilon(z), \varepsilon'(z) > 0$ and $u_z, v_z : t \in (0, T_1) \mapsto \varepsilon \bar{u}\left(\frac{t}{T_1}\right), \varepsilon' \bar{v}\left(\frac{t}{T_1}\right)$. Then, with the Magnus formula,

$$\mathbb{P}(x(T_1;(u_z,v_z),0)) = \mathbb{P}\left(\sum_{b\in\mathcal{B}_1}\right) + \xi_{\tilde{b}}(T_1,(u_z,v_z)) + \textit{remainders},$$

$$\mathbb{P}(x(T_1;(u_z,v_z),0)) = \varepsilon\varepsilon' T_1^{|\tilde{b}|}\xi_{\tilde{b}}(1,(\bar{u},\bar{v})) + O\left(\varepsilon\varepsilon' T_1^{|\tilde{b}|+1} + (\varepsilon+\varepsilon')^3 T_1^3\right).$$

Taking $\varepsilon = sgn(z)|z|^{\sigma_1}$, $\varepsilon' = |z|^{\sigma_2}$, and $T_1 = \varepsilon = |z|^{\sigma_3}$, and $\bar{u}, \bar{v}$ such that $\xi_{\tilde{b}}(1,(\bar{u},\bar{v})) = 1$, one has :

$$\mathbb{P}(x(T_1;(u_z,v_z),0)) = z + O\left(|z|^{1+\beta}\right).$$

Step 2 : Using the explicit form of $\mathcal{B}_1$, one prove that the new step doesn't destroy the first step.

We consider the following PDE :

$$\begin{cases} i\partial_t \psi = -\partial^2_{xx}\psi - \left(u(t)\mu_1(x) + v(t)\mu_2(x)\right)\psi, & (t,x) \in (0,T)\times(0,1) \\ \psi(t,0) = \psi(t,1) = 0, & t \in (0,T) \\ \psi(0,x) = \psi_0(x), & x \in (0,1) \end{cases}$$

Functional analysis : $A := -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$, $D(A) = H^2(0,1) \cap H^1_0(0,1)$.

1. eigenvalues : $\lambda_j = (j\pi)^2$, $j \geqslant 1$.
2. eigenvectors : $\varphi_j := \sqrt{2}\sin(j\pi\cdot)$, $j \geqslant 1$.
3. $(\varphi_j)_{j\geqslant 1}$ orthonormal basis of $L^2(0,1)$.

Ground state : $\psi_1(t,x) := \varphi_1(x)e^{-i\lambda_1 t} = \psi(t;(0,0),\varphi_1)$.

Theorem (Linear Test, K. Beauchard, C. Laurent (2010))

Let $\mu_1, \mu_2 \in H^3((0,1),\mathbb{R})$ such that

$$\exists c > 0, \quad \forall j \in \mathbb{N}^*, \quad \left\| ((\mu_i \varphi_1, \varphi_j))_{1 \leqslant i \leqslant 2} \right\| \geqslant \frac{c}{j^3}.$$

The bilinear Schrödinger equation is L^2-STLC in $H^3_{(0)}(0,1)$:

$$\forall T > 0, \forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } \forall \psi_f \in \mathcal{S} \cap H^3_{(0)}(0,1) \text{ with } \|\psi_f - \psi_1(T)\|_{H^3} \leqslant \delta,$$

$$\exists (u,v) \in L^2((0,T)\mathbb{R})^2 \text{ s.t. } \psi(T;(u,v),\varphi_1) = \psi_f \text{ and } \|(u,v)\|_{L^2} \leqslant \varepsilon.$$

Mégane Bournissou : Quadratic obstructions in the bilinear Schrödinger equation with **single-input system**.

