Ex sheet 2

Ex 1: 1. $(|x\rangle)_{x\in X}$ are linearly independent.

Indeed, if $\sum_{x\in X} \underbrace{\alpha_x}_{\in\mathbb{C}} |x\rangle = 0$ then evaluating at $y\in X$ gives $\alpha_y = 0$

Let us show that this family span $\mathbb{C}^X$. Let $\varphi\in\mathbb{C}^X$.

Then $\forall y\in X,\ \varphi(y) = \left(\sum_{x\in X} \varphi(x)\,|x\rangle\right)(y)$

ie $\varphi = \sum_{x\in X}\varphi(x)\,|x\rangle \in \mathrm{Span}_{\mathbb{C}}\left((|x\rangle)_{x\in X}\right)$.

2. $\dim(\mathbb{C}^X) = |X|$

3. $\alpha_x = \varphi(x)$.

Ex 2. Assume that $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ can be written as $\varphi\otimes\psi$.

Write $\varphi = a|0\rangle + b|1\rangle$; $\psi = c|0\rangle + d|1\rangle$.

Then $\varphi\otimes\psi = ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

So $\underbrace{ac = bd = \frac{1}{\sqrt{2}}}$ and $\underbrace{ad = bc = 0}$

$\Downarrow$ $\qquad\qquad\qquad\qquad$ $\Downarrow$

$a,b,c,d \neq 0$ $\qquad\qquad$ a or $d = 0$, b or $c = 0$ $\Big]$ Contradiction!

Ex 3: Assume that U exists. Then for all $|\varphi\rangle, |\psi\rangle \in Q_n$, all $a,b\in\mathbb{C}$, we would have

$$U((a|\varphi\rangle + b|\psi\rangle)\otimes|0\rangle) = (a|\varphi\rangle + b|\psi\rangle)\otimes(a|\varphi\rangle + b|\psi\rangle)$$
$$= a^2|\varphi\rangle\otimes|\varphi\rangle + ab\,|\varphi\rangle\otimes|\psi\rangle + ab|\psi\rangle\otimes|\varphi\rangle + b^2|\psi\rangle\otimes|\psi\rangle$$
$$\|$$
$$U(a|\varphi\rangle\otimes|0\rangle + b|\psi\rangle\otimes|0\rangle) = aU(|\varphi\rangle\otimes|0\rangle) + bU(|\psi\rangle\otimes|0\rangle) = a|\varphi\rangle\otimes|\varphi\rangle + b|\psi\rangle\otimes|\psi\rangle$$

Therefore it suffices to take $|\varphi\rangle = |x\rangle$, $|\psi\rangle = |y\rangle$ for $x, y \in B_n$ **distinct**.

then $|\varphi\rangle\otimes|\varphi\rangle$, $|\varphi\rangle\otimes|\psi\rangle$, $|\psi\rangle\otimes|\varphi\rangle$, $|\psi\rangle\otimes|\psi\rangle$ are distinct vectors of the canonical basis $\mathcal{B}_{2n}$ of $\mathcal{B}_n$, so that the equality

$$a^2|\varphi\rangle\otimes|\varphi\rangle + ab|\varphi\rangle\otimes|\psi\rangle + ab|\psi\rangle\otimes|\varphi\rangle + b^2|\psi\rangle\otimes|\psi\rangle$$

$$= a|\varphi\rangle\otimes|\varphi\rangle + b|\psi\rangle\otimes|\psi\rangle$$

implies $ab = 0$, hence a or $b = 0$. But the equality should be true for any $a, b \in \mathbb{C}$, hence a contradiction $\square$

Ex4:

