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Dilatations d'opérateurs et projections L^p

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Introduction (en français)

Les objets phares de cette thèse sont les dilatations unitaires et les L^p-projections.

Dilatations unitaires. La théorie des dilatations est une collection de résultats, d'outils, et de techniques en théorie des opérateurs et en algèbres d'opérateurs, qui sont regroupées sous l'idée que l'on peut apprendre énormément d'un opérateur ou d'une famille d'opérateurs en les interprétant comme la compression (i.e. "un morceau de") d'un autre opérateur, quant à lui déjà bien connu. Historiquement, le premier résultat en théorie des dilatations a été obtenu par Béla Sz.-Nagy en 1953 : toute contraction peut être dilatée (ainsi que ses puissances) en un opérateur unitaire. En partant de ce résultat relativement élémentaire, toute une théorie des opérateurs non-normaux fut développée, en prenant le nom suggestif d'analyse harmonique de contractions sur des espaces de Hilbert. Une monographie conséquente du même titre fut publié en 1970 by B. Sz.-Nagy et C. Foias ; une deuxième édition revue et complètement mise à jour par Sz.-Nagy, Foias, H. Bercovici et L. Kérchy. a été publiée par Springer in 2010.

L'on connaît de multiples applications à l'existence d'une dilatation unitaire (minimale) pour une contraction sur un espace de Hilbert donnée. La première application, démontrée à nouveau par Sz.-Nagy dans son article de 1953, est une nouvelle preuve de la renommée inégalité de von Neumann. Ce résultat énonce que pour un opérateur T sur un espace de Hilbert, on a :

$$\text{Si } \|T\| \leq 1 \text{ alors } \|P(T)\| \leq \|P\|_{L^\infty(\mathbb{D})} \text{ pour tout polynôme } P \in \mathbb{C}[Z].$$

La dilatation unitaire minimale peut aussi être utilisée pour définir un calcul fonctionnel amélioré sur les contractions, peut servir à analyser des semi-groupes d'opérateurs à un paramètre, et fournit un "modèle fonctionnel" à travers duquel l'on peut analyser les contractions et les classifier, ce qui a amené à de considérables avancées dans l'études des sous-espaces invariants pour certaines familles d'opérateurs.

Pour esquisser une des applications précédentes, intéressons-nous brièvement au calcul fonctionnel pour les contractions sur un espace de Hilbert. On note $H^\infty = H^\infty(\mathbb{D})$ l'algèbre des fonctions holomorphes bornées sur le disque unité ouvert $\mathbb{D}$. Pour un opérateur $T \in \mathcal{B}(H)$ donné, on souhaite définir un calcul fonctionnel $f \mapsto f(T)$ pour tout $f \in H^\infty$. Si le spectre de T est contenu dans $\mathbb{D}$, on peut alors appliquer le calcul fonctionnel holomorphe à T pour obtenir un morphisme d'algèbres $f \mapsto f(T)$ défini sur l'algèbre $\mathrm{Hol}(\mathbb{D})$ des fonctions holomorphes sur $\mathbb{D}$. En fait, pour $f \in \mathrm{Hol}(\mathbb{D})$ et $\sigma(T) \subset \mathbb{D}$, on peut directement injecter T dans la décomposition en série entière de f. Ainsi, on sait dans ce cas définit $f(T)$ pour toute fonction $f \in H^\infty$.

Maintenant, prenons T une contraction sont le spectre $\sigma(T)$ n'est pas contenu dans le disque ouvert $\mathbb{D}$. Pour une fonction holomorphe bornée $f \in H^\infty$, comment peut-on définir $f(T)$? Le calcul fonctionnel holomorphe ne peut pas être utilisé car $\sigma(T)$ contient des points du cercle unité, là où certaines fonctions $f \in H^\infty$ ne se prolongent pas en une fonction holomorphe sur un voisinage du disque unité fermé.

Un sous-cas que l'on peut traiter aisément est pour les fonctions f appartenant à l'*algèbre du disque* $A(\mathbb{D}) \subset H^\infty$, qui 'est l'algèbre des fonctions holomorphes bornées sur le disque unité qui s'étendent continûment au disque unité fermé $\overline{\mathbb{D}}$. Ce cas peut être traité en utilisant l'inégalité de von Neumann. Il n'est en effet pas trop difficile de montrer que les fonctions polynômiales sont denses dans $A(\mathbb{D})$ pour la norme infinie $\|f\|_\infty = \sup_{|z|\leq 1} |f(z)|$. Si $(P_n)_n$ est une suite de polynômes qui converge univormément sur $\overline{\mathbb{D}}$ vers f, et si T est une contraction, alors l'inégalité de von Neumann implique que $P_n(T)$ est une suite de Cauchy, ce qui permet de définir $f(T)$ comme $\lim_n P_n(T)$. Il n'est pas difficile de montrer que le calcul fonctionnel ainsi défini sur $A(\mathbb{D})$ vérifie toutes les propriétés que l'on pourrait demander : c'est un morphisme d'algèbre étendant le calcul fonctionnel holomorphe, il est continu, et il coïncide avec le calcul fonctionnel continu sur $\overline{\mathbb{D}}$ si T est normal.

Traiter le cas de H^∞ est plus délicat, mais les dilatations unitaires apportent une façon de procéder. L'idée générale est de regarder la dilatation unitaire minimale U associée à T, et d'utiliser la théorie spectrale pour analyser ce que l'on peut faire pour U. Si la mesure spectrale associée à U est absolument continue par rapport à la mesure de Lebesgue du cercle unité, alors il s'avère que l'on peut définir $f(U)$ pour tout $f \in H^\infty$, et l'on peut simplement définir $f(T)$ comme la compression de $f(U)$ à H. Dans ce cas le calcul fonctionnel $f \mapsto f(T)$ est un morphisme d'algèbres qui étend le calcul fonctinnel holomorphe, qui est continu, et qui est en accord avec le calcul fonctionnel Borélien lorsque T est normal. Bien entendu, cela n'est utile que si l'on peut trouver des conditions garantissant que la dilatation unitaire minimale pour T a une mesure spectrale absolument continue. Une contraction T est dite *complètement non-unitaire* (c.n.u.) si elle ne possède aucun sous-espace stable F tel que la restriction $T\big|_F$ soit unitaire. Toute contraction se décompose en une somme directe $T = T_0 \oplus T_1$, où T_1 is c.n.u. et T_0 est unitaire. Sz.-Nagy and Foias ont montré que si T est c.n.u., alors la mesure spectrale de sa dilatation unitaire minimale est absolument continue.

Si la mesure spectrale pour U n'est pas absolument continue par rapport à la mesure de Lebesgue, alors il existe une sous-algèbre H^∞_U de H^∞ pour laquelle il existe un calcul fonctionnel $f \mapsto f(U)$, que l'on peut alors comprimer pour obtenir $f(T)$; il a été montré que H^∞_U est précisément la sous-algèbre des fonctions de H^∞ pour lesquelles $f \mapsto f(T)$ est un morphisme bien défini. Voir [SNBFK10, Chapitre III] pour tous les détails nécessaires.

Ma contribution sur ce thème dans cette thèse est au sujet de dilatations sur l'étude de plusieurs classes d'opérateurs généralisant le théorème de dilatation de Sz. Nagy's. Un objet d'étude central est la classe $C_{(\rho_n)}$ associée à une suite (ρ_n) de nombres complexes non-nuls donnée. Par définition, $T \in \mathcal{L}(H)$ possède une (ρ_n)-*dilation*, ou de façon équivalente appartient à la classe $C_{(\rho_n)}$, s'il existe un espace de Hilbert K et un opérateur unitaire $U \in \mathcal{L}(K)$ avec $H \subset K$ tels que $T^n = \rho_n P_H U^n|_H$ pour tout $n \geq 1$, où $P_H \in \mathcal{L}(K)$ est la projection orthogonale sur le sous-espace fermé H. Diverses propriétés spectrales sur les opérateurs appartenant à ces classes sont étudiées dans cette thèse.

L^p-projections. La géométrie des espaces de Hilbert est, en un certain sens, celle qui est la plus proche de la géométrie euclidienne en dimension finie. Tout sous-espace fermé d'un espace de Hilbert possède un supplémentaire orthogonal, et toute décomposition de cette forme est associée à une projection orthogonale. Les projections orthogonales sur les espaces de Hilbert sont ainsi des objets d'étude de base dans la théorie des opérateurs sur un espace de Hilbert.

Dans les espaces de Banach, différentes versions de "projections orthogonales" ont été envisagées : idempotents de norme 1, projections orthogonales, projections hermitiennes, etc. La notion de L^p-projection a été introduite par Cunningham [Cun53] en 1953, la même année où Sz.-Nagy prouva son théorème de dilatation. Une L^p-projections sur un espace de Banach X, pour $1 \leq p \leq +\infty$, est un opérateur idempotent P vérifiant $\|f\|_X = \|(\|P(f)\|_X, \|(I-P)(f)\|_X)\|_{\ell_p}$ pour tout $f \in X$. Ceci est une version L^p de l'égalité $\|f\|^2 = \|Q(f)\|^2 + \|(I-Q)(f)\|^2$, vérifiée par les projections orthogonales sur un espace de Hilbert.

La motivation principale à l'introduction de cette notion vient du développement de certains chapitres de géométrie des espaces de Banach qui peuvent maintenant être regroupés sous l'en-tête "structure L^p". Soit X un espace de Banach et soit p avec $1 \leq p \leq +\infty$. Deux sous-espaces fermés de X, J et $J^\perp$, sont dits complémentaires L^p-*sommants* si X est la somme algébrique de J et de $J^\perp$ et si pour tous $x \in J$, $x^\perp \in J^\perp$ on a

$$\|x + x^\perp\|^p = \|x\|^p + \|x^\perp\|^p \quad (\text{ si } 1 \leq p < +\infty)$$

ou

$$\|x + x^\perp\| = \max\Big(\|x\|, \|x^\perp\Big) \quad (\text{ si } p = +\infty).$$

Par conséquent les éléments de J et de $J^\perp$ se comportent comme des éléments à support disjoint dans un espace L^p. La projection de X sur J parallèlement à $J^\perp$ est précisément une L^p-projection.

Bien que les L^1- and L^∞-sommants et leurs projections associées ont été étudiés en premier par Cunningham et d'autres chercheurs, le moment charnière dans l'histoire de la "structure L^p" fut l'article [AE72] (découpé en deux morceaux) d'Eric Alfsen et Edward G. Effros, publié dans Annals of Mathematics. Les résultats probablement les plus importants de cet article concernent la caractérisation des M-idéaux en utilisant une propriété d'intersection et via l'introduction d'une topologie de structure. On rappelle que, par définition, un M-idéal est un sous-espace fermé dont la décomposition polaire est un L^1-sommant dans l'espace dual.

Les L^p-projections ont été étudiées, principalement pour $p = 1$ et $p = +\infty$, dans les articles [Cun53, Cun60, Cun67, CER73]. Le cas plus général $1 < p < +\infty$ fut étudié Alfsen-Effros dans leur article mentionné précédemment [AE72], ainsi que par Sullivan [Sul70] et Fakhoury [Fak74]. Les principaux résultats de caractérisation, qui ont été obtenus en 1973-1976 par Alfsen-Effros, Behrends, Fakhoury, Sullivan et d'autres, ont été compilés dans le livre [BDE+77, Ch.1,2,6]. Une caractérisation intéressante des espaces L^p réels via leur norme a été obtenue dans [Sul68] : Sullivan a prouvé qu'un espace L^p réel X est caractérisé par des inégalités de Clarkson pour X et son dual X^* et par l'existence de suffisamment de L^p-projections.

On mentionne aussi que chaque L^p-projection est hermitienne. On rappelle (voir par exemple [BS74]) qu'une projection Q est *hermitienne* si $\|e^{i\alpha Q}\| = 1$ pour tout $\alpha \in \mathbb{R}$ et qu'une projection Q est hermitienne si et seulement si $Q + \lambda(I - Q)$ est une isométrie pour tout $\lambda \in \partial\mathbb{D}$ ou, de façon équivalente, si $\lambda Q + \gamma(I-Q)$ est une isométrie pour tous $\lambda, \gamma \in \partial\mathbb{D}$. Pour voir qu'une L^p-projection P est une projection hermitienne, on peut constater que la condition de L^p-projection est équivalente à

$$\|f + g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p}, \text{ pour tout } f \in \text{Im}(P),\ g \in \text{Ker}(P),$$

où Im et Ker sont respectivement l'image et le noyau de P. Ainsi, pour tous $\lambda, \gamma \in \partial\mathbb{D}$, on a

$$\|\lambda f + \gamma g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p} = \|f + g\|_X, \text{ pour tous } f \in \text{Im}(P),\ g \in \text{Ker}(P),$$

ce qui montre que $\lambda P + \gamma(I - P)$ est une isométrie sur X et ainsi que P est une projection hermitienne.

Ma contribution sur ce sujet dans cette thèse porte sur l'étude des L^p-projections sur les sous-espaces et les quotients d'espaces de Banach complexes. J'introduis une notion de p-orthogonalité pour deux vecteurs x, y en demandant que $\mathrm{Vect}(x, y)$ admette une L^p-projection séparant x et y. J'introduis aussi la notion de L^p-projection maximale pour X, c'est-à-dire une L^p-projection sur un sous-espace G de X qui ne peut pas être étendue comme L^p-projection sur un sous-espace plus large. Je démontre des résultats concernant les L^p-projections et la p-orthogonalité sur des espaces de Banach généraux ainsi que que des espaces de Banach ayant des propriétés supplémentaires. La généralisation de certains de ces résultats à des espaces $L^p(\Omega, X)$ ainsi que des résultats au sujet de L^q-projections sur des sous-espaces de quotients de $L^p(\Omega)$ $(p \neq q)$ sont aussi présents.

Chapitre 1

Les classes C_ρ ont été introduites par B. Sz-Nagy et C. Foias [SNF66] en 1966. Pour un espace de Hilbert complexe H et un nombre réel $\rho > 0$, une application linéaire continue $T \in \mathcal{L}(H)$ est dans la classe $C_\rho(H)$ si toutes les puissances de T peuvent être dilatées en les puissances d'un opérateur unitaire sur un espace de Hilbert K, contenant H comme sous-espace fermé. Les classes $C_{(\rho_n)}$ sont une généralisation des classes C_ρ. Elles sont définies de la façon suivante :

Définition 1.2.1. (Classes $C_{(\rho_n)}$) *Soit $(\rho_n)_{n\geq 1}$ une suite de nombres complexes non-nuls. Soit H un espace de Hilbert complexe. On définit la classe*

$$C_{(\rho_n)}(H) := \{T \in \mathcal{L}(H) : \text{ il existe un Hilbert } K \text{ et un opérateur unitaire } U \in \mathcal{L}(K) \text{ avec } H \subset K \text{ tels que } T^n = \rho_n P_H U^n|_H,\ \forall n \geq 1\},$$

où $P_H \in \mathcal{L}(K)$ est la projection orthogonale dans K sur le sous-espace fermé H.
On dit alors qu'un operateur $T \in C_{(\rho_n)}(H)$ possède une (ρ_n)-dilatation.

Ces classes apparaissent brièvement dans les articles [Rác74, Bad03], concernant des résultats autour de la similitude à une contraction. Certaines propriétés des opérateurs dans ces classes ont été étudiées dans [SZ16] pour $\rho_n \in \mathbb{R}_+^*$.

Ce premier chapitre commence en généralisant plusieurs résultats connus pour les classes C_ρ aux classes $C_{(\rho_n)}$, ce qui fournit des outils fort pratiques pour les étudier. L'outil le plus central est une caractérisation de ces classes reposant sur la positivité d'opérateurs auto-adjoints spécifiques.

Théorème 1.2.8. *Soit $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ avec $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Soit $T \in \mathcal{L}(H)$. Alors, les assertions suivantes sont équivalentes :*

(i) $T \in C_{(\rho_n)}$;

(ii) $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$ *et, pour* $f_{(\rho_n)}(zT) := \sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$, *on a*

$$I + Re(f_{(\rho_n)}(zT)) \geq 0, \forall z \in \mathbb{D}.$$

La fonction holomorphe $f_{(\rho_n)}$, et son rayon de convergence valant $\liminf_n(|\rho_n|^{\frac{1}{n}})$, a un rôle central dans la structure des classes $C_{(\rho_n)}$.

Ces classes sont aussi en lien avec la quantité suivante

Définition 1.2.11. ((ρ_n)-rayon) *Soit H un espace de Hilbert, $T \in \mathcal{L}(H)$, et $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$. Le* (ρ_n)-rayon *de T est défini comme*

$$w_{(\rho_n)}(T) := \inf\{u > 0 : \frac{T}{u} \in C_{(\rho_n)}(H)\} \in [0, +\infty].$$

Lorsque $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$, on montre que le (ρ_n)-rayon est une quasi-norme sur $\mathcal{L}(H)$ qui est équivalente à $\|\cdot\|$, dont sa boule unité fermée est la classe $C_{(\rho_n)}(H)$. Par conséquent, on a

$$w_{(\rho_n)}(T) \leq 1 \Leftrightarrow T \in C_{(\rho_n)}(H).$$

Ainsi, plusieurs questions au sujet des classes $C_{(\rho_n)}(H)$ peuvent être reformulées à l'aide de leurs rayons, offrant parfois un point de vue radicalement différent.