Framework of the article : $\exists K \geqslant 2$ such that $\langle \mu_1\varphi_1, \varphi_K\rangle = \langle \mu_2\varphi_1, \varphi_K\rangle = 0$.
$\rightarrow$ use quadratic expansion of the solution to recover this direction

Theorem (T.G. (2024))

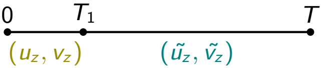
One considers μ_1, μ_2 such that :

1. $\mu_1, \mu_2 \in H^3((0,1), \mathbb{R})$.
2. $\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0$.
3. $\exists c > 0, \quad \forall j \in \mathbb{N}^* \setminus \{K\}, \quad \left\| \left((\mu_i \varphi_1, \varphi_j) \right)_{1 \leqslant i \leqslant 2} \right\| \geqslant \frac{c}{j^3}$.
4. $A_1^1 := \langle [\mu_1, [\mu_1, \Delta]] \varphi_1, \varphi_K \rangle = 0$.
5. $A_1^2 := \langle [\mu_2, [\mu_2, \Delta]] \varphi_1, \varphi_K \rangle = 0$.
6. $\gamma_1 := \langle [\mu_2, [\mu_1, \Delta]] \varphi_1, \varphi_K \rangle \neq 0$.

The Schrödinger equation is L^2-STLC around the ground state in $H^3_{(0)}(0,1)$.

- Point 1 : well-posedness.
- Point 3 : related to control in projection.
- Point 4 and 5 : prevents the system from a drift.
- Point 6 : allows us to use the bracket to recover the direction.

Idea of proof :



The proof is divised in two steps :

1. We construct (u_z, v_z) such that :

$$\mathbb{P}(x(T_1; (u_z, v_z), 0)) = iz + O\left(|z|^{\frac{13}{12}}\right)$$

$$\text{and } x(T_1; (u_z, v_z), 0) = O(|z|^s), \text{ with } s > \frac{1}{2}.$$

2. STLC in $\overline{\mathrm{Span}_{\mathbb{C}}\left(\varphi_j,\ j \in \mathbb{N}^* \setminus \{K\}\right)}$ + Brouwer fixed-point theorem.

Step 1 : Let $\bar{u}, \bar{v} \in L^2((0,1), \mathbb{R})$ be such that, $\int_0^1 \bar{u}(t)\mathrm{d}t = \int_0^1 \bar{v}(t)\mathrm{d}t = 0$. Let $T_1(z) > 0$, $\varepsilon(z), \varepsilon'(z) > 0$ and $u_z, v_z : t \in (0, T_1) \mapsto \varepsilon \bar{u}'\left(\frac{t}{T_1}\right), \varepsilon' \bar{v}'\left(\frac{t}{T_1}\right)$. Then,

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = \mathcal{F}_{T_1}(u_z) + \mathcal{G}_{T_1}(u_z, v_z) + \mathcal{F}_{T_1}(v_z) + O\left(\|(u_z, v_z)\|_{L^2}^3\right).$$

A direct computation gives :

$$\mathcal{F}_{T_1}(u_z) = -i\varepsilon^2 T_1^3 A_1^1 \int_0^1 \bar{u}(t)^2 \mathrm{d}t + O\left(\varepsilon^2 T_1^4\right) = O\left(\varepsilon^2 T_1^4\right).$$

Similarly, $\mathcal{F}_{T_1}(v_z) = O\left(\varepsilon'^2 T_1^4\right)$.Moreover,

$$\mathcal{G}_{T_1}(u_z, v_z) = i\varepsilon\varepsilon' T_1^3 \gamma_1 \int_0^1 \bar{u}(t)\bar{v}(t)\mathrm{d}t + O\left(\varepsilon\varepsilon' T_1^4\right).$$

Thus,

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = i\varepsilon\varepsilon' T_1^3 \gamma_1 \int_0^1 \bar{u}(t)\bar{v}(t)\mathrm{d}t + O\left((\varepsilon + \varepsilon')^2 T_1^4 + \left(\varepsilon^3 + \varepsilon'^3\right) T_1^{\frac{3}{2}}\right).$$

Let $\rho > 0$ and $z \in (-\rho, \rho)$. With $\varepsilon = sgn(z)|z|^{\frac{3}{8}}$, $\varepsilon' = |z|^{\frac{3}{8}}$ and $T_1 = |z|^{\frac{1}{12}}$, $(\bar{u}, \bar{v}) \in \mathcal{C}_c^\infty(0,1)^2$ such that $\int_0^1 \bar{u}(t)\bar{v}(t)\mathrm{d}t = \frac{1}{\gamma_1}$, one obtains :

$$\langle \psi(T_1; (u_z, v_z), \varphi_1), \psi_K(T_1) \rangle = iz\gamma_1 \int_0^1 \bar{u}\bar{v}' + O\left(|z|^{\frac{13}{12}}\right) = iz + O\left(|z|^{\frac{13}{12}}\right).$$