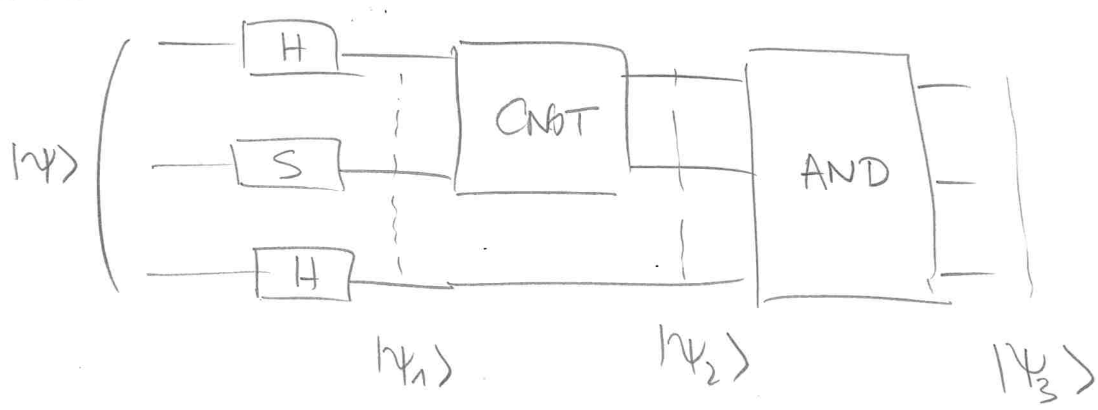
$$f : B_1 \longrightarrow B_1, \quad 0 \longmapsto 1, \quad 1 \longmapsto 1$$

Then $\tilde{f} : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$

$$|00\rangle \longmapsto |0 \;\; 0+f(0)\rangle = |0\,1\rangle$$
$$|01\rangle \longmapsto |0 \;\; 1+f(0)\rangle = |0\,0\rangle$$
$$|10\rangle \longmapsto |1 \;\; f(1)\rangle = |1\,1\rangle$$
$$|11\rangle \longmapsto |1 \;\; 1+f(1)\rangle = |1\,0\rangle$$

So $M_{\mathcal{C}an}(U_f) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Ex5:



$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes |0\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

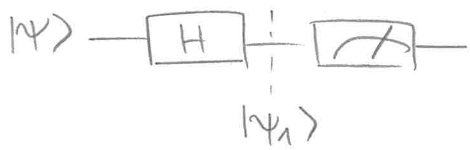
$$= \frac{1}{\sqrt{2}}(|00\rangle+|10\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$= \frac{1}{2}(|000\rangle+|001\rangle+|110\rangle+|111\rangle)$$

$$|\psi_3\rangle = \frac{1}{2}(|000\rangle+|001\rangle+|111\rangle+|110\rangle)$$

Ex6:



- if $|\psi\rangle = |0\rangle$, then $H(|\psi\rangle) = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ so the outcome of the measure is either 0 or 1, each with probability $\frac{1}{2}$.

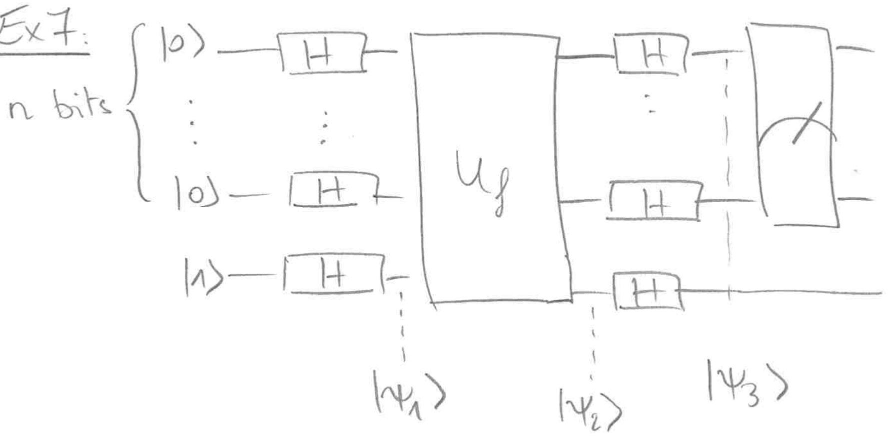
- if $|\psi\rangle = |1\rangle$, then $H(|\psi\rangle) = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ so the outcome of the measure is 0 or 1, each with probability $\frac{1}{2}$

- if $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$ then $H(|\psi\rangle) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+\cancel{|1\rangle}) + \frac{1}{\sqrt{2}}(|0\rangle-\cancel{|1\rangle})\right)$

$$= |0\rangle$$

So the outcome of the measure is 0 with probability 1.