La section 1.3 se concentre sur le cas où la suite $(\rho_n)_n$ est constante et égale à $\rho \in \mathbb{C}^*$. Dans ce cadre, la plupart des caractérisations obtenues prennent une forme plus simple, ce qui permet de relier ces classes aux classes C_τ pour $\tau > 0$. Ce phénomène est mis en évidence par le résultat principal de cette section.

Proposition 1.3.3. *Soient $\rho \neq 0$ et $\alpha > 0$. Soit $T \in \mathcal{L}(H)$.*
Les assertions suivantes sont équivalentes :

(i) $w_{(\rho)}(T) \leq \alpha$;

(ii) $((\rho-1)zT - \rho\alpha I)$ *est inversible et* $\|(zT)((\rho-1)zT - \rho\alpha I)^{-1}\| \leq 1$, $\forall z \in \mathbb{D}$;

(iii) $((\rho-1)T - \rho w I)$ *est inversible et* $\|T((\rho-1)T - \rho w I)^{-1}\| \leq 1$, $\forall |w| > \alpha$;

(iv) $\|T(h)\| \leq \|(\rho-1)T(h) - \rho w h\|$, $\forall h \in H$, $\forall |w| > \alpha$.

Par conséquent, on a :

$$|\rho| w_{(\rho)}(T) = (1 + |\rho - 1|) w_{1+|\rho-1|}(T). \tag{0.0.1}$$

Ainsi, la fonction $\rho \in \mathbb{C}^* \mapsto |\rho| w_{(\rho)}(T)$ est constante sur les cercles de centre 1, est continue sur $\mathbb{C}^*$, et peut être étendue continûment à $2w_{(2)}(T)$ en 0.

Ce résultat permet de calculer le (ρ)-rayon d'opérateurs T vérifiant $T^2 = aT$ ou $T^2 = bI$, et fournit plusieurs façons d'utiliser le (ρ)-rayon pour ρ complexe. Une expression du (ρ)-rayon pour les opérateurs T vérifiant $(T - aI)^2 = 0$ (Proposition 1.3.9) est aussi obtenue, ce qui agrandit l'ensemble des matrices $M \in M_2(\mathbb{C})$ pour lesquelles une expression de $w_\rho(M)$ est connue (le cas général étant encore ouvert).

La section 1.4 revient au cas général, en étudiant des propriétés supplémentaires des (ρ_n)-rayons ainsi que des relations entre les (ρ_n) et $(\tau_n)_n$-rayons. La section précédente nous a motivés à considérer les $(z\rho_n)$-rayons, pour une suite (ρ_n) donnée et pour $z \in \mathbb{C}^*$, afin de se retrouver dans un contexte d'étude de familles à 1 paramètre. Un travail autour de la fonction $z \mapsto w_{(z\rho_n)}(T)$ a amené au résultat suivant.

Proposition 1.4.8. *Soit $T \in \mathcal{L}(H)$. Soit $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ telle que $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. On a :*

(i) $z \mapsto w_{(z\rho_n)}(T)$ *est uniformément continue sur* $\mathbb{C} \setminus \mathbb{D}(0, \epsilon)$*, pour tout* $\epsilon > 0$*. Cette fonction tend vers* $+\infty$ *quand* $|z| \to 0$*, et vers* $\frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}$ *quand* $|z| \to +\infty$*;*

(ii) Pour tout $t \in \mathbb{R}$*, la fonction* $r \mapsto w_{(re^{it}\rho_n)}(T)$ *est log-convexe sur* $]0, +\infty[$.

Il s'avère aussi que ces familles peuvent se comporter significativement différemment de la famille des classes $C_{(z)}$, $z \in \mathbb{C}^*$. On peut par exemple trouver des suites (ρ_n) et des opérateurs T pour lesquels $w_{(z\rho_n)}(T)$ est constant lorsque $|z|$ est suffisamment grand (see Proposition 1.4.16). Il existe aussi des suites (ρ_n) pour lesquelles $\bigcup_{r>0} C_{(r\rho_n)}$ contient tous les opérateurs à puissances bornées (see Corollary 1.4.18), alors que $\bigcup_{\rho>0} C_{(\rho)}$ est strictement contenue dans l'ensemble des opérateurs semblables à une contraction. Dans cette lignée, cette section se conclut par une éude du $(z\rho_n)$-rayon de I pour deux familles spécifiques, où la fonction $f_{(\rho_n)}$ est respectivement liée à log et à exp.

Chapitre 2

Les classes C_A sont une autre généralisation des classes C_ρ qui ont été définies par H. Langer (voir [SNBFK10, p.53] et ses références), puis étudiées ensuite par Suen [Sue98a] en 1998.

Définition 2.1.1. (Classes C_A) *Soit* H *un espace de Hilbert complexe. Soit* $A \in \mathcal{L}(H)$ *un opérateur auto-adjoint positif inversible. On définit la classe* $C_A(H)$ *comme*

$$C_A(H) := \{T \in \mathcal{L}(H) : \exists K \text{ Hilbert et } U \in \mathcal{L}(K) \text{ unitaire avec } H \subset K \text{ tel que } A^{-\frac{1}{2}}T^n A^{-\frac{1}{2}} = P_H U^n|_H, \forall n \geq 1\}$$

où P_H *est la projection orthogonale sur* H*. Lorsque l'espace de Hilbert sous-jacent* H *n'est pas ambigu, les classes* $C_A(H)$ *peuvent être abrégées en* C_A*.*

De façon similaire au Chapitre 1, les résultats initiaux de ce second chapitre, qui peuvent aussi être trouvés dans [Sue98a], et sont ensuite utilisés pour obtenir quelques améliorations (voir Propositions 2.2.12 et 2.3.1), ainsi que de nouveaux résultats (du Lemme 2.3.3 à la Proposition 2.3.9). La structure des classes C_A a beaucoup de points communs avec celle des classes $C_{(\rho_n)}$, étant donné que les outils principaux pour les étudier sont les suivants.

Proposition 2.1.4. *Soit* H *un espace de Hilbert. Soient* $A, T \in \mathcal{L}(H)$ *avec* $A > 0$*. Les assertions suivantes sont équivalentes.*

(i) $T \in C_A(H)$*;*

(ii) $r(T) \leq 1$ *et* $I + Re(2 \sum_{n\geq 1} A^{-\frac{1}{2}}(zT)^n A^{-\frac{1}{2}}) \geq 0, \forall z \in \mathbb{D}$*;*

(iii) $r(T) \leq 1$ *et* $A + Re(2 \sum_{n\geq 1}(zT)^n) \geq 0, \forall z \in \mathbb{D}$*;*

(iv) $r(T) \leq 1$ *et* $A - 2Re(z(A - I)T) + |z|^2 T^*(A - 2I)T \geq 0, \forall z \in \mathbb{D}$*;*

Définition 2.2.1. *Soient* $A, T \in \mathcal{L}(H)$ *avec* $A > 0$*. On définit*

$$w_A(T) := \inf(\{r > 0 \colon \frac{1}{r}T \in C_A(H)\}),$$

le A-rayon *de* T*.*

Proposition 2.2.4. *Soit $A \in \mathcal{L}(H)$, avec $A > 0$. Alors w_A est une quasi-norme qui est équivalente à la norme d'opérateur $\|\cdot\|$, dont la boule unité fermée est la classe C_A.*

Ces outils sont utilisés pour déterminer le A-rayon de I (Proposition 2.2.12), pour améliorer des résultats liant $w_A(CTC^*)$ and $w_A(T)$ (Proposition 2.3.1), ainsi que pour donner des calculs de $w_A(T)$ lorsque $T^2 = 0$ dans certains contextes (Propositions 2.3.4 et 2.3.5).

Chapitre 3

Ce chapitre traite de deux objets différents. Sa première section étudie les opérateurs algébriques, tandis que les deux suivantes se concentrent sur des classes de projections. Un opérateur algébrique est défini comme suit.

Définition 3.1.3. (Opérateurs algébriques) *Soit X un espace de Banach. Un opérateur $T \in \mathcal{L}(X)$ est* algébrique *s'il existe un polynôme $Q \in \mathbb{C}[Z]$ tel que $Q(T) = 0$.*

Dans la Section 3.1, nous nous concentrons sur l'étude d'opérateurs algébriques par rapport à la similitude à une contraction ainsi qu'à des conditions plus faibles (polynomialement borné, à puissances bornées,...). On prouve que la majorité de ces conditions sont équivalentes pour les opérateurs algébriques. Cela fournit aussi des moyens pratiques pour déterminer si un opérateur algébrique T donné est semblable à une contraction ou non.
Comme un opérateur algébrique possède une décomposition spectrale via ses projections caractéristiques, l'étude se déplace dans la Section 3.2 vers des classes de projections afin de voir comment celles-ci se comportent les unes par rapport aux autres. Cela va de la classe des projections de norme 1 jusqu'aux classes de L^p-projections. Voici un détail des classes que l'on considèrera.

Définition 3.2.1. (Classes de projections) *Soit X un espace de Banach complexe. Soient X_1, X_2 des sous-espaces fermés de X en somme directe. Soit $P \in \mathcal{L}(X)$ la projection sur X_1 parallèlement à X_2. On définit les propriétés suivantes pour P.*

(P_1) $\|P\| = 1$;

(P_2) $\|P\| = \|I - P\| = 1$;

(P_3) *P est* bi-circulaire généralisée*: Il existe $\lambda \in \partial\mathbb{D} \setminus \{1\}$ tel que $\lambda P + (I - P)$ est une isométrie surjective.*

(P_4) *P est* hermitienne*: Pour tout $\alpha \in \mathbb{R}$, $e^{i\alpha P}$ est une isométrie surjective.*

(P_5) *Pour tout $\lambda_i \in \mathbb{C}$, tous $x_i \in X_i$, $1 \le i \le 2$, on a*

$$\|\lambda_1 x_1 + \lambda_2 x_2\| \le \max_i(|\lambda_i|)\|x_1 + x_2\|.$$

(P'_5) *Pour tous $\lambda \in \partial\mathbb{D}$, $x_i \in X_i$, $1 \le i \le 2$, on a*

$$\|x_1 + \lambda x_2\| = \|x_1 + x_2\|.$$

(P_6) *Pour tous* $x_i, y_i \in X_i$, $1 \le i \le 2$, *avec* $\|x_2\| = \|y_2\|$, *on a*

$$\|x_1 + x_2\| = \|x_1 + y_2\|.$$

(P_7) *Pour tous sous-espaces fermés* $V_i \subset X_i$, *tous* $C_i \in \mathcal{L}(X_i)$, *et tous* $x_i \in V_i$, $1 \le i \le 2$, *on a*

$$\|C_1(x_1) + C_2(x_2)\| \le \max_{i=1,2}(\sup_{y_i \in V_i,\ \|y_i\|=1}(\|C_i(y_i)\|))\|x_1 + x_2\|.$$

(P_8) *Pour tous* $x_i \in X_i \setminus \{0\}$, *tous* $C_i \in \mathcal{L}(X_i)$, $1 \le i \le 2$, *on a*

$$\|C_1(x_1) + C_2(x_2)\| \le \max_i(\frac{\|C_i(x_i)\|}{\|x_i\|})\|x_1 + x_2\|.$$

(P'_8) *Pour tous* $x_i, y_i \in X_i$ *avec* x_i *non nul*, $1 \le i \le 2$, *on a*

$$\|y_1 + y_2\| \le \max_i(\frac{\|y_i\|}{\|x_i\|})\|x_1 + x_2\|.$$

(P_9) *Pour tous* $x_i, y_i \in X_i$ *avec* $\|x_i\| = \|y_i\|$, $1 \le i \le 2$, *on a*

$$\|y_1 + y_2\| = \|x_1 + x_2\|.$$

(P_{10}) *Il existe* $1 \le p \le +\infty$ *tel que* P *est une* L^p-projection*: Pour tous* $x_i \in X_i$ *on a*

$$\|x_1 + x_2\| = \|(\|x_1\|, \|x_2\|)\|_{\ell^p}.$$

Comme les propriétés définissant certaines de ces classes concernent la norme de vecteurs dans une somme directe de deux sous-espaces, il est possible de définir des généralisations pour des sommes directes d'un nombre fini de sous-espaces, ce qui est abordé dans la Sous-section 3.3.B.

Définition 3.2.9. *Soit* X *un espace de Banach*, $r \ge 2$, *et* $X_1, .., X_r$ *des sous-espaces fermés tels que* $X = X_1 \oplus X_2 \oplus ... \oplus X_r$. *On définit les propriétés suivantes :*

($P_{8,r}$) *Pour tous* $x_i \in X_i$ *et* $C_i \in \mathcal{L}(X_i)$, $1 \le i \le r$, *on a*

$$\|C_1(x_1) + ... + C_r(x_r)\| \le \max_i(a(C_i, x_i))\|x_1 + ... + x_r\|,$$

où $a(C_i, x_i) := \begin{cases} \frac{\|C_i(x_i)\|}{\|x_i\|} & si\ x_i \neq 0 \\ 0 & sinon \end{cases}$

($P_{9,r}$) *Pour tous* $x_i, y_i \in X_i$ *avec* $\|x_i\| = \|y_i\|$, $1 \le i \le r$, *on a*

$$\|x_1 + ... + x_r\| = \|y_1 + ... + y_r\|.$$

($P_{5,r}$) *Pour tous* $x_i \in X_i$ *et* $\lambda_i \in \mathbb{C}$, $1 \le i \le r$, *on a*

$$\|\lambda_1 x_i + ... + \lambda_r x_r\| \le \max_i(|\lambda_i|)\|x_1 + ... + x_r\|.$$

($P'_{5,r}$) Pour tous $x_i \in X_i$ et $\lambda_i \in \partial\mathbb{D}$, $1 \leq i \leq r$, on a

$$\|\lambda_1 x_i + ... + \lambda_r x_r\| = \|x_1 + ... + x_r\|.$$

Travailler avec des sommes directes finies apporte des questions supplémentaires par rapport à ces propriétés car certaines peuvent par exemple être vraies sur tout $X_i \oplus X_j$ mais pas sur $X_1 \oplus ... \oplus X_r$ (voir Lemme 3.2.11).

La Section 3.3 regarde le cas spécifique où X est un $L^p(\Omega)$ ou un sous-espace fermé de $L^p(\Omega)$, avec des projections qui sont soit hermitiennes, soit des L^p-projections. Ces projections peuvent se définir avec une condition qui doit être vérifiée pour tout $z \in \mathbb{C}$, ainsi nous cherchons à voir si une condition plus faible (pour tout $z \in \partial\mathbb{D}$, ou pour $z = \pm 1$) donnerait forcément les mêmes objets. De même, les L^p-projections étant un cas particulier de projections hermitiennes, nous essayons aussi de regarder comment seraient des sous-espaces de L^p sur lesquels toute projection hermitienne est une L^p-projection, ou bien de trouver une condition qui assure l'existence de contre-exemples.

Dans la Sous-section 3.3.B nous étudions le cas $p = 2n$, où la relation $|f+g|^{2n} = (f+g)^n(\overline{f}+\overline{g})^n$ nous permet d'obtenir une propriété supplémentaire assez utile.

Proposition 3.3.9. *Soit $(\Omega, \mathcal{F}, \mu)$ un espace mesuré. Soit $n \geq 1$. Soit $X = L^{2n}(\Omega, \mathcal{F}, \mu)$. Soient $f, g \in X$. Les assertions suivantes sont équivalentes.*

(i) $\|f + \lambda g\|_{2n} = \|f + g\|_{2n}$, $\forall \lambda \in \partial\mathbb{D}$;

(ii) $\|f + zg\|_{2n} = \|f + |z|g\|_{2n}$, $\forall z \in \mathbb{C}$;

(iii) $\sum\limits_{k=0}^{n-l} \binom{n}{k}\binom{n}{l+k} \int\limits_{\Omega} (g^l |g|^{2k} \bar{f}^l |f|^{2(n-l-k)})(x) d\mu(x) = 0$, $\forall 1 \leq l \leq n$;

(iv) $\|f + zg\|_{2n}^{2n} = \sum_{k=0}^{n} |z|^{2k} \binom{n}{k}^2 \int\limits_{\Omega} (|f|^{2(n-k)}|g|^{2k})(x) d\mu(x)$, $\forall z \in \mathbb{C}$.

En utilisant cette proposition, nous avons pu facilement construire des sous-espaces de $l^{2n}(\{0, .., n\})$ possédant une projection hermitienne qui n'est pas une L^{2n}-projection (voir Proposition 3.3.10), et étendre ce résultat à d'autres espaces L^{2n}.

Chapitre 4

Le chapitre final de cette thèse porte sur la caractérisation et les propriétés des L^p-projections. Ces opérateurs furent introduits par Cunningham [Cun53] en 1953.

Définition 4.1.1. (L^p-projections) *Soit X un espace de Banach complexe, et soit $1 \leq p \leq +\infty$. Une projection P ($P^2 = P$) de $\mathcal{L}(X)$ est une* L^p-projection *si elle vérifie la condition*

$$\|f\|_X = \|(\|P(f)\|, \|(I-P)(f)\|)\|_p, \text{ pour tout } f \in X.$$

Cela équivaut à

$$\begin{cases} \|f\|_X^p = \|P(f)\|_X^p + \|(I-P)(f)\|_X^p, \forall f \in X & \text{quand } 1 \leq p < +\infty. \\ \|f\|_X = \max(\|P(f)\|_X, \|(I-P)(f)\|_X), \forall f \in X & \text{quand } p = +\infty. \end{cases}$$

On note $\mathcal{P}_p(X)$ l'ensemble des L^p-projections sur X.