Step 2 : let $\tilde{u}_z, \tilde{v}_z$, given by **control in projection theorem**, such that :

$$\mathcal{P}_{\mathcal{H}}\left(\psi(T,(\tilde{u}_z,\tilde{v}_z),\psi(T_1,(u_z,v_z),\varphi_1))\right)=\psi_1(T),$$

with

$$\mathcal{H} := \overline{\mathrm{Span}_{\mathbb{C}}\left(\varphi_j,\ j\in\mathbb{N}^*\setminus\{K\}\right)}.$$

Finally, let $U_z = u_z \sharp \tilde{u}_z$, and $V_z = v_z \sharp \tilde{v}_z$, then

$$\|\psi(T;(U_z,V_z),\varphi_1)-\psi_1(T)-iz\psi_K(T)\|_{H^3_{(0)}} =$$

$$|\langle\psi(T;(U_z,V_z),\varphi_1),\psi_K(T)\rangle - iz| \leqslant \underbrace{|\langle\psi(T_1;(u_z,v_z),\varphi_1),\psi_K(T_1)\rangle - iz|}_{\leqslant C|z|^{\frac{13}{12}} \text{ by first step}}$$

$$+\underbrace{|\langle\psi(T;(U_z,V_z),\varphi_1),\psi_K(T)\rangle - \langle\psi(T_1;(u_z,v_z),\varphi_1),\psi_K(T_1)\rangle|}_{\leqslant C|z|^{\frac{61}{60}} \text{ thanks to weak estimates on the control}}.$$

Finally,

$$\|\psi(T;(U_z,V_z),\varphi_1)-\psi_1(T)-iz\psi_K(T)\|_{H^3_{(0)}} = O\left(|z|^{\frac{61}{60}}\right).$$

Theorem (T.G. (2024))

Let $n \geqslant 1$, $m, p \geqslant 0$, $K \geqslant 2$ such that $\lfloor \frac{n}{2} \rfloor \leqslant p$. Let μ_1, μ_2 such that :

1. $\mu_1, \mu_2 \in H^{2(p+m)+3}((0,1), \mathbb{R})$ with $\mu^{(2k+1)} \big|_{\{0,1\}} = 0$, for $0 \leqslant k \leqslant p-1$.
2. $\langle \mu_1 \varphi_1, \varphi_K \rangle = \langle \mu_2 \varphi_1, \varphi_K \rangle = 0$.
3. $\exists c > 0, \quad \forall j \in \mathbb{N}^* \setminus \{K\}, \quad \left\| ((\mu_i \varphi_1, \varphi_j))_{1 \leqslant i \leqslant 2} \right\| \geqslant \frac{c}{j^{2p+3}}$.
4. $\forall k \in [\![1, \lfloor \frac{n+1}{2} \rfloor]\!]$, $A_k^1 := \left\langle [ad_\Delta^{k-1}(\mu_1), ad_\Delta^k(\mu_1)]\varphi_1, \varphi_K \right\rangle = 0$.
5. $\forall k \in [\![1, \lfloor \frac{n+1}{2} \rfloor]\!]$, $A_k^2 := \left\langle [ad_\Delta^{k-1}(\mu_2), ad_\Delta^k(\mu_2)]\varphi_1, \varphi_K \right\rangle = 0$.
6. $\gamma_n := \left\langle [ad_\Delta^{\lfloor \frac{n+1}{2} \rfloor}(\mu_1), ad_\Delta^{\lfloor \frac{n}{2} \rfloor}(\mu_2)]\varphi_1, \varphi_K \right\rangle \neq 0$.

The equation is H_0^m−STLC around the ground state in $H_{(0)}^{2(p+m)+3}(0,1)$.

Perspectives :

1. Several lost directions (as in finite dimension) ? An infinite number ?
2. **Obstruction for STLC with multi-input systems**
3. Other equations ? KdV ?

Thank you for your attention !