Ex 7:



$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

$$= \left(\frac{1}{2^{n/2}} \sum_{x\in B_n} |x\rangle \right) \otimes \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$$

$$|\psi_2\rangle = \frac{1}{2^{\frac{n+1}{2}}} \sum_{x\in B_n} \underbrace{|x \; f(x)\rangle - |x \; 1+f(x)\rangle}$$

↳ if $f(x)=0$, it is $|x\rangle \otimes (|0\rangle - |1\rangle)$

if $f(x)=1$, it is $|x\rangle \otimes (|1\rangle - |0\rangle)$

So it is $(-1)^{f(x)} |x\rangle \otimes (|0\rangle - |1\rangle)$

hence $$|\psi_2\rangle = \frac{1}{2^{\frac{n+1}{2}}} \sum_{x\in B_n} (-1)^{f(x)} |x\rangle \otimes (|0\rangle - |1\rangle)$$

Now since we need to apply $H^{\otimes (n+1)}$ to the qubit $|\psi_2\rangle$ above, it will be convenient to have a formula for $H^{\otimes n}(|x\rangle)$ for all $x \in B_n$.

Write $|x\rangle = |x_1 \ldots x_n\rangle$. Then $H^{\otimes n}(|x\rangle) = H(|x_1\rangle) \otimes \cdots \otimes H(|x_n\rangle)$

But for $x_i \in \{0,1\}$, $H(|x_i\rangle) = (|0\rangle + (-1)^{x_i}|1\rangle)\frac{1}{\sqrt{2}}$

So $$H^{\otimes n}(|x\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_1}|1\rangle) \otimes \cdots \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_n}|1\rangle)$$

we develop this product:

$$= \frac{1}{2^{n/2}} \sum_{y \in B_n} \prod_{i=1}^{n} (-1)^{x_i y_i} |y_i\rangle = \frac{1}{2^{n/2}} \sum_{y \in B_n} (-1)^{\langle x, y\rangle} |y\rangle$$

Applying this $H^{\otimes n} \otimes H$ to

$$|\psi_2\rangle = \frac{1}{2^{n/2}} \sum_{x \in B_n} (-1)^{f(x)} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

we get $|\psi_3\rangle = \frac{1}{2^{n/2}} \sum_{x \in B_n} (-1)^{f(x)} H^{\otimes n}(|x\rangle) \otimes \underbrace{H\left(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right)}_{\downarrow}$

$$= \frac{1}{2^{n/2}} \sum_{x \in B_n} (-1)^{f(x)} \frac{1}{2^{n/2}} \sum_{y \in B_n} (-1)^{\langle x, y\rangle} |y\rangle \otimes |1\rangle$$

$$= \frac{1}{2^n} \sum_{x, y \in B_n} (-1)^{f(x) + \langle x, y\rangle} |y\rangle \otimes |1\rangle$$

Finally, when we measure the first n-bits, we get y with probability

$$p(y) = \frac{1}{2^{2n}} \left| \sum_{x \in B_n} (-1)^{f(x) + \langle x, y\rangle} \right|^2$$

- If f is constant then $p(y) = \frac{1}{2^{2n}} \left| \sum_{x \in B_n} (-1)^{\langle x, y\rangle} \right|^2$ because $|(-1)^{f(x)}| = 1$.

  We note that for $y = |0\dots0\rangle$, $p(y) = 1$. So the outcome of the measure is $|0\dots0\rangle$ with probability 1.

- If f is balanced, then $p(|0\dots0\rangle) = \frac{1}{2^{2n}} \left| \sum_{x \in B_n} (-1)^{f(x)} \right|^2 = 0$

  because $|f^{-1}(\{0\})| = |f^{-1}(\{1\})|$. So we never measure $|0\dots0\rangle$.

$\square$