Les articles de Cunningham [Cun53, Cun60, Cun67, CER73] se sont principalement concentrés sur les cas $p = 1$ et $p = +\infty$, là où le cas général commença à être étudié par Alfsen-Effros [AE72], Sullivan [Sul70] et Fakhoury [Fak74]. Les premiers résultats de caractérisation des L^p-projections furent obtenus en 1973-1976 par Behrends, Greim & al, et sont compilés dans le livre [BDE+77, Ch.1,2,6]. Les préliminaires de la Section 4.1 rappellent certains de ces résultats. Le théorème fondamental que l'on utilisera fréquemment est le suivant.

Théorème 4.1.8. *Soit X un espace de Banach, et $1 \leq p \leq +\infty$, $p \neq 2$. On a :*

(i) Les éléments de $\mathcal{P}_p(X)$ commutent deux à deux.

(ii) L'ensemble $\mathcal{P}_p(X)$ est une algèbre de Boole commutative pour les opérations $(P,Q) \mapsto PQ$, $(P,Q) \mapsto P + Q - PQ$ et $P \mapsto (I - P)$.

(iii) La relation $P \leq Q \;\Leftrightarrow\; PQ = P$ est une relation d'ordre sur $\mathcal{P}_p(X)$.

(iv) Lorsque $p \neq +\infty$, toute famille totalement ordonnée $(P_i)_{i\in I}$ de $\mathcal{P}_p(X)$ admet un inf $P = \inf_{i\in I}(P_i)$, vers lequel elle converge ponctuellement.

(v) Si $p \neq +\infty$, l'algèbre de Boole $\mathcal{P}_p(X)$ est complète : Tout sous-ensemble $\{P_i,\, i \in I\}$ possède un infimum $\inf_{i\in I}(P_i)$ in $\mathcal{P}_p(X)$. De plus, $Im(\inf_{i\in I}(P_i)) = \bigcap_{i\in I} Im(P_i)$.

Comme presque tous les énoncés de ce théorème sont valables pour $1 \leq p \leq +\infty$, $p \neq 2$, la majorité de nos résultats dans ce chapitre seront vrais pour $p \neq 2$. Cependant les L^∞-projections ont une définition légèrement différente des autres L^p-projections et ne vérifient pas tous les énoncés du Théorème 4.1.8 ; ainsi leur étude sera en partie traitée comme un cas particulier (voir Sous-section 4.2.C). Enfin, comme nous considèrerons des quotients d'espaces de Banach dans la seconde moitié de ce chapitre, les cas $p = 1, +\infty$ seront exclus pour la majorité des résultats de la Section 4.3 à cause de comportements gênants des normes L^1 et L^∞ par rapport aux quotients d'espaces vectoriels.

Notre objectif initial était de caractériser les L^p-projections sur des sous-espaces, quotients, et sous-espaces de quotients d'espaces L^p (aussi appelés espaces SQ^p). Nous avons travaillé dans un cadre plus général afin de mieux entrevoir les résultats qu'il était possible d'obtenir sur un espace de Banach X général ou bien sur un espace de Banach X possédant des propriétés issues de celles des espaces L^p. A cet effet, nous avons commencé par définir une relation d'orthogonalité liée aux L^p-projections, à savoir :

Propriété 4.2.1. (p-orthogonalité) *Soit X un espace de Banach, et $1 \leq p \leq +\infty$. Soient $f, g \in X$. Les vecteurs f et g sont dits p-*orthogonaux*, noté par $f \perp_p g$, si*

$$\begin{cases} \|f + zg\|^p = \|f\|^p + |z|^p \|g\|^p,\ \forall z \in \mathbb{C},\ \textit{quand } p < +\infty; \\ \|f + zg\| = \max(\|f\|, |z|\|g\|),\ \forall z \in \mathbb{C},\ \textit{quand } p = +\infty. \end{cases}$$

Lorsque $f \neq 0$ et $g \neq 0$, cette condition est équivalente au fait que Vect(f,g) est de dimension 2 et que la projection sur Vect(f) parallèlement à Vect(g) dans Vect(f,g) est une L^p-projection.

Cette relation est symétrique, homogène, définie, mais pas linéaire en général. Elle nous permet aussi de définir le p-orthogonal d'un ensemble E de la même façon que les orthogonaux dans le cas des espaces munis d'un produit scalaire. Cette relation a une caractérisation élémentaire dans les espaces L^p et sur les sous-espaces de L^p (voir Corollaire 4.2.3), ce qui permet alors de caractériser aisément les L^p-projections sur les sous-espaces de $L^p(\Omega)$. Il s'avère que si l'on considère un espace de Banach X vérifiant les deux propriétés suivantes

Propriété 4.2.7. (Extension de la p-orthogonalité à X) *Pour tous $f, g \in X$ tels que $f \perp_p g$, il existe $P \in \mathcal{P}_p(X)$ telle que $P(f) = f$ et $P(g) = 0$.*

Propriété 4.2.8. (Linearité de la p-orthogonality sur X) *Pour tous $f, g, h \in X$ tels que $f \perp_p g$ et $f \perp_p h$, on a $f \perp_p (g + h)$.*

alors la description des comportements sur les sous-espaces de $L^p(\Omega)$ peut se généraliser aux sous-espaces de X. Cela donne par exemple le résultat suivant.

Proposition 4.2.10. (L^p-projections sur des sous-espaces de X) *Soit X un espace de Banach, et $1 \leq p < +\infty$, $p \neq 2$. Les assertions suivantes sont équivalentes.*

(i) X vérifie la Propriété 4.2.7 pour p;

(ii) Pour tous sous-espaces E_1, E_2 de X, avec $f \perp_p g$ pour tous $f \in E_1$, $g \in E_2$, il existe $P \in \mathcal{P}_p(X)$ telle que $P(E_1) = E_1$ et $P(E_2) = \{0\}$.

De plus, si l'une d'entre elles est vraie, alors pour n'importe quel sous-espace F de X et pour n'importe quelle L^p-projection $P \in \mathcal{P}_p(F)$, il existe $Q \in \mathcal{P}_p(X)$ telle que $P = Q|_F$.

Nous avons aussi dédié la Sous-section 4.2.B à des contre-exemples élémentaires montrant des comportements différents lorsque les propriétés précédentes ne sont pas vérifiées par X.

Comme il s'avère que la majorité des sous-espaces de $L^p(\Omega)$ ne vérifie que la Propriété Property 4.2.8, nous avons introduit un ensemble plus important de projections dans la Sous-section 4.2.D.

Définition 4.2.30. (L^p-projections maximales) *Soit X un espace de Banach, et $1 \leq p \leq +\infty$. Soit F un sous-espace fermé de X, et soit $P \in \mathcal{P}_p(F)$. La L^p-projection P est dite* maximale pour *X s'il n'existe aucun sous-espace G ni aucune L^p-projection Q sur G tels que $F \subsetneq G$ et $Q|_F = P$.*
On définit alors $\alpha(F) := Card(\{P : P$ est une L^p-projection maximale pour $F\})$.

Ces L^p-projections maximales nous permettent d'étudier la structue de la relation de p-orthogonalité sur le sous-espace F considéré. Nous donnons plusieurs résultats autour du comportement général des L^p-projections maximales, ainsi que leur comportement lorsque F est un sous-espace d'un espace de Banach X vérifiant les Propriétés 4.2.7 et 4.2.8.

Nous nous intéressons aussi à déterminer un majorant du nombre de L^p-projections maximales qu'un tel sous-espace F peut posséder lorsqu'il est de dimension finie. Nous avons d'abord travaillé sur un exemple générique.

Proposition 4.2.44. *Soient $1 \leq p < +\infty$, $p \neq 2$, $n \geq 4$ et $X = \ell^p(\mathbb{C}^n)$. On note $(e_i)_i$ la base canonique X. Pour $1 \leq i < n$, soit $f_i = e_i + e_{i+1}$. On prend $F = Span(f_i,\ 1 \leq i < n)$. Alors, on a $dim(F) = n - 1$, et*

$$\alpha(F) = Card(\{P\colon P \text{ est une } L^p\text{-projection maximale pour } F\}) = 2^n - 2n.$$

Ainsi, pour $n \geq 5$ il y a strictement plus de $2^{dim(F)}$ L^p-projections qui sont maximales pour F.

Nos résultats pour ce sous-espace ainsi que dans un autre contexte précis (voir Proposition 4.2.46) nous ont amené à conjecturer un majorant pour le cas général (voir Conjecture 4.2.49 et Proposition 4.2.50). Cette conjecture est malheureusement encore ouverte lors de la rédaction de ce manuscrit.

Après une étude focalisée sur les L^p-projections dans les sous-espaces d'un espace de Banach X, nous abordons en Section 4.3 ces projections dans les quotients et sous-espaces de quotients d'un espace de Banach donné. La plupart des résultats obtenus reposent fortement sur ce lemme clé.

Lemme 4.3.3. *Soit $1 < p \leq +\infty$. Soit X un espace de Banach et soit F un sous-espace fermé de X. Soientt $x \in X$ et G un sous-espace de X contenant F et x. Soit $P \in \mathcal{P}_p(G)$ telle que $P(x) = x$. Les assertions suivantes sont équivalentes.*

(i) $\inf_{a\in F} \|x - a\| = \|x\|$;

(ii) $\inf_{a\in F} \|x - P(a)\| = \|x\|$.

Si les projections métriques sur F et $P(F)$ sont bien définies, alors on a l'équivalence suivante.

(1) $Proj(x, F) = 0$;

(2) $Proj(x, P(F)) = 0$.

Même avec ce résultat, la relation p-orthogonalité peut ne pas se comporter assez bien pour mettre en lien les L^p-projections sur X avec celles sur un quotient de X. Nous introduisons à cet effet la propriété suivante entre X et un sous-espace F.

Propriété 4.3.9. *Pour tous $\overline{x}, \overline{y} \in X/F$ tels que $\overline{x} \perp_p \overline{y}$, il existe $x, y \in X$ des représentants de $\overline{x}, \overline{y}$ de norme minimale tels que $x \perp_p y$.*

Ces éléments nous permettent alors de caractériser la p-orthogonalité ainsi que les L^p-projections sur des sous-espaces de quotients de L^p, et de généraliser ces caractérisations à une classe plus large d'espaces de Banach.

Nos résultats principaux concernant les L^p-projections sur un quotient d'un espace de Banach sont les suivants.

Proposition 4.3.13. *Soit X un espace de Banach, et soit $1 \leq p < +\infty$, $p \neq 2$. Soit F un sous-espace fermé de X, et soit $P \in \mathcal{P}_p(X)$ telle que $P(F) \subset F$. Alors, on a :*

(i) $X/F \simeq P(X)/P(F) \oplus_p (I - P)(X)/(I - P)(F)$;
Si la projection métrique sur F est bien définie, alors

$$Rep(X/F) = Rep(P(X)/P(F)) \oplus_p Rep((I - P)(X)/(I - P)(F)).$$

(ii) *Il existe une L^p-projection P' sur X/F telle que $P'(\overline{x}) = \overline{P(x)}$;*

(iii) *P' est non triviale si et seulement si $P(F) \neq P(X)$ et $(I - P)(F) \neq (I - P)(X)$;*

(iv) *Soit $P_F \in \mathcal{P}_p(X)$ la L^p-projection ayant la plus grande image telle que $Im(P_F) \subset F$. Alors, X/F est isométriquement isomorphe à $(I - P_F)(X)/(I - P_F)(F)$.*

(v) *On pose* $\phi : P \in \{Q \in \mathcal{P}_p(X)\colon Q(F) \subset F\} \mapsto P' \in \mathcal{P}_p(X/F)$. *Alors* ϕ *est un morphisme d'algèbres de Boole commutatives, et* $Ker(\phi) = \mathcal{P}_p(X) \circ P_F$. *Ainsi, on a* $\phi(P_1) = \phi(P_2)$ *si et seulement si* $(I - P_F)P_1 = (I - P_F)P_2$, *et* ϕ *est injective si et seulement si* $P_F = 0$. *En général, on a* $Im(\phi) = \phi(\{Q \in \mathcal{P}_p(X)\colon QP_F = 0,\ P(F) \subset F\})$ *et* ϕ *est injective sur cet ensemble.*

Proposition 4.3.16. *Soit* $1 < p < +\infty$, $p \neq 2$. *Soit* X *un espace de Banach vérifiant la Propriété 4.2.7. Soit* F *un sous-espace fermé de* X *tel que tout élément de* X/F *possède un unique représentant de norme minimale. On suppose que la Propriété4.3.9 est vérifiée pour* X,F *et* p. *On note* $\phi : P \in \{Q \in \mathcal{P}_p(X)\colon Q(F) \subset F\} \mapsto \phi(P) \in \mathcal{P}_p(X/F)$ *le morphisme d'algèbres de Boole commutatives de la Proposition 4.3.13, avec* $\phi(P)$ *vérifiant* $\phi(P)(\overline{x}) = \overline{P(x)}$ *pour tout* $x \in X$. *Alors, on a :*

(i) *Le morphisme* ϕ *est surjectif ; toute* L^p*-projection de* X/F *peut être associée à une* L^p*-projection* P *de* X *telle que* $P(F) \subset F$.

(ii) *L'algèbre de Boole* $\mathcal{P}_p(X/F)$ *est isomorphe à* $\{P \in \mathcal{P}_p(X)\colon PP_F = 0,\ P(F) \subset F\}$.

(iii) *On note* P_F *la* L^p*-projection de* X *de plus grande image telle que* $Im(P_F) \subset F$. *Le quotient* X/F *possède des* L^p*-projections non triviales si et seulement s'il existe des* L^p*-projections* P *telles que* $P_F < P < I$ *et* $P(F) \subset F$.

Dans la Sous-section 4.3.B, ces résultats sont poursuivis vers la caractérisation de L^p-projections sur des sous-espaces de quotients d'un espace de Banach X vérifiant les propriétés 4.2.7,4.2.8 et 4.3.9 (voir Proposition 4.3.19).

Nous terminons ce chapitre sur une Section où nous regardons des espaces $L^p(\Omega, X)$ pour lesquels les résultats précédents s'appliquent (voir Sous-section 4.4.A), et où nous étudions l'éventuelle existence de L^q-projections non-triviales sur un sous-espace, quotient, ou sous-espace de quotient de L^p, pour $q \neq p$ (voir Sous-section 4.4.B).

Introduction (in English)

The main characters of this thesis are unitary dilations and L^p-projections.

Unitary dilations. Dilation theory is a collection of results, tools and techniques in operator theory and operator algebras, that fall under the unifying idea that one can learn a lot about an operator, or a family of operators, by viewing it as a compression of (i.e., "a part of") another, well understood operator. Historically, the first result of dilation theory was proved by Béla Sz.-Nagy in 1953: every contraction can be (power) dilated to a unitary operator. Based on this relatively simple fact an entire theory of non-normal operators has been developed, under the suggestive name of harmonic analysis of Hilbert space contractions. An influential monograph with the same title has been published in 1970 by B. Sz.-Nagy and C. Foias ; a fully updated and revised second edition has been published by Springer in 2010 by Sz.-Nagy, Foias, H. Bercovici and L. Kérchy.

There are several applications for the existence of a (minimal, power) unitary dilation for a given Hilbert space contraction. The first application, proved again by Sz.-Nagy in his 1953 paper, is a new proof of the celebrated von Neumann inequality. This assertion says that for a bounded linear operator T on a Hilbert space, we have:

$$\text{If } \|T\| \leq 1 \text{ then } \|P(T)\| \leq \|P\|_{L^\infty(\mathbb{D})} \text{ for every polynomial } P \in \mathbb{C}[Z].$$

The minimal unitary dilation can also be used to define a refined functional calculus on contractions, it can be employed to analyse one-parameter semigroups of operators, it provides a "functional model" by which we can analyse contractions and by which they can be classified, and it has led to considerable progress in the study of invariant subspaces.

To sketch just one of the above applications, let us briefly consider the functional calculus for Hilbert space contractions. We let $H^\infty = H^\infty(\mathbb{D})$ denote the algebra of bounded analytic functions on the open unit disc $\mathbb{D}$. Given an operator $T \in \mathcal{B}(H)$, we wish to define a functional calculus $f \mapsto f(T)$ for all $f \in H^\infty$. If the spectrum of T is contained in $\mathbb{D}$, then we can apply the holomorphic functional calculus to T to define a homomorphism $f \mapsto f(T)$ from the algebra $\mathrm{Hol}(\mathbb{D})$ of analytic functions on $\mathbb{D}$ into $\mathcal{B}(H)$. In fact, if $f \in \mathrm{Hol}(\mathbb{D})$ and $\sigma(T) \subset \mathbb{D}$, then we can simply plug T into the power series of f. Thus, in this case we know how to define $f(T)$ for all $f \in H^\infty$.

Now, suppose that T is a contraction, but that $\sigma(T)$ is not contained in the *open* disc $\mathbb{D}$. Given a bounded analytic function $f \in H^\infty$, how can we define $f(T)$? Note that the holomorphic functional calculus cannot be used, because $\sigma(T)$ contains points on the circle $\mathbb{T} = \partial\mathbb{D}$, while not all $f \in H^\infty$ extend to a holomorphic function on a neighbourhood of the closed disc.

The first case that we can treat easily is the case when f belongs to the *disc algebra* $A(\mathbb{D}) \subset H^\infty$, which is the algebra of all bounded analytic functions on the open unit disc that extend continuously to the closure $\overline{\mathbb{D}}$. This case can be handled using von Neumann's inequality. Indeed, it is not very hard to show that $A(\mathbb{D})$ is the closure of the polynomials with respect to the supremum norm $\|f\|_\infty = \sup_{|z|\le 1}|f(z)|$. If $(P_n)_n$ is a sequence of polynomials that converges uniformly on $\overline{\mathbb{D}}$ to f, and T is a contraction, then von Neumann's inequality implies that $P_n(T)$ is a Cauchy sequence, so we can define $f(T)$ to be $\lim_n P_n(T)$. It is not hard to show that the functional calculus $A(\mathbb{D}) \ni f \mapsto f(T)$ has all the properties one wishes for: it is a homomorphism extending the polynomial functional calculus, it is continuous, and it agrees with the continuous functional calculus if T is normal.

Defining a functional calculus for H^∞ is a more delicate matter, but here again the unitary dilation leads to a resolution. The rough idea is that we can look at the minimal unitary dilation U of T, and use spectral theory to analyse what can be done for U. If the spectral measure of U is absolutely continuous with respect to Lebesgue measure on the unit circle, then it turns out that we can define $f(U)$ for all $f \in H^\infty$, and then we can simply define $f(T)$ to be the compression of $f(U)$ to H. In this case the functional calculus $f \mapsto f(T)$ is a homomorphism that extends the polynomial and holomorphic functional calculi, it is continuous in the appropriate sense, and it agrees with the Borel functional calculus when T is normal. Of course, this only becomes useful if one can find conditions that guarantee that the minimal unitary dilation of T has absolutely continuous spectral measure. A contraction T is said to be *completely non-unitary* (c.n.u.) if it has no reducing subspace M such that the restriction $T\big|_M$ is unitary. Every contraction splits as a direct sum $T = T_0 \oplus T_1$, where T_1 is c.n.u. and T_0 is unitary. Sz.-Nagy and Foias have shown that if T is c.n.u., then the spectral measure of its minimal unitary dilation is absolutely continuous.

If the spectral measure of U is not absolutely continuous with respect to Lebesgue measure, then there exits a sub-algebra H^∞_U of H^∞ for which there exists a functional calculus $f \mapsto f(U)$, and then one can compress to get $f(T)$; it was shown that H^∞_U is precisely the sub-algebra of functions in H^∞ on which $f \mapsto f(T)$ is a well defined homomorphism. See [SNBFK10, Chapter III] for precise details.

Our contribution in this thesis is about skew dilations and the study of several classes of operators generalizing Sz. Nagy's dilation theorem. One central object of study is the class $C_{(\rho_n)}$ associated with given a sequence (ρ_n) of complex numbers with $\rho_n \neq 0$ for each n. By definition, $T \in \mathcal{L}(H)$ is said to possess a (ρ_n)-*dilation*, or that it belongs to the classe $C_{(\rho_n)}$, if there exists a Hilbert space K and a unitary operator $U \in \mathcal{L}(K)$ with $H \subset K$ and $T^n = \rho_n P_H U^n|_H$ for every $n \ge 1$, where $P_H \in \mathcal{L}(K)$ is the orthogonal projection from K onto its closed subspace H. Several spectral properties of operators belonging to this class are studied in this thesis.

L^p-projections. The geometry of Hilbert spaces is, in a certain sense, the closest possible to the usual one of a (finite dimensional) Euclidean space. Every closed subspace in Hilbert space has an orthogonal complement, and all such decompositions lead to an orthogonal projection. Orthogonal projections in Hilbert space are thus basic objects of study in Hilbert space operator theory. In Banach spaces different versions of "orthogonal projections" have been considered: idempotents of norm one, orthogonal projections, Hermitian projections, etc. The notion of L^p-projection has been introduced by Cunningham [Cun53] in 1953, the same year that Sz.-Nagy proved his dilation theorem. An L^p-projection on a Banach space X, for $1 \le p \le +\infty$, is an idempotent operator P satisfying $\|f\|_X = \|(\|P(f)\|_X, \|(I-P)(f)\|_X)\|_{\ell_p}$ for all $f \in X$. This is an L^p version of the equality $\|f\|^2 = \|Q(f)\|^2 + \|(I-Q)(f)\|^2$, valid for orthogonal projections on Hilbert spaces.

The main motivation for the introduction of this notion came from the development of some chapters of the geometry of Banach spaces which can now be grouped under the heading "L^p-structure". Let X be a real Banach space and let p be a real number with $1 \leq p \leq +\infty$. Two closed subspaces, J and $J^\perp$, of X are called complementary *L^p-summands* if X is the algebraic sum of J and $J^\perp$ and for every $x \in J$, $x^\perp \in J^\perp$ we have

$$\|x + x^\perp\|^p = \|x\|^p + \|x^\perp\|^p \quad (\text{ if } 1 \leq p < +\infty)$$

and

$$\|x + x^\perp\| = \max\Big(\|x\|, \|x^\perp\Big) \quad (\text{ if } p = +\infty).$$

Therefore the elements of J and $J^\perp$ behave like disjoints elements in an L^p-space. The projection from X onto J corresponding to this decomposition is precisely an L^p-projection.

While L^1- and L^∞- summands and the corresponding projections have been first studied by Cunningham and others, the turning point in the history of the "L^p-structure" theory was the paper [AE72] (split in two parts) by Eric Alfsen and Edward G. Effros published in Annals of Mathematics. Probably the most important results in [AE72] are the characterization of M-ideals by means of the intersection property and the introduction of the structure topology. We recall here that, by definition, an M-ideal is a closed subspace whose polar is an L^1-summand in the dual space.

L^p-projections were studied, mainly in the cases $p = 1$ and $p = +\infty$, in the papers [Cun53, Cun60, Cun67, CER73]. The general case $1 < p < +\infty$ has been studied by Alfsen-Effros in the above-cited paper [AE72], Sullivan [Sul70] and Fakhoury [Fak74]. The main characterization results, which were obtained in 1973-1976 by Alfsen-Effros, Behrends, Fakhoury, Sullivan and others, were compiled in the book [BDE+77, Ch.1,2,6]. An interesting norm characterization of real L^p-spaces has been obtained in [Sul68]: Sullivan proved that a real L^p space X is characterized by Clarkson's inequalities for X and its dual X^* and by the existence of enough L^p-projections.

We also mention that every L^p-projection is Hermitian. We recall (see for instance [BS74]) that a projection Q is *Hermitian* if $\|e^{i\alpha Q}\| = 1$ for each $\alpha \in \mathbb{R}$ and that a projection Q is Hermitian if and only if $Q + \lambda(I - Q)$ is an isometry for every $\lambda \in \partial\mathbb{D}$ or, equivalently, if $\lambda Q + \gamma(I - Q)$ is an isometry, for any $\lambda, \gamma \in \partial\mathbb{D}$. To see that an L^p-projection P is Hermitian, we note that the L^p-projection condition is also equivalent to

$$\|f + g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p}, \text{ for all } f \in \text{Ran}(P),\ g \in \text{Ker}(P),$$

where *Ran* and *Ker* denote the range and respectively the kernel. Hence, for any $\lambda, \gamma \in \partial\mathbb{D}$, we have

$$\|\lambda f + \gamma g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p} = \|f + g\|_X, \text{ for all } f \in \text{Ran}(P),\ g \in \text{Ker}(P).$$

Therefore, $\lambda P + \gamma(I - P)$ is an isometry on X, and the L^p-projection P is a Hermitian projection.

Our contribution in this thesis is to study L^p-projections on subspaces and quotients of complex Banach spaces. We introduce a notion of p-orthogonality for two elements x, y by requiring that $\text{Span}(x, y)$ admits an L^p-projection separating x and y. We also introduce the notion of maximal L^p-projections for X, that is L^p-projections defined on a subspace G of X that cannot be extended to L^p-projections on larger subspaces. We prove results concerning L^p-projections and p-orthogonality on general Banach spaces or on Banach spaces with additional properties. Generalizations of some results to spaces $L^p(\Omega, X)$ as well as some results about L^q-projections on subspaces of $L^p(\Omega)$ are also discussed.

Chapter 1

Classes C_ρ have been introduced by B. Sz-Nagy and C. Foias [SNF66] in 1966. For a complex Hilbert space H and a real number $\rho > 0$, a bounded linear operator $T \in \mathcal{L}(H)$ is said to be in the class $C_\rho(H)$ if all powers of T can be skew-dilated to powers of a unitary operator on a Hilbert space K, containing H as a closed subspace. Classes $C_{(\rho_n)}$ are a generalization of these classes C_ρ. They are defined as follows.

Definition 1.2.1. (Classes $C_{(\rho_n)}$) *Let $(\rho_n)_{n\geq 1}$ be a sequence of complex numbers, with $\rho_n \neq 0$ for each n. Let H be a complex Hilbert space. We define the class*

$$C_{(\rho_n)}(H) := \{T \in \mathcal{L}(H) : \text{ there exists a Hilbert space } K \text{ and a unitary operator } U \in \mathcal{L}(K) \\ \text{with } H \subset K \text{ and } T^n = \rho_n P_H U^n|_H,\ \forall n \geq 1\},$$

where $P_H \in \mathcal{L}(K)$ is the orthogonal projection from K onto its closed subspace H. An operator $T \in C_{(\rho_n)}(H)$ is said to possess a (ρ_n)-dilation.

These classes appeared briefly in [Rác74, Bad03], in results about similarity to a contraction. Some of the properties of operators in these classes were studied in [SZ16] for $\rho_n \in \mathbb{R}_+^*$.

This first chapter starts by generalizing many results known for classes C_ρ to classes $C_{(\rho_n)}$, which provides useful tools for their study. The main such useful tool is a characterization of these classes using the positivity of some self-adjoint operators.

Theorem 1.2.8. *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Let $T \in \mathcal{L}(H)$. Then, the following assertions are equivalent:*

(i) $T \in C_{(\rho_n)}$;

(ii) $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$ *and, for* $f_{(\rho_n)}(zT) := \sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$, *we have*

$$I + Re(f_{(\rho_n)}(zT)) \geq 0, \forall z \in \mathbb{D}.$$

The holomorphic map $f_{(\rho_n)}$, and its convergence radius of $\liminf_n(|\rho_n|^{\frac{1}{n}})$, has a central role in the structure of the class $C_{(\rho_n)}$.

These classes are also related to the following quantity

Definition 1.2.11. ((ρ_n)-radius) *Let H be a Hilbert space, $T \in \mathcal{L}(H)$, and $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$. The* (ρ_n)-radius *of T is defined as*

$$w_{(\rho_n)}(T) := \inf\{u > 0 \colon \frac{T}{u} \in C_{(\rho_n)}(H)\} \in [0, +\infty].$$

When $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$, we prove that the (ρ_n)-radius is a quasi-norm on $\mathcal{L}(H)$ which is equivalent to $\|\cdot\|$, and its closed unit ball is the class $C_{(\rho_n)}(H)$. Therefore, we have

$$w_{(\rho_n)}(T) \leq 1 \Leftrightarrow T \in C_{(\rho_n)}(H).$$

Hence many questions regarding classes $C_{(\rho_n)}(H)$ can be reformulated using their radii, sometimes giving a completely different point of view.

Section 1.3 focuses on the case where the sequence $(\rho_n)_n$ is constant and equal to $\rho \in \mathbb{C}^*$. In this context many characterizations take a simpler form, allowing us to link these classes to classes C_τ for $\tau > 0$. This behaviour is emphasized by the main result of this section.

Proposition 1.3.3. *Let $\rho \neq 0$ and $\alpha > 0$ be two scalars. Let $T \in \mathcal{L}(H)$. The following assertions are equivalent:*

(i) $w_{(\rho)}(T) \leq \alpha$;

(ii) $((\rho - 1)zT - \rho\alpha I)$ *is invertible and* $\|(zT)((\rho-1)zT - \rho\alpha I)^{-1}\| \leq 1$, $\forall z \in \mathbb{D}$;

(iii) $((\rho - 1)T - \rho w I)$ *is invertible and* $\|T((\rho-1)T - \rho w I)^{-1}\| \leq 1$, $\forall |w| > \alpha$;

(iv) $\|T(h)\| \leq \|(\rho-1)T(h) - \rho w h\|$, $\forall h \in H$, $\forall |w| > \alpha$.

Furthermore, we have:

$$|\rho| w_{(\rho)}(T) = (1 + |\rho - 1|) w_{1+|\rho-1|}(T). \tag{0.0.2}$$

Hence, the map $\rho \in \mathbb{C}^* \mapsto |\rho| w_{(\rho)}(T)$ *is constant on circles of center* 1*, is continuous on* $\mathbb{C}^*$ *and can be extended continuously to* $2w_{(2)}(T)$ *at 0.*

This result allows for computations of (ρ)-radii for operators T satisfying $T^2 = aT$ or $T^2 = bI$, and provides several ways to use (ρ)-radii for complex ρ. We also obtain a computation of (ρ)-radii for operators T satisfying $(T - aI)^2 = 0$ (Proposition 1.3.9).

Section 1.4 goes back to the general case, studying additional properties of (ρ_n)-radii as well as relationships between (ρ_n) and $(\tau_n)_n$-radii. The previous section inspired us into considering $(z\rho_n)$-radii, for a given sequence (ρ_n) and for $z \in \mathbb{C}^*$, in order to bring back the study to 1-parameter families. Looking at the map $z \mapsto w_{(z\rho_n)}(T)$ leads to the following result.

Proposition 1.4.8. *Let $T \in \mathcal{L}(H)$. Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that* $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$*. We have:*

(i) $z \mapsto w_{(z\rho_n)}(T)$ *is uniformly continuous on* $\mathbb{C} \setminus \mathbb{D}(0, \epsilon)$*, for all* $\epsilon > 0$*. This maps tends to* $+\infty$ *as* $|z| \to 0$*, and to* $\frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}$ *as* $|z| \to +\infty$;

(ii) For any $t \in \mathbb{R}$*, the map* $r \mapsto w_{(re^{it}\rho_n)}(T)$ *is log-convex on* $]0, +\infty[$.

It also turns out that these families can behave differently from the family of classes $C_{(z)}$, $z \in \mathbb{C}^*$. We can for example find sequences (ρ_n) and operators T for which $w_{(z\rho_n)}(T)$ is constant when $|z|$ is large enough (see Proposition 1.4.16). There are also sequences (ρ_n) for which $\bigcup_{r>0} C_{(r\rho_n)}$ contains all power-bounded operators (see Corollary 1.4.18), whereas $\bigcup_{\rho>0} C_{(\rho)}$ is strictly included in the set of operators similar to a contraction. In this regard, we end this section with a study of the $(z\rho_n)$-radius of I for two families, where the map $f_{(\rho_n)}$ is respectively related to log and to exp.

Chapter 2

Classes C_A are a different generalization of classes C_ρ that were defined by H. Langer (see [SNBFK10, p.53] and its references), and then studied by Suen [Sue98a] in 1998.

Definition 2.1.1. (Classes C_A) *Let H be a Hilbert space. Let $A \in \mathcal{L}(H)$ be a self-adjoint positive operator that is invertible. We define the class $C_A(H)$ as*

$$C_A(H) := \{T \in \mathcal{L}(H)\colon \exists K \text{ Hilbert and } U \in \mathcal{L}(K) \text{ unitary such that } H \subset K \text{ and } A^{-\frac{1}{2}} T^n A^{-\frac{1}{2}} = P_H U^n|_H,\ \forall n \geq 1\}$$

where P_H is the orthogonal projection onto H. When the underlying Hilbert space H is not ambiguous, classes $C_A(H)$ will be abbreviated as C_A.

Similarly to Chapter 1, the initial results of this second chapter, that can also be found in [Sue98a], are then used to obtain a few improvements (see Propositions 2.2.12 and 2.3.1), as well as new results (from Lemma 2.3.3 to Proposition 2.3.9). The structure of classes C_A also shares many similarities with classes $C_{(\rho_n)}$, as the two main tools to study these classes are the following ones.

Proposition 2.1.4. *Let H be a Hilbert space. Let $A, T \in \mathcal{L}(H)$ be such that $A > 0$. The following are equivalent*

(i) $T \in C_A(H)$;

(ii) $r(T) \leq 1$ *and* $I + Re(2 \sum\limits_{n\geq 1} A^{-\frac{1}{2}}(zT)^n A^{-\frac{1}{2}}) \geq 0$, $\forall z \in \mathbb{D}$;

(iii) $r(T) \leq 1$ *and* $A + Re(2 \sum_{n\geq 1}(zT)^n) \geq 0$, $\forall z \in \mathbb{D}$;

(iv) $r(T) \leq 1$ *and* $A - 2Re(z(A-I)T) + |z|^2 T^*(A-2I)T \geq 0$, $\forall z \in \mathbb{D}$;

Definition 2.2.1. *Let $A, T \in \mathcal{L}(H)$ be such that $A > 0$. We define*

$$w_A(T) := \inf(\{r > 0\colon \frac{1}{r}T \in C_A(H)\}),$$

which is called the A-radius *of T.*

Proposition 2.2.4. *Let $A \in \mathcal{L}(H)$, with $A > 0$. Then w_A is a quasi-norm that is equivalent to the operator norm $\|\cdot\|$, and whose closed unit ball is the class C_A.*

These tools are used to compute the A-radius of I (Proposition 2.2.12), to improve results linking $w_A(CTC^*)$ and $w_A(T)$ (Proposition 2.3.1), and to give computations of $w_A(T)$ when $T^2 = 0$ in some contexts (Propositions 2.3.4 and 2.3.5).

Chapter 3

This chapter deals with two different objects. Its first section studies algebraic operators, while the two others study classes of projections. An algebraic operator is defined as follows.

Definition 3.1.3. (Algebraic operators) *Let X be a complex Banach space. An operator $T \in \mathcal{L}(X)$ is said to be* algebraic *if there exists a polynomial $Q \in \mathbb{C}[Z]$ such that $Q(T) = 0$.*

In Section 3.1 we focus on the study of algebraic operators regarding similarity to a contraction and weaker conditions (polynomially bounded, power-bounded,...). We prove that most of these conditions are equivalent for algebraic operators. This also gives useful ways to determine if a given algebraic operator T is similar to a contraction or not.
With the use of the kernel lemma and minimal polynomials, an algebraic operator possesses a spectral decomposition through its characteristic projections. This fact led our study to shift in Section 3.2 to classes of projections in order to see how they behave with respect to each other. They range from the class of norm one projections to classes of L^p-projections. Here are the classes we will be considering.

Definition 3.2.1. (Classes of projections) *Let X be a complex Banach space. Let X_1, X_2 be closed subspaces of X that are in direct sum. Let $P \in \mathcal{L}(X)$ be the projection onto X_1 parallel to X_2. We define the following properties for P:*

(P_1) $\|P\| = 1$;

(P_2) $\|P\| = \|I - P\| = 1$;

(P_3) P is generalized bicircular*: There is $\lambda \in \partial\mathbb{D} \setminus \{1\}$ such that $\lambda P + (I - P)$ is a surjective isometry.*

(P_4) P is Hermitian*: For every $\alpha \in \mathbb{R}$, $e^{i\alpha P}$ is a surjective isometry.*

(P_5) For every $\lambda_i \in \mathbb{C}$, every $x_i \in X_i$, $1 \leq i \leq 2$, we have

$$\|\lambda_1 x_1 + \lambda_2 x_2\| \leq \max_i(|\lambda_i|)\|x_1 + x_2\|.$$

(P_5') For every $\lambda \in \partial\mathbb{D}$, $x_i \in X_i$, $1 \leq i \leq 2$, we have

$$\|x_1 + \lambda x_2\| = \|x_1 + x_2\|.$$

(P_6) For every $x_i, y_i \in X_i$, $1 \leq i \leq 2$, with $\|x_2\| = \|y_2\|$, we have

$$\|x_1 + x_2\| = \|x_1 + y_2\|.$$

(P_7) For every closed subspaces $V_i \subset X_i$, every $C_i \in \mathcal{L}(X_i)$, and every $x_i \in V_i$, $1 \leq i \leq 2$, we have

$$\|C_1(x_1) + C_2(x_2)\| \leq \max_{i=1,2}(\sup_{y_i \in V_i,\ \|y_i\|=1}(\|C_i(y_i)\|))\|x_1 + x_2\|.$$

(P_8) For every $x_i \in X_i \setminus \{0\}$, every $C_i \in \mathcal{L}(X_i)$, $1 \leq i \leq 2$, we have

$$\|C_1(x_1) + C_2(x_2)\| \leq \max_i(\frac{\|C_i(x_i)\|}{\|x_i\|})\|x_1 + x_2\|.$$

(P_8') For every $x_i, y_i \in X_i$ with x_i non-zero, $1 \leq i \leq 2$, we have

$$\|y_1 + y_2\| \leq \max_i(\frac{\|y_i\|}{\|x_i\|})\|x_1 + x_2\|.$$

(P_9) For every $x_i, y_i \in X_i$ with $\|x_i\| = \|y_i\|$, $1 \leq i \leq 2$, we have

$$\|y_1 + y_2\| = \|x_1 + x_2\|.$$

(P_{10}) There exists $1 \leq p \leq +\infty$ such that P is a L^p-projection*: For every $x_i \in X_i$ we have*

$$\|x_1 + x_2\| = \|(\|x_1\|, \|x_2\|)\|_{\ell^p}.$$

As the properties defining some of these classes are mainly about the norm of vectors in a direct sum of two subspaces, we make some generalizations for direct sums of a finite number of subspaces in Subsection 3.3.*B*.

Definition 3.2.9. *Let X be a Banach space, $r \geq 2$, and $X_1, .., X_r$ be closed subspaces such that $X = X_1 \oplus X_2 \oplus ... \oplus X_r$. We define the following properties*

($P_{8,r}$) *For every $x_i \in X_i$ and $C_i \in \mathcal{L}(X_i)$, $1 \leq i \leq r$, we have*

$$\|C_1(x_1) + ... + C_r(x_r)\| \leq \max_i(a(C_i, x_i))\|x_1 + ... + x_r\|,$$

where $a(C_i, x_i) := \begin{cases} \frac{\|C_i(x_i)\|}{\|x_i\|} & \textit{if } x_i \neq 0 \\ 0 & \textit{else} \end{cases}$

($P_{9,r}$) *For every $x_i, y_i \in X_i$ such that $\|x_i\| = \|y_i\|$, $1 \leq i \leq r$, we have*

$$\|x_1 + ... + x_r\| = \|y_1 + ... + y_r\|.$$

($P_{5,r}$) *For every $x_i \in X_i$ and $\lambda_i \in \mathbb{C}$, $1 \leq i \leq r$, we have*

$$\|\lambda_1 x_i + ... + \lambda_r x_r\| \leq \max_i(|\lambda_i|)\|x_1 + ... + x_r\|.$$

($P'_{5,r}$) *For every $x_i \in X_i$ and $\lambda_i \in \partial\mathbb{D}$, $1 \leq i \leq r$, we have*

$$\|\lambda_1 x_i + ... + \lambda_r x_r\| = \|x_1 + ... + x_r\|.$$

Working with direct sums brings additional questions regarding these new properties as some may be true for every $X_i \oplus X_j$ but not for $X_1 \oplus ... \oplus X_r$ for example (see Lemma 3.2.11).

Section 3.3 looks at the specific case where X is equal to $L^p(\Omega)$ or to some subspace of $L^p(\Omega)$, for projections that are either Hermitian or L^p-projections. These projections come with a property that must be satisfied for every $z \in \mathbb{C}$, hence we try to see if a weaker condition (for every $z \in \partial\mathbb{D}$, for $z = \pm 1$) would give the same results. As L^p-projections are a specific case of Hermitian projections, we also try to look at subspaces of L^p for which every Hermitian projection is an L^p-projections, or at conditions that ensure the contrary.

In Subsection 3.3.B we study the case $p = 2n$, where the relationship $|f+g|^{2n} = (f+g)^n(\overline{f}+\overline{g})^n$ allows us to obtain a useful additional property.

Proposition 3.3.9. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $n \geq 1$. Let $X = L^{2n}(\Omega, \mathcal{F}, \mu)$. Let $f, g \in X$. The following are equivalent*

(i) $\|f + \lambda g\|_{2n} = \|f + g\|_{2n}$, $\forall \lambda \in \partial\mathbb{D}$;

(ii) $\|f + zg\|_{2n} = \|f + |z|g\|_{2n}$, $\forall z \in \mathbb{C}$;

(iii) $\sum_{k=0}^{n-l} \binom{n}{k}\binom{n}{l+k} \int_\Omega (g^l|g|^{2k}\overline{f}^l|f|^{2(n-l-k)})(x)d\mu(x) = 0$, $\forall 1 \leq l \leq n$;

(iv) $\|f + zg\|_{2n}^{2n} = \sum_{k=0}^{n} |z|^{2k}\binom{n}{k}^2 \int_\Omega (|f|^{2(n-k)}|g|^{2k})(x)d\mu(x)$, $\forall z \in \mathbb{C}$.

Using this Proposition we are easily able to build subspaces of $l^{2n}(\{0, .., n\})$ possessing a Hermitian projection that is not a L^{2n}-projection (see Proposition 3.3.10), and extend this result to many other L^{2n} spaces.

Chapter 4

The final chapter in this thesis deals with the characterization and properties of L^p-projections. These operators were introduced by Cunningham [Cun53] in 1953.

Definition 4.1.1. (L^p-projections) *Let X be a complex Banach space, and let $1 \leq p \leq +\infty$. A projection P ($P^2 = P$) in $\mathcal{L}(X)$ is an* L^p-projection *if it satisfies the condition*

$$\|f\|_X = \|(\|P(f)\|, \|(I-P)(f)\|)\|_p, \text{ for all } f \in X.$$

This means that

$$\begin{cases} \|f\|_X^p = \|P(f)\|_X^p + \|(I-P)(f)\|_X^p, \ \forall f \in X & \text{when } 1 \leq p < +\infty. \\ \|f\|_X = \max(\|P(f)\|_X, \|(I-P)(f)\|_X), \ \forall f \in X & \text{when } p = +\infty. \end{cases}$$

We denote by $\mathcal{P}_p(X)$ the set of L^p-projections on X.

Cunningham's papers [Cun53, Cun60, Cun67, CER73] mainly focused on the cases $p = 1$ and $p = +\infty$, and the general case was first studied by Alfsen-Effros [AE72], Sullivan [Sul70] and Fakhoury [Fak74]. The main characterization results for L^p-projections were obtained in 1973-1976 by Behrends, Greim & al, and are compiled in the book [BDE+77, Ch.1,2,6]. These results are recalled in the Preliminaries of Section 4.1. The main theorem that we will frequently use is the following one.

Theorem 4.1.8. *Let X be a Banach space, and $1 \leq p \leq +\infty$, $p \neq 2$. We then have*

(i) All elements of $\mathcal{P}_p(X)$ commute with each other.

(ii) The set $\mathcal{P}_p(X)$ is a commutative Boolean algebra for the operations $(P,Q) \mapsto PQ$, $(P,Q) \mapsto P + Q - PQ$ and $P \mapsto (I - P)$.

(iii) The relationship $P \leq Q \ \Leftrightarrow \ PQ = P$ is an order relationship on $\mathcal{P}_p(X)$.

(iv) When $p \neq +\infty$, every decreasing filtrating net $(P_i)_{i\in I}$ in $\mathcal{P}_p(X)$ is pointwise convergent to an L^p-projection P, with $P = \inf_{i\in I}(P_i)$.

(v) When $p \neq +\infty$, the Boolean algebra $\mathcal{P}_p(X)$ is complete: Every subset $\{P_i,\, i \in I\}$ admits an infimum $\inf_{i\in I}(P_i)$ in $\mathcal{P}_p(X)$. Furthermore, $Ran(\inf_{i\in I}(P_i)) = \bigcap_{i\in I} Ran(P_i)$.

As almost all the statements of this theorem are valid for $1 \leq p \leq +\infty$, $p \neq 2$, most of our results in this chapter will be true for $p \neq 2$. As the L^∞-projections have a slightly different definition from other L^p-projections and do not satisfy every statement in Theorem 4.1.8, we will need to study them as a special case (see Subsection 4.2.C). Lastly, as we will be considering quotients of Banach spaces, the cases $p = 1, +\infty$ will be dropped for most of the results in Section 4.3 due to some hindering behaviours of L^1 and L^∞ norms regarding quotients.

Our initial goal was to characterize L^p-projections on subspaces and quotients of L^p spaces (also called SQ^p spaces). We first worked on a larger frame in order to clear out the results we could get on any Banach space X. For that matter, we first defined an orthogonality relationship related to L^p-projections, that is

Property 4.2.1. (p-orthogonality) *Let X be a Banach space, and $1 \leq p \leq +\infty$. Let $f, g \in X$. The elements f and g are said to be* p-orthogonal, *denoted by $f \perp_p g$, if*

$$\begin{cases} \|f + zg\|^p = \|f\|^p + |z|^p \|g\|^p, \ \forall z \in \mathbb{C}, \text{ when } p < +\infty; \\ \|f + zg\| = \max(\|f\|, |z|\|g\|), \ \forall z \in \mathbb{C}, \text{ when } p = +\infty. \end{cases}$$

When $f \neq 0$ and $g \neq 0$, this condition is equivalent to the fact that Span(f, g) has dimension 2 and that the projection on Span(f) parallel to Span(g) is an L^p-projection on Span(f, g).

This relationship is symmetric, homogeneous, definite, but not linear in general. It also allows us to define the p-orthogonal of a set E similarly to orthogonals in the case of inner product spaces. This relationship has an easy characterization on L^p spaces and on subspaces of L^p (see Corollary 4.2.3), which allows in turn to easily characterize L^p-projections on subspaces of $L^p(\Omega)$. It turned out that if we take any Banach space X satisfying the two following properties, namely

Property 4.2.7. (Extension of p-orthogonality to X) *For every $f, g \in X$ such that $f \perp_p g$, there exists $P \in \mathcal{P}_p(X)$ such that $P(f) = f$ and $P(g) = 0$.*

Property 4.2.8. (Linearity of p-orthogonality on X) *For every $f, g, h \in X$ such that $f \perp_p g$ and $f \perp_p h$, we have $f \perp_p (g + h)$.*

then we are able to generalize the behaviours we can find on $L^p(\Omega)$ and its subspaces. This gives for example the following result.

Proposition 4.2.10. (L^p-projections on subspaces of X) *Let X be a Banach space, and $1 \leq p < +\infty$, $p \neq 2$. The following are equivalent*

(i) X satisfies Property 4.2.7 for p;

(ii) For any subsets E_1, E_2 of X, such that $f \perp_p g$ for every $f \in E_1$, $g \in E_2$, there exists $P \in \mathcal{P}_p(X)$ such that $P(E_1) = E_1$ and $P(E_2) = \{0\}$.

Furthermore, if one of them is true, then for any subspace F of X and for any $P \in \mathcal{P}_p(F)$, there exists $Q \in \mathcal{P}_p(X)$ such that $P = Q|_F$.

We also dedicated Subsection 4.2.B to elementary counter-examples showing different behaviour when the previous properties are not true for X.

As it turns out that most of the subspaces of $L^p(\Omega)$ only satisfy Property 4.2.8, we introduced a larger class of projections in Subsection 4.2.D.

Definition 4.2.30. (Maximal L^p-projections) *Let X be a Banach space, and $1 \leq p \leq +\infty$. Let F be a closed subspace of X, and let $P \in \mathcal{P}_p(F)$. The L^p-projection P is said to be* maximal for *X if there exists no subspaces G and L^p-projection Q on G such that $F \subsetneq G$ and $Q|_F = P$.*
We also define $\alpha(F) := Card(\{P : P \text{ is a maximal } L^p\text{-projection for } F\})$.

These maximal L^p-projections allow us to study the structure of the p-orthogonality on the subspace F. We give several results about the main behaviour of maximal L^p-projections, as well as their behaviour when F is a subspace of a Banach space X satisfying Properties 4.2.7 and 4.2.8.

We also focus on giving an upper bound to the number of maximal L^p-projections such a subspace F can possess when it has a finite dimension. We first worked out on a generic example.

Proposition 4.2.44. *Let $1 \leq p < +\infty$, $p \neq 2$, $n \geq 4$ and $X = \ell^p(\mathbb{C}^n)$. Denote $(e_i)_i$ the canonical basis of X. For $1 \leq i < n$, denote $f_i = e_i + e_{i+1}$. Take $F = Span(f_i,\, 1 \leq i < n)$. Then, we have $dim(F) = n - 1$, and*

$$\alpha(F) = Card(\{P\colon\ P \text{ is a maximal } L^p\text{-projection for } F\}) = 2^n - 2n.$$

Thus, for $n \geq 5$, there is strictly more than $2^{dim(F)}$ L^p-projections that are maximal for F.

Our findings in this case and in another specific context (see Proposition 4.2.46) led us to consider a conjecture for the general case (see Conjecture 4.2.49 and Proposition 4.2.50). This conjecture is unfortunately open as the time of this writing.

With this thorough study of L^p-projections for subspaces of a Banach space, we went in Section 4.3 into considering quotients and subspaces of quotients of Banach spaces. Most of the results rely heavily on the following key lemma.

Lemma 4.3.3. *Let $1 < p \leq +\infty$. Let X be a Banach space and let F be a closed subspace of X. Let $x \in X$, let G be a subspace of X containing F and x, and let $P \in \mathcal{P}_p(G)$ be such that $P(x) = x$. Then the following assertions are equivalent*

(i) $\inf_{a \in F} \|x - a\| = \|x\|$;

(ii) $\inf_{a \in F} \|x - P(a)\| = \|x\|$.

If the metric projections on F and $P(F)$ are well-defined, then we also have the equivalence :

(1) $Proj(x, F) = 0$;

(2) $Proj(x, P(F)) = 0$.

Even with this result, the p-orthogonality does not behave well enough for some results, which require the following additional property on a Banach space X with a subspace F.

Property 4.3.9. *For every $\overline{x}, \overline{y} \in X/F$ such that $\overline{x} \perp_p \overline{y}$, there exists $x, y \in X$ representatives of $\overline{x}, \overline{y}$ of minimal norm such that $x \perp_p y$.*

With these elements we are able to characterize the p-orthogonality relationship as well as the L^p-projections on subspaces of quotients of L^p and to generalize this characterization to a broader class of Banach spaces.

Our main results for L^p-projections on a quotient of a Banach space are the following

Proposition 4.3.13. *Let X be a Banach space and let $1 < p < +\infty$, $p \neq 2$. Let F be a closed subspace of X, and let $P \in \mathcal{P}_p(X)$ be such that $P(F) \subset F$. Then,*

(i) $X/F \simeq P(X)/P(F) \oplus_p (I - P)(X)/(I - P)(F)$;
If the metric projection on F is well-defined, then

$$Rep(X/F) = Rep(P(X)/P(F)) \oplus_p Rep((I - P)(X)/(I - P)(F)).$$

(ii) *There exists an L^p-projection P' on X/F such that $P'(\overline{x}) = \overline{P(x)}$;*

(iii) *P' is non-trivial if and only if $P(F) \neq P(X)$ and $(I - P)(F) \neq (I - P)(X)$;*

(iv) *Let* $P_F \in \mathcal{P}_p(X)$ *be the maximal* L^p*-projection such that* $Ran(P_F) \subset F$*. Then* X/F *is isometrically isomorphic to* $(I - P_F)(X)/(I - P_F)(F)$*.*

(v) *Denote* $\phi : P \in \{Q \in \mathcal{P}_p(X)\colon Q(F) \subset F\} \mapsto P' \in \mathcal{P}_p(X/F)$*. Then* ϕ *is a morphism of commutative Boolean algebras, and* $Ker(\phi) = \mathcal{P}_p(X) \circ P_F$*. Hence,* $\phi(P_1) = \phi(P_2)$ *if and only if* $(I - P_F)P_1 = (I - P_F)P_2$*, and* ϕ *is injective if and only if* $P_F = 0$*.*
In general, $Ran(\phi) = \phi(\{Q \in \mathcal{P}_p(X)\colon QP_F = 0,\, P(F) \subset F\})$ *and* ϕ *is injective on this set.*

Proposition 4.3.16. *Let* $1 < p < +\infty$*,* $p \neq 2$*. Let* X *be a Banach space satisfying Property 4.2.7. Let* F *be a closed subspace of* X *such that every element of* X/F *admits a unique representative of minimal norm. Suppose that Property 4.3.9 is satisfied for* X*,*F *and* p*. Denote* $\phi : P \in \{Q \in \mathcal{P}_p(X)\colon Q(F) \subset F\} \mapsto \phi(P) \in \mathcal{P}_p(X/F)$ *the morphism of commutative Boolean algebras from Proposition 4.3.13, with* $\phi(P)$ *satisfying* $\phi(P)(\overline{x}) = \overline{P(x)}$ *for every* $x \in X$*. Then,*

(i) *The morphism* ϕ *is surjective, every* L^p*-projection of* X/F *can be associated to an* L^p*-projection* P *on* X *such that* $P(F) \subset F$*;*

(ii) *The Boolean algebra* $\mathcal{P}_p(X/F)$ *is isomorphic to* $\{P \in \mathcal{P}_p(X)\colon PP_F = 0,\, P(F) \subset F\}$*;*

(iii) *Denote* P_F *the maximal* L^p*-projection of* X *such that* $Ran(P_F) \subset F$*. The space* X/F *admits non-trivial* L^p*-projections if and only if there exist* L^p*-projections* P *such that* $P_F < P < I$ *and* $P(F) \subset F$*.*

In Subsection 4.3.B we refine these results by characterizing L^p-projections on subspaces of quotients of a Banach space X that satisfies Properties 4.2.7,4.2.8 and 4.3.9 (see Proposition 4.3.19).

We end this chapter on a Section where we look at spaces $L^p(\Omega, X)$ for which the previous results would apply (see Subsection 4.4.A) and where we try to see if a subspace, quotient, or subspace of quotient of L^p can possess non-trivial L^q-projections for any $q \neq p$ (see Subsection 4.4.B).

Chapter 1

Hilbert space operators with unitary skew-dilations: Classes $C_{(\rho_n)}$

The aim of this chapter is to study, for a given sequence $(\rho_n)_{n\geq 1}$ of complex numbers, the class of Hilbert space operators possessing (ρ_n)-unitary dilations. This is the class of bounded linear operators T acting on a Hilbert space H, whose iterates T^n can be represented as $T^n = \rho_n P_H U^n|_H$, $n \geq 1$, for some unitary operator U acting on a larger Hilbert space, containing H as a closed subspace. Here P_H is the projection from this larger space onto H. The case when all ρ_n's are equal to a positive real number ρ leads to the class C_ρ introduced in the 1960s by Foias and Sz.-Nagy, while the case when all ρ_n's are positive real numbers has been previously considered by several authors. Some applications and examples of operators possessing (ρ_n)-unitary dilations, showing a different behaviour from the classical case, are given in this chapter.

In Section 1.1 and 1.2 we introduce our classes and lay out the main results that allow us to characterize them and work with them. For each class $C_{(\rho_n)}$, we also define a (ρ_n)-radius map from $\mathcal{L}(H)$ to $[0, +\infty]$. This map turns out to be a quasi-norm that is equivalent to the operator norm, and whose closed unit ball is the class $C_{(\rho_n)}$. The multiple properties of the (ρ_n)-radii allow us to obtain additional information on the classes $C_{(\rho_n)}$.
For Section 1.3 we focus on the case where the sequence $(\rho_n)_n$ is constant and equal to $\rho \in \mathbb{C}^*$. In this context many characterizations take a simpler form, allowing us to link these classes to classes C_τ for $\tau > 0$. This allows us to generalize the computations of (ρ)-radii for operator T satisfying either $T^2 = aI$ or $T^2 = bT$. We were also able go further and compute the (ρ)-radii of operators T satisfying $(T - aI)^2 = 0$ (see Prop.1.3.9).
Section 1.4 goes back to the general case by studying additional properties of (ρ_n)-radii as well as relationships between (ρ_n) and $(\tau_n)_n$-radii. The previous section inspired us into considering $(z\rho_n)$-radii, for a given sequence (ρ_n) and for $z \in \mathbb{C}^*$, in order to bring back the study to 1-parameter families. Looking at the map $z \mapsto w_{(z\rho_n)}(T)$ leads to maps that can behave differently from what can be seen with classes $C_{(z)}$, $z \in \mathbb{C}^*$. We can for example find sequences (ρ_n) and operators T for which $w_{(z\rho_n)}(T)$ is constant when $|z|$ is large enough (see Proposition 1.4.16). There are also sequences (ρ_n) for which $\bigcup_{r>0} C_{(r\rho_n)}$ contains all power-bounded operators (see Corollary 1.4.18), whereas $\bigcup_{\rho>0} C_{(\rho)}$ is strictly included in the set of operators similar to a contraction. In this regard, we end this section with a study of the $(z\rho_n)$-radius of I for two families, where the map $f_{(\rho_n)}$ is respectively related to log and to exp.

1.1 Introduction

Classes C_ρ have been introduced by B. Sz-Nagy and C. Foias [SNF66] in 1966. For a complex Hilbert space H and a real number $\rho > 0$, a bounded linear operator $T \in \mathcal{L}(H)$ is said to be in the class $C_\rho(H)$ if all powers of T can be skew-dilated to powers of a unitary operator on a Hilbert space K, containing H as a closed subspace. This means that

$$T^n = \rho P_H U^n|_H, \text{ for all } n \geq 1,$$

where $U \in \mathcal{L}(K)$ is a suitable unitary operator, and $P_H \in \mathcal{L}(K)$ denotes the orthogonal projection onto H. Such an operator T is called a ρ-contraction, while the unitary operator U is called a ρ-*dilation*, or a ρ-unitary dilation, of T.

The famous Sz.-Nagy dilation theorem (see [SNF66]) shows that $C_1(H)$ is exactly the class of all Hilbert space contractions *i.e.*, operators of norm no greater than one. It is also known (see [BS67]) that the class $C_2(H)$ coincides with the class of all operators T with numerical range $W(T)$ included in the closed unit disk; equivalently, those T satisfying $w(T) \leq 1$. Here the *numerical range* $W(T)$ and the *numerical radius* $w(T)$ of T are defined by

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\} \quad ; \quad w(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

Let T be an operator in the class C_ρ. Then

(i) T is power-bounded. More precisely, we have $\|T^n\| \leq \max(1, \rho)$, for all $n \geq 0$. In particular, the spectral radius $r(T)$ of T satisfies $r(T) \leq 1$;

(ii) T^k is in $C_\rho(H)$ for every $k \geq 1$;

(iii) For a closed subspace, F, of H which is stable by T (*i.e.*, $T(F) \subset F$), the restriction $T\,|_F$ is in $C_\rho(F)$;

(iv) The functional calculus map $f \mapsto f(T)$ that sends a polynomial f into $f(T)$ can be extended in a well-defined manner to the disk algebra $\mathbb{A}(\mathbb{D}) := C^0(\overline{\mathbb{D}}) \cap \mathrm{Hol}(\mathbb{D})$. It is a morphism of Banach algebras, and satisfies

$$\|f(T)\| \leq \max(1, \rho)\|f\|_{L^\infty(\mathbb{D})};$$

(v) T is similar to a contraction: there exists an invertible operator $L \in \mathcal{L}(H)$ such that $\|LTL^{-1}\| \leq 1$.

We refer the reader to [Hol68, Hol71, SNBFK10, Rác74, AO75] for proofs of these results, which mainly use several characterizations of classes $C_\rho(H)$. We record the principal ones in the following theorem.

Theorem. — Let T be an operator in $\mathcal{L}(H)$ and let $\rho > 0$. The following are equivalent:

(i) $T \in C_\rho(H)$;

(ii) $r(T) \leq 1$ and, for all $z \in \mathbb{D}$, we have $(1 - \frac{2}{\rho})I + \frac{2}{\rho}\mathrm{Re}((I - zT)^{-1}) \geq 0$;

(iii) For all $z \in \mathbb{D}$ and all $h \in H$ we have $(\frac{2}{\rho} - 1)\|zTh\|^2 + (2 - \frac{2}{\rho})\langle zTh, h\rangle \leq \|h\|^2$.

We remark that these characterization can be expressed in terms of classes of operator-valued holomorphic functions. For instance, (ii) says that the map $z \mapsto (1-\frac{2}{\rho})I + \frac{2}{\rho}(I-zT)^{-1}$ is in the Caratheodory class of operator-valued holomorphic functions on $\mathbb{D}$, having all real parts positive-definite operators. Item (iii) can be equivalently expressed by the membership of $z \mapsto zT((\rho-1)zT-\rho I)^{-1}$ to the Schur class of holomorphic maps $f : \mathbb{D} \to \mathcal{L}(H)$ having all norms no greater than one (i.e., $\|f(z)\| \leq 1$ for every $z \in \mathbb{D}$).

J.A.R. Holbrook [Hol68] and J.P. Williams [Wil68] introduced the notion of *ρ-radius* of an operator $T \in \mathcal{L}(H)$ as follows:

$$w_\rho(T) := \inf\{u > 0\colon\ \frac{1}{u}T \in C_\rho(H)\}.$$

This ρ-radius is a quasi-norm on the Banach space $\mathcal{L}(H)$, equivalent to the operator norm, whose closed unit ball is exactly $C_\rho(H)$. Recall ([Kal03]) that a *quasi-norm* satisfies all properties of a norm, except that the triangular inequality holds true up to a multiplicative constant. For $\rho > 2$, the quasi-norm w_ρ satisfies ([SNBFK10, AO76])

$$w_\rho(T_1 + T_2) \leq \rho\left(w_\rho(T_1) + w_\rho(T_2)\right).$$

Therefore the ρ-contractions are exactly the contractions for the ρ-radius, and many relationships between classes C_ρ can be expressed more easily using the associated ρ-radii. The ρ-radius is a usual Banach-space norm for $0 < \rho \leq 2$.

Some generalizations of classes C_ρ have been studied, like classes $C_A(H)$ introduced by H. Langer (see [SNBFK10, p.53] and its references, and [Sue98a]), or the classes $C_{(\rho_n)}(H)$ considered by several authors (see [Rác74, Bad03, SZ16]). This latter generalization will be the main topic of study in this chapter, with the novelty that we consider the general case when the ρ_n's are non-zero complex scalars. This will lead to classes of operators with several new features and different behaviour.

1.2 Hilbert Space Operators with (ρ_n)-Dilations

1.2.A Definition and first properties.

In light of the preceding discussion we introduce the following definition.

Definition 1.2.1 (Classes $C_{(\rho_n)}$)**.** — Let $(\rho_n)_{n\geq 1}$ be a sequence of complex numbers, with $\rho_n \neq 0$ for each n. We write $(\rho_n)_{n\geq 1} \in (\mathbb{C}^*)^{\mathbb{N}^*}$. Let H be a complex Hilbert space. Define now

$$C_{(\rho_n)}(H) := \{T \in \mathcal{L}(H) : \text{ there exists a Hilbert space } K \text{ and a unitary operator } U \in \mathcal{L}(K) \text{ with } H \subset K \text{ and } T^n = \rho_n P_H U^n|_H,\ \forall n \geq 1\}.$$

Here $P_H \in \mathcal{L}(K)$ is the orthogonal projection from K onto its closed subspace H. We say in this case that T possesses (ρ_n)-*dilations*.

In other words, an operator T is in the class $C_{(\rho_n)}(H)$ if and only if all its powers admit dilations of the form $\rho_n U^n$ for a certain unitary operator U acting on a larger Hilbert space. For the rest of this chapter, we will suppose that the Hilbert space H on which T acts is fixed.

If there is no ambiguity, $C_{(\rho_n)}(H)$ will be abbreviated as $C_{(\rho_n)}$. Note also that the sequence $(\rho_n) = (\rho_n)_{n\geq 1}$ starts at $n = 1$: for $n = 0$ we have of course $T^0 = I_H = P_H U^0|_H$.

In the papers [Rác74, Bad03, SZ16], the case when the ρ_n's are non-negative real numbers is considered. We went for a broader choice of sequences as the main ideas do not rely heavily on the fact that ρ_n are in $\mathbb{R}_+^*$ and as this eventually allows for some interesting new phenomena for the classes $C_{(\rho_n)}$. One first difference is recorded in the following remark.

Remark 1.2.2. — The definition of $C_{(\rho_n)}$ easily gives that $T \in C_{(\rho_n)}$ if and only if $T^* \in C_{(\overline{\rho_n})}$. Therefore, when the ρ_n are real scalars, the class $C_{(\rho_n)}$ is stable under the adjoint map $T \mapsto T^*$. This is no longer true in the general case. This may be one of the main reasons explaining the previous interest about the real positive case.

Remark 1.2.3. — As another basic remark, we note that if T is in $C_{(\rho_n)}$, then we have $\|T^n\| \leq |\rho_n|$. Thus, $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$. This relationship implies that two different cases appear in the study of the classes $C_{(\rho_n)}$:

(i) $0 < \liminf_n(|\rho_n|^{\frac{1}{n}}) \leq +\infty$;

(ii) $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 0$.

Although many of the proofs below work the same way in both cases, most of the results will be stated in the case (i). The study of the case (ii) is more problematic. Indeed, in case (ii), the class $C_{(\rho_n)}$ will only contain quasi-nilpotent operators, that is operators whose spectra reduce to $\{0\}$.
We also note that when $\liminf_n(|\rho_n|^{\frac{1}{n}}) = +\infty$, we trivially have $r(T) < \liminf_n(|\rho_n|^{\frac{1}{n}})$ for every operator T. We also note that the condition $\liminf_n(|\rho_n|^{\frac{1}{n}}) = +\infty$ leads to small changes in the proofs below: the main difference between this condition and $\liminf_n(|\rho_n|^{\frac{1}{n}}) < +\infty$ in case (i) is the fact that the quantity $\frac{1}{\liminf_n(|\rho_n|^{\frac{1}{n}})}$, which exists when $\liminf_n(|\rho_n|^{\frac{1}{n}}) \in]0, +\infty[$, has to be replaced by 0 when $\liminf_n(|\rho_n|^{\frac{1}{n}}) = +\infty$. This motivates the following convention.

Convention. For the rest of this chapter, we assume that

$$\frac{1}{\liminf_n(|\rho_n|^{\frac{1}{n}})} = 0 \quad \text{whenever} \quad \liminf_n(|\rho_n|^{\frac{1}{n}}) = +\infty. \tag{1.2.1}$$

One of the main tools to characterize the classes $C_{(\rho_n)}$ is the following Herglotz-type theorem.

Theorem 1.2.4. — *Let H a Hilbert space. Let $F : \mathbb{D} \to H$ be an analytic function such that:*

(i) $F(0) = I$

(ii) $Re(F(z)) \geq 0$, $\forall z \in \mathbb{D}$.

Then, there exists a Hilbert space K containing H and $U \in \mathcal{L}(K)$ an unitary operator such that

$$F(z) = P_H(I + zU)(I - zU)^{-1}|_H,\ \forall z \in \mathbb{D}$$

A proof of this theorem can be found in [Fil70, p.65-69]. This theorem is the key element that allows us to obtain a very useful characterization for classes $C_{(\rho_n)}$ in terms of positivity of certain operators.

Definition 1.2.5. — For $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and for w in a complex Banach algebra, $f_{(\rho_n)}$ denotes the entire series given by $f_{(\rho_n)}(w) = \sum_{n\geq 1} \frac{2w^n}{\rho_n}$.
For $a \in \mathbb{R}$, we denote $Re_{\geq a}$ the half-plane $\{z \in \mathbb{C}, \operatorname{Re}(z) \geq a\}$, while $Re_{>a}$ is the half-plane $\{z \in \mathbb{C}, \operatorname{Re}(z) > a\}$.

Proposition 1.2.6. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and let $T \in \mathcal{L}(H)$. The following are equivalent:*

(i) $T \in C_{(\rho_n)}$*;*

(ii) The series
$$f_{(\rho_n)}(zT) = \sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$$
is absolutely convergent in $\mathcal{L}(H)$ and
$$I + Re(f_{(\rho_n)}(zT)) \geq 0, \text{ for all } z \in \mathbb{D}.$$

Proof. $(i) \Rightarrow (ii)$ Let U be an unitary operator on a Hilbert space K, with K containing H as a closed subspace, such that
$$T^n = \rho_n P_H U^n|_H, \ \forall n \geq 1.$$
For every polynomial $P(X) = a_0 + \cdots + a_n X^n$ and every $z \in \mathbb{D}$, we have
$$a_0 I + \frac{a_1}{\rho_1} zT + \cdots + \frac{a_n}{\rho_n}(zT)^n = P_H(a_0 I + a_1 zU + \cdots + a_n(zU)^n)|_H = P_H\, P(zU)|_H.$$
Since the series $1 + \sum_{n\geq 1} 2w^n$ converges absolutely to $f(w) = \frac{1+w}{1-w}$ for all $w \in \mathbb{D}$, and since U is unitary, the series $I + \sum_{n\geq 1} 2(zU)^n$ converges in norm to
$$f(zU) = (I + zU)(I - zU)^{-1}, \text{ for all } z \in \mathbb{D}.$$
Thus, as
$$\|\frac{T^n}{\rho_n}\| = \|P_H U^n|_H\| \leq \|U^n\| \leq 1,$$
the series $I_H + \sum_{n\geq 1} \frac{2}{\rho_n}(zT)^n$ is absolutely convergent and converges to $P_H[(I+zU)(I-zU)^{-1}]|_H$ for all $z \in \mathbb{D}$. As U is unitary, $f(zU)$ is normal, so the closure of its numerical range $W(f(zU))$ is the convex hull of its spectrum. We have
$$\sigma(f(zU)) = f(\sigma(zU)) \subset f(\mathbb{D}) \subset Re_{>0}.$$
Thus,
$$W((I + zU)(I - zU)^{-1}) = W(f(zU)) \subset Hull(\sigma(f(zU))) \subset Re_{\geq 0}.$$
Furthermore, $W(P_H f(zU)|_H) \subset W(f(zU))$, so the numerical range of $I_H + f_{(\rho_n)}(zT)$ is included in $Re_{\geq 0}$. This is equivalent to $\operatorname{Re}(I_H + f_{(\rho_n)}(zT)) \geq 0$, so (ii) is true.
- $(ii) \Rightarrow (i)$ We define $F(z) := I_H + f_{(\rho_n)}(zT)$. Thus, F is analytic on $\mathbb{D}$, $F(0) = I_H$, and $\operatorname{Re}(F(z)) \geq 0$ for all $z \in \mathbb{D}$. By applying Theorem 1.2.4, we obtain a Hilbert space K and a unitary operator $U \in \mathcal{L}(K)$, such that $F(z) = P_H(I + zU)(I - zU)^{-1}|_H$, for all $z \in \mathbb{D}$. By developing both analytic expressions in entire series, and identifying their coefficients, we obtain $\frac{2}{\rho_n} T^n = 2P_H U^n|_H$ for all $n \geq 1$. Therefore $T \in C_{(\rho_n)}$. □

We will obtain most of the following results by applying Proposition 1.2.6. We can directly see by applying this proposition that any class $C_{(\rho_n)}$ contains 0, the null operator, so none of these classes is empty.

One remark is in order. We did not consider the case where $\rho_n = 0$ for some n in Definition 1.2.1. Indeed, this condition does not behave well with computations similar to the ones in the proof of Proposition 1.2.6. Having $\rho_n = 0$ implies $T^n = 0$, but it does not give any information on $P_H U^n|_H$. This prevents us from showing that certain sums of powers of T and T^* are positive, which is a crucial tool when dealing with operators in the class $C_{(\rho_n)}$.

If we were to denote $m := \inf(\{n\colon \rho_n = 0\})$, then any operator T in $C_{(\rho_n)}$ would need to be nilpotent of order at most m. The following Corollary treats this nilpotent case and gives a characterization that was the one we expected in the case $\rho_m = 0$. See also [BC02, Proposition 6.1] for another use of the positivity condition (ii) below.

Corollary 1.2.7. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and $m \geq 1$. Let $T \in \mathcal{L}(H)$ be such that $T^m = 0$. Then, the following are equivalent:*

(i) $T \in C_{(\rho_n)}$;

(ii) $I + Re(\sum_{n=1}^{m-1} z^n \frac{2}{\rho_n} T^n) \geq 0$ *for all* $z \in \mathbb{D}$.

Thus, for any sequence (τ_n) such that $\rho_k = \tau_k$, for all $1 \leq k < m$, we have $T \in C_{(\tau_n)}$ if and only if $T \in C_{(\rho_n)}$.

Proof. A direct application of Proposition 1.2.6 with the extra condition $T^m = 0$ gives the equivalence. □

Now we come back to Proposition 1.2.6. When $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$, we can see that the series $\sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$ is absolutely convergent if and only if $|z| r(T) < \liminf_n(|\rho_n|^{\frac{1}{n}})$. We can thus reformulate Proposition 1.2.6 as follows.

Theorem 1.2.8. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Let $T \in \mathcal{L}(H)$. Then, the following assertions are equivalent:*

(i) $T \in C_{(\rho_n)}$;

(ii) $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$ *and, for* $f_{(\rho_n)}(zT) := \sum_{n=1}^{\infty} \frac{2}{\rho_n} z^n T^n$, *we have*

$$I + Re(f_{(\rho_n)}(zT)) \geq 0, \forall z \in \mathbb{D}.$$

Remark 1.2.9. — Replacing the condition of absolute convergence of a series by a condition concerning the spectral radius of T is useful in several instances. We can first notice that if we take $v > 0$ small enough, then vT will satisfy the spectral radius condition. However, if $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 0$, this condition must be replaced by $\limsup_n(\frac{\|T^n\|}{|\rho_n|}^{\frac{1}{n}}) \leq 1$, which can only be satisfied by certain quasi-nilpotent operators. Hence, aside from nilpotent operators and Corollary 1.2.7, knowing which operators can be "near" operators belonging to a class $C_{(\rho_n)}$ is a difficult problem. In this case, the map $f_{(\rho_n)}$ also has convergence radius 0, so we cannot use analytic or geometric properties related to the images of certain disks by $f_{(\rho_n)}$.

Many of the following results, related to specific operators or to $f_{(\rho_n)}$ will have no meaning in this case, but others will be true under the additional condition

$$\limsup_n \left(\frac{\|T^n\|}{|\rho_n|}\right)^{\frac{1}{n}} \leq 1.$$

We look now at the closure of the class $C_{(\rho_n)}$ for the operator norm.

Corollary 1.2.10. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Then the class $C_{(\rho_n)}$ is closed for the operator norm: if $(T_m)_m$ is a sequence of operators converging in $\mathcal{L}(H)$ to T, such that $T_m \in C_{(\rho_n)}$, then $T \in C_{(\rho_n)}$.*

Proof. Let $(T_m)_m$ be a sequence of operators converging to T such that $T_m \in C_{(\rho_n)}$. We have

$$\|T^n\| = \lim_m(\|T_m^n\|) \leq |\rho_n|, \text{ so } r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}}).$$

Thus, for any $z \in \mathbb{D}$, the series $f_{(\rho_n)}(zT)$ converges absolutely and $f_{(\rho_n)}(zT) = \lim_m f_{(\rho_n)}(zT_m)$. Hence, for any $h \in H$, we have

$$\text{Re}(\langle (I + f_{(\rho_n)}(zT))h, h\rangle) = \text{Re}[\lim_m \langle (I + f_{(\rho_n)}(zT_m))h, h\rangle] \geq 0.$$

This implies that $I + \text{Re}(f_{(\rho_n)}(zT)) \geq 0$, and the proof is complete by using Theorem 1.2.8. □

1.2.B Operator radii.

The condition in Theorem 1.2.8 will be useful when studying the (ρ_n)-radius, which is introduced in the following definition.

Definition 1.2.11. — Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$. Let $T \in \mathcal{L}(H)$. We define the (ρ_n)-*radius* of T as:

$$w_{(\rho_n)}(T) := \inf\{u > 0\colon \frac{T}{u} \in C_{(\rho_n)}\} \in [0, +\infty].$$

The definition of the (ρ_n)-radius is similar to the definition of the ρ-radius that can be found in [Hol68, AL10, AO97, AN73]. As the classes $C_{(\rho_n)}$ and C_ρ share the same type of definition, the (ρ_n)-radius and the ρ-radius will share the same role with some slight different variations.

We will for now focus on properties of the (ρ_n)-radius.

Lemma 1.2.12. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Then, the map $T \mapsto w_{(\rho_n)}(T)$ takes values in $[0, +\infty[$, is a quasi-norm, is equivalent as a quasi-norm to the operator norm $\|\cdot\|$, and its closed unit ball is the class $C_{(\rho_n)}$.*
We also have

$$w_{(\rho_n)}(T) \geq \left(\frac{\|T^m\|}{|\rho_m|}\right)^{\frac{1}{m}} \textit{ for every } m \geq 1, \textit{ and } w_{(\rho_n)}(T) \geq \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}.$$

Proof. We start off by obtaining the inequalities concerning the (ρ_n)-radius, to then show its finiteness while obtaining its equivalence with the operator norm $\|\cdot\|$. Let $T \in \mathcal{L}(H)$. If $w_{(\rho_n)}(T) = +\infty$ then the inequalities of Lemma 1.2.12 are true. Else, let $u > 0$ be such that $\frac{T}{u} \in C_{(\rho_n)}$. For any $m \geq 1$, we have $\frac{\|T^m\|}{u^m} \leq |\rho_m|$, that is

$$u \geq \left(\frac{\|T^m\|}{|\rho_m|}\right)^{\frac{1}{m}}.$$

Therefore, by taking the infimum over u such that $\frac{T}{u} \in C_{(\rho_n)}$, we get

$$w_{(\rho_n)}(T) \geq \left(\frac{\|T^m\|}{|\rho_m|}\right)^{\frac{1}{m}}.$$

For $m = 1$ we obtain $w_{(\rho_n)}(T) \geq (\frac{\|T\|}{|\rho_1|})$. If we also take the lim sup of the right-hand side quantity, we get

$$w_{(\rho_n)}(T) \geq \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}.$$

Let us now prove the finiteness of $w_{(\rho_n)}(T)$. Let $r < \liminf_n(|\rho_n|^{\frac{1}{n}})$. Therefore, the series $f_{(|\rho_n|)}(rz) := \sum_{n=1}^{\infty} \frac{2}{|\rho_n|} r^n z^n$ is absolutely convergent for all $z \in \mathbb{D}$, thus analytic on $\mathbb{D}$. Since $f_{(|\rho_n|)}(0) = 0$, there is a radius r_0, with $1 > r_0 > 0$, such that $|f_{(|\rho_n|)}(r_0 w)| \leq 1$ for all $|w| \leq r$. Let $u > 0$ be such that $\frac{\|T\|}{u} < r_0 r$. Thus, we have

$$r(\frac{T}{u}) < r_0 r < \liminf_n(|\rho_n|^{\frac{1}{n}}),$$

and for all $z \in \mathbb{D}$ we have

$$\|f_{(\rho_n)}(z\frac{T}{u})\| \leq \sum_{n=1}^{\infty} \frac{2}{|\rho_n|}|z|^n(\frac{\|T\|}{u})^n \leq \sum_{n=1}^{\infty} \frac{2}{\rho_n}|z|^n(r_0 r)^n = |f_{(|\rho_n|)}(r_0|z|r)| \leq 1.$$

We recall that for any $B \in \mathcal{L}(H)$ we have

$$\operatorname{Re}(B) \geq -\|\operatorname{Re}(B)\|I = -\|\frac{B+B^*}{2}\|I \geq -\|B\|I.$$

Thus, for any $z \in \mathbb{D}$, $f_{(\rho_n)}(z\frac{T}{u})$ converges absolutely and we have

$$I + \operatorname{Re}(f_{(\rho_n)}(z\frac{T}{u})) \geq I - \|f_{(\rho_n)}(z\frac{T}{u})\|I \geq 0.$$

This means that $\frac{T}{u} \in C_{(\rho_n)}$ according to Proposition 1.2.6, so $w_{(\rho_n)}(T) \leq u < +\infty$. Furthermore, since $\frac{T}{u} \in C_{(\rho_n)}$ for every u such that $u > \frac{\|T\|}{r_0 r}$, we get $w_{(\rho_n)}(T) \leq \frac{\|T\|}{r_0 r}$. Hence, we have

$$\frac{\|T\|}{|\rho_1|} \leq w_{(\rho_n)}(T) \leq \frac{\|T\|}{r_0 r}.$$

With these inequalities we immediately get

$$w_{(\rho_n)}(T) = 0 \Leftrightarrow T = 0.$$

These inequalities also imply that, for $S, T \in \mathcal{L}(H)$, we have

$$w_{(\rho_n)}(S+T) \leq \frac{\|S+T\|}{r_0 r} \leq \frac{\|S\|+\|T\|}{r_0 r} \leq \frac{|\rho_1|}{r_0 r}(w_{(\rho_n)}(S)+w_{(\rho_n)}(T)).$$

In order to show that $w_{(\rho_n)}(\cdot)$ is a quasi-norm, we still have to show that it is homogeneous, that is $w_{(\rho_n)}(zT) = |z|w_{(\rho_n)}(T)$ for any $z \in \mathbb{C}$. Let $z \in \mathbb{C}$. The cases $z = 0$ and $T = 0$ have been treated, so we now consider $z = e^{it}|z| \neq 0$ and $T \neq 0$. Let $u \geq w_{(\rho_n)}(zT)$ be such that $\frac{zT}{u} \in C_{(\rho_n)}$. Denote $u' = \frac{u}{|z|}$. We can see that $r(\frac{zT}{u}) = r(\frac{T}{u'})$ and that $f_{(\rho_n)}(w\frac{zT}{u}) = f_{(\rho_n)}(e^{it}w\frac{T}{u'})$ for any $w \in \mathbb{D}$. Thus, the series $f_{(\rho_n)}(e^{it}w\frac{T}{u'})$ converges absolutely and $I+\mathrm{Re}(f_{(\rho_n)}(e^{it}w\frac{T}{u'})) \geq 0$, for any $w \in \mathbb{D}$. Hence $\frac{T}{u'} \in C_{(\rho_n)}$, so

$$u' = \frac{u}{|z|} \geq w_{(\rho_n)}(T).$$

Thus, by taking the infimum for $u \geq w_{(\rho_n)}(zT)$, we get

$$w_{(\rho_n)}(zT) \geq |z|w_{(\rho_n)}(T).$$

Applying the same result to $T' = zT$ and $z' = \frac{1}{z}$, we obtain

$$w_{(\rho_n)}(T) = w_{(\rho_n)}(z'T') \geq |z'|w_{(\rho_n)}(T') = \frac{1}{|z|}w_{(\rho_n)}(zT),$$

which proves the desired equality.
We will now prove that the closed unit ball for the (ρ_n)-radius is exactly $C_{(\rho_n)}$. Notice again that $w_{(\rho_n)}(T) = 0$ reduces to $T = 0$. If $T \in C_{(\rho_n)}$, then $w_{(\rho_n)}(T) \leq \frac{1}{1} = 1$. Conversely, suppose that $w_{(\rho_n)}(T) \leq 1$ and let $(u_m)_m$ be a sequence, with $u_m > 0$, converging to $w_{(\rho_n)}(T)$ such that $\frac{T}{u_m} \in C_{(\rho_n)}$. Using the fact that the class $C_{(\rho_n)}$ is closed for the operator norm, as proved in Corollary 1.2.10, we get $\frac{T}{w_{(\rho_n)}(T)} \in C_{(\rho_n)}$. Therefore, we have

$$r(T) \leq r(\frac{T}{w_{(\rho_n)}(T)}) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$$

and

$$I + \mathrm{Re}(f_{(\rho_n)}(zT)) \geq 0, \text{ for every } z \text{ with } |z| \leq \frac{1}{w_{(\rho_n)}(T)}.$$

Since $\frac{1}{w_{(\rho_n)}(T)} \geq 1$, we can conclude that $T \in C_{(\rho_n)}$. The proof is now complete. □

Remark 1.2.13. In the case when $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 0$, we have $w_{(\rho_n)}(T) = +\infty$ unless T is quasi-nilpotent and the sequence of $\|T^n\|^{\frac{1}{n}}$ decreases to 0 fast enough.

Remark 1.2.14. Since the (ρ_n)-radius is homogeneous and

$$w_{(\rho_n)}(T) \leq 1 \Leftrightarrow T \in C_{(\rho_n)},$$

whenever $T \neq 0$, we have

$$\{u > 0\colon \frac{T}{u} \in C_{(\rho_n)}\} = [w_{(\rho_n)}(T), +\infty[.$$

Corollary 1.2.15. — *Let* $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ *with* $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$*. Let* $T \in \mathcal{L}(H)$*. We have*

(i) *For any* $z \neq 0$, $\frac{1}{|z|}w_{(\rho_n)}(T) = w_{(\rho_n)}(\frac{1}{z}T) = w_{(z^n\rho_n)}(T)$*;*

(ii) *If* $T \in C_{(\rho_n)}(H)$*, then* $T^k \in C_{(\rho_{kn})}(H)$*, for all* $k \geq 1$*;*

(iii) $w_{(\rho_{kn})_n}(T^k) \leq w_{(\rho_n)}(T)^k$*, for all* $k \geq 1$*;*

(iv) $w_{(\rho_n)}(T) = w_{(\overline{\rho_n})}(T^*)$*.*

Proof. (i) The left-hand equality is given by the homogeneity of $w_{(\rho_n)}(\cdot)$. For the right-hand one, we can see that

$$(\frac{T}{z})^n = \rho_n P_H U^n|_H \text{ if and only if } T^n = z^n\rho_n P_H U^n|_H.$$

Thus $\frac{T}{z} \in C_{(\rho_n)}$ if and only if $T \in C_{(z^n\rho_n)}$. Lemma 1.2.12 implies that

$$w_{(\rho_n)}(\frac{1}{z}T) = w_{(z^n\rho_n)}(T).$$

- (ii) By definition of the class $C_{(\rho_n)}$, if $T \in C_{(\rho_n)}$, then

$$(T^k)^m = \rho_{km} P_H (U^k)^m|_H,$$

so $T^k \in C_{(\rho_{kn})}(H)$.
- (iii) The result is true when $T = 0$. When $T \neq 0$, consider $T' = \frac{T}{w_{(\rho_n)}(T)}$. By homogeneity of $w_{(\rho_n)}(\cdot)$, we have $w_{(\rho_n)}(T') = 1$, so $T' \in C_{(\rho_n)}$ according to Lemma 1.2.12. Thus, for any $k \geq 1$, $(T')^k \in C_{(\rho_{kn})}(H)$. Using again the homogeneity of the (ρ_n)-radius, we obtain

$$\frac{w_{(\rho_{kn})_n}(T^k)}{w_{(\rho_n)}(T)^k} = w_{(\rho_{kn})}((T')^k) \leq 1.$$

This completes the proof.
- (iv) We use Remark 1.2.2 and Lemma 1.2.12 to obtain the equivalence

$$w_{(\rho_n)}(T) \leq 1 \Leftrightarrow w_{(\overline{\rho_n})}(T^*) \leq 1.$$

Since the (ρ_n)-radii are homogeneous, these quantities must be equal. □

Corollary 1.2.16. — *Let* $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ *with* $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$*. Let* $T \in \mathcal{L}(H)$*. The following assertions are true:*

(i) *Let* F *be an invariant closed subspace of* T*. Then* $w_{(\rho_n)}(T|_F) \leq w_{(\rho_n)}(T)$*;*

(ii) *For any isometry* V *we have* $w_{(\rho_n)}(VTV^*) \leq w_{(\rho_n)}(T)$*, with equality if* V *is unitary;*

(iii) *For a Hilbert space* K *we have* $w_{(\rho_n)}(T \otimes I_K) = w_{(\rho_n)}(T)$*;*

(iv) *For* $T_m \in \mathcal{L}(H_m)$*, with* $\sup_m(\|T_m\|) < +\infty$*, we have*

$$w_{(\rho_n)}(\oplus_{m\geq 1} T_m) = \sup_{m\geq 1}(w_{(\rho_n)}(T_m));$$

(v) If $T^{(\infty)}$ denotes the countable orthogonal sum $T \oplus T \oplus \cdots$, then $w_{(\rho_n)}(T^{(\infty)}) = w_{(\rho_n)}(T)$.

Proof. - (i) We have $r(T|_F) \leq r(T)$. If $I + \mathrm{Re}(f_{(\rho_n)}(zT))$ is positive, then $I + \mathrm{Re}(f_{(\rho_n)}(zT|_F))$ is positive too. Thus, by using Lemma 1.2.12 we obtain

$$w_{(\rho_n)}(T) \leq 1 \Rightarrow w_{(\rho_n)}(T|_F) \leq 1.$$

The homogeneity of the (ρ_n)-radius gives the result.
- (ii) We have $r(VTV^*) \leq r(T)$ and $(VTV^*)^n = VT^nV^*$. Thus, $f_{(\rho_n)}(zVTV^*) = Vf_{(\rho_n)}(zT)V^*$. Hence, for any $h \in H$ and any $z \in \mathbb{D}$, we have

$$\mathrm{Re}(\langle (I + f_{(\rho_n)}(zVTV^*))h, h\rangle) = \mathrm{Re}(\langle (I + f_{(\rho_n)}(zT))V^*h, V^*h\rangle).$$

By applying Theorem 1.2.8 and Lemma 1.2.12, we get

$$w_{(\rho_n)}(T) \leq 1 \Rightarrow w_{(\rho_n)}(VTV^*) \leq 1.$$

The homogeneity of the (ρ_n)-radii gives the desired inequality. When the isometry V is also invertible, the converse inequality is true, so both quantities are equal.
- (iii) Since $\|T^n\| = \|(T \otimes I_K)^n\|$, we have $r(T) = r(T \otimes I_K)$. Let $u > 0$ be such that $u \geq \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}$. Thus the series $f_{(\rho_n)}(z\frac{T\otimes I_K}{u})$ is absolutely convergent for all $z \in \mathbb{D}$, and $f_{(\rho_n)}(z\frac{T\otimes I_K}{u}) = f_{(\rho_n)}(z\frac{T}{u}) \otimes I_K$. Since for any $h_1 \otimes k_1, h_2 \otimes k_2 \in H \otimes K$ we have

$$\langle h_1 \otimes k_1, h_2 \otimes k_2\rangle = \langle h_1, h_2\rangle\langle k_1, k_2\rangle,$$

we can see that the condition

$$\langle (I + \mathrm{Re}(f_{(\rho_n)}(z\frac{T \otimes I_K}{u})))(h \otimes k), h \otimes k\rangle \geq 0, \, \forall h \otimes k \in H \otimes K,$$

is equivalent to

$$\langle (I + \mathrm{Re}(f_{(\rho_n)}(z\frac{T}{u})))(h), h\rangle \geq 0, \, \forall h \in H.$$

Hence, $\frac{T\otimes I_K}{u} \in C_{(\rho_n)}(H \otimes K)$ is equivalent to $\frac{T}{u} \in C_{(\rho_n)}(H)$, which implies that $w_{(\rho_n)}(T) = w_{(\rho_n)}(T \otimes I_K)$.
- (iv) Since $\sup_m(\|T_m\|) < +\infty$, the linear map $T = \oplus_{m\geq 1}T_m$ is bounded on the Hilbert space $H = \oplus_{m\geq 1}H_m$, and $\|T\| = \sup_m(\|T_m\|)$. Thus, $r(T) = \sup_m(r(T_m))$. Let $u > 0$ be such that $u \geq \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}$. We have

$$r(\frac{T_m}{u}) \leq r(\frac{T}{u}) \leq \liminf_n(|\rho_n|^{\frac{1}{n}}).$$

Thus, the series $f_{(\rho_n)}(z\frac{T}{u})$ and $f_{(\rho_n)}(z\frac{T_m}{u})$ are absolutely convergent for all $z \in \mathbb{D}$, and

$$f_{(\rho_n)}(z\frac{T}{u}) = \oplus_{m\geq 1}f_{(\rho_n)}(z\frac{T_m}{u}).$$

Since for any $h = (h_m)_m \in H$, we have

$$[I + \mathrm{Re}(f_{(\rho_n)}(z\frac{T}{u}))](h) = ((I + \mathrm{Re}(f_{(\rho_n)}(z\frac{T_m}{u})))(h_m))_m,$$

this implies that

$$\langle (I + \mathrm{Re}(f_{(\rho_n)}(z\frac{T}{u})))(h), h\rangle \geq 0, \forall h \in H,$$

is equivalent to

$$\langle (I + \mathrm{Re}(f_{(\rho_n)}(z\frac{T_m}{u})))(h_m), h_m\rangle \geq 0, \forall h_m \in H_m, \forall m \geq 1.$$

Hence, the assertion $\frac{T}{u} \in C_{(\rho_n)}(H)$ is equivalent to $\frac{T_m}{u} \in C_{(\rho_n)}(H_m)$, $\forall n \geq 1$, which implies that $w_{(\rho_n)}(T) = \sup_m(w_{(\rho_n)}(T_m))$.
- (v) The proof is a consequence of item (iii) and [BC02, Remark 1.1]. □

The items (i) and (ii) of this Corollary show that the classes $C_{(\rho_n)}$ are unitarily invariant, and stable under the restriction to an invariant closed subspace. The item (iv) is a generalization of a known property of direct sums of operators in the class $C_{(\rho)}$. Items $(i), (ii)$ and (v) show that, under the condition of Corollary 1.2.16, the radius $w_{(\rho_n)}$ is an *admissible radius* in the terminology of [BC02, Definition 1.1]. Thus, all the results proved in [BC02] for admissible radii are valid for $w_{(\rho_n)}$ when $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. In particular, the following result is true.

Corollary 1.2.17. — *Let $T \in \mathcal{L}(H)$, with $\|T\| \leq 1$ and $T^n = 0$ for some $n \geq 2$. Then, for each polynomial $P \in \mathbb{C}[X]$, we have*

$$w_{(\rho_n)}(P(T)) \leq w_{(\rho_n)}(P(S_n)).$$

Here S_n is the nilpotent Jordan cell on $\mathbb{C}^n$

$$S_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

Some other consequences of the condition $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$ are proved in the next Proposition.

Proposition 1.2.18. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. The following assertions are true:*

(i) We have

$$w_{(\rho_n)}(I) = \min(\{r \geq \liminf_n(|\rho_n|^{\frac{1}{n}})^{-1} \colon f_{(\rho_n)}(\mathbb{D}(0, \frac{1}{r})) \subset Re_{\geq -1}\});$$

(ii) For any $T \in \mathcal{L}(H)$, we have $w_{(\rho_n)}(T) \geq r(T)w_{(\rho_n)}(I)$;

(iii) If T is normal, then $w_{(\rho_n)}(T) = \|T\|w_{(\rho_n)}(I)$.

Proof. (i) Take $u = w_{(\rho_n)}(I)$ such that $\frac{I}{u} \in C_{(\rho_n)}$. We have $r(\frac{I}{u}) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$, so $\frac{1}{u}$ is no greater than the convergence radius of $f_{(\rho_n)}$. For any $z \in \mathbb{D}$, we have $f_{(\rho_n)}(z\frac{I}{u}) = f_{(\rho_n)}(\frac{z}{u})I$. Thus, $I + \mathrm{Re}(f_{(\rho_n)}(z\frac{I}{u})) \geq 0$ for any $z \in \mathbb{D}$ if and only if $f_{(\rho_n)}(\mathbb{D}(0, \frac{1}{u})) \in Re_{\geq -1}$.

- (ii) Let $T \in \mathcal{L}(H)$. There is nothing to prove if $T = 0$ or $r(T) = 0$. Otherwise, let $u = w_{(\rho_n)}(T)$ be such that $\frac{T}{u} \in C_{(\rho_n)}$ (cf. Lemma 1.2.12). Since $I + \text{Re}(f_{(\rho_n)}(z\frac{T}{u})) \geq 0$, the spectrum of $I + f_{(\rho_n)}(z\frac{T}{u})$ lies in $Re_{\geq 0}$. This spectrum is the set $\{1 + f_{(\rho_n)}(zw),\ w \in \sigma(\frac{T}{u})\}$. The union of these spectra, when z describes $\mathbb{D}$, is $\{1 + f_{(\rho_n)}(w),\ |w| < \frac{r(T)}{u}\}$. Since $\frac{r(T)}{u} > 0$, we obtain from item (i) that $\frac{u}{r(T)} \geq w_{(\rho_n)}(I)$. Hence $w_{(\rho_n)}(T) \geq r(T)w_{(\rho_n)}(I)$.
- (iii) Let T be a normal operator with $T \neq 0$. For $u = \|T\|.w_{(\rho_n)}(I)$, we have

$$r(\frac{T}{u}) = \frac{\|T\|}{u} = \frac{1}{w_{(\rho_n)}(I)} \leq \liminf_n (|\rho_n|^{\frac{1}{n}}).$$

Thus, we obtain that

$$\bigcup_{z \in \mathbb{D}} \sigma(I + f_{(\rho_n)}(z\frac{T}{u})) = \{1 + f_{(\rho_n)}(w),\ |w| < \frac{1}{w_{(\rho_n)}(I)}\}.$$

Item (i) of this Proposition tells us that this set is included in $Re_{\geq 0}$. As T is normal, $I + f_{(\rho_n)}(z\frac{T}{u})$ is also normal, so

$$W(I + f_{(\rho_n)}(z\frac{T}{u})) \subset Hull(\sigma(I + f_{(\rho_n)}(z\frac{T}{u}))) \subset Re_{\geq 0},\ \forall z \in \mathbb{D}.$$

Hence, $I + \text{Re}(f_{(\rho_n)}(z\frac{T}{u})) \geq 0$, and $\frac{T}{u} \in C_{(\rho_n)}$. By Lemma 1.2.12, we then have

$$w_{(\rho_n)}(T) \leq u = \|T\| w_{(\rho_n)}(I).$$

The inequality of item (ii) provides the desired equality. □

Remark 1.2.19. Since we also have

$$w_{(\rho_n)}(I) \geq \frac{1}{\liminf_n (|\rho_n|^{\frac{1}{n}})},$$

the inequality in Proposition 1.2.18 is better than the last one of Lemma 1.2.12. Thus, if there is T such that $w_{(\rho_n)}(T) = \frac{r(T)}{\liminf_n (|\rho_n|^{\frac{1}{n}})}$, the same must be true for the identity operator I. In the case when $\rho_n = \rho$, $\rho > 0$, this can only happen when $\rho \geq 1$.

1.3 Classes $C_{(\rho)}$ for $\rho \neq 0$

In this section, we will focus on the case where $\rho_n = \rho$, for some $\rho \in \mathbb{C}^*$. This is an intermediate class between the classical case considered by Sz.-Nagy and Foias (classes C_τ for $\tau > 0$) and the general $C_{(\rho_n)}$-classes. Thus the obtained results are already known when $\rho > 0$, but the generalization to the case $\rho \in \mathbb{C}^*$ seems to be new. Nevertheless, we acknowledge the influence of [SNBFK10, AL10, AN73, AO76] for the results of this section. The results obtained here will turn out to be useful when we will look again at $C_{(\rho_n)}$-classes in the next section.

1.3.A Some characterizations.

Lemma 1.3.1. — *Let $\rho \neq 0$ and $\rho_n = \rho$, $\forall n \geq 1$. Let $T \in \mathcal{L}(H)$.*
The following are equivalent:

(i) $T \in C_{(\rho)}(H)$;

(ii) $r(T) \leq 1$ *and* $Re((1 - \frac{2}{\rho})I + \frac{2}{\rho}(I - zT)^{-1}) \geq 0$, $\forall z \in \mathbb{D}$;

(iii) $r(T) \leq 1$ *and* $Re(\frac{2}{\rho}(I - zT)) + Re(1 - \frac{2}{\rho})(I - zT)^*(I - zT) \geq 0$, $\forall z \in \mathbb{D}$;

(iv) $Re(\frac{2}{\rho}(I - zT)) + Re(1 - \frac{2}{\rho})(I - zT)^*(I - zT) \geq 0$, $\forall z \in \mathbb{D}$.

Proof. $(i) \Leftrightarrow (ii)$ We have $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 1$. When $r(T) \leq 1$, for $z \in \mathbb{D}$, we have

$$I + \sum_{n \geq 1} \frac{2}{\rho}(zT)^n = (1 - \frac{2}{\rho})I + \frac{2}{\rho}(I - zT)^{-1}.$$

Apply now Proposition 1.2.6.
- $(ii) \Leftrightarrow (iii)$ We will use several times the known fact that for $A, B \in \mathcal{L}(H)$, with A invertible,

$$\mathrm{Re}(B) \geq 0 \Leftrightarrow \mathrm{Re}(A^*BA) \geq 0.$$

We obtain the equivalence $(ii) \Leftrightarrow (iii)$ by choosing

$$A = (I - zT),\ B = (1 - \frac{2}{\rho})I + \frac{2}{\rho}(I - zT)^{-1}$$

and by rearranging the expression, using that $(I - zT)^*(I - zT)$ is a positive self-adjoint operator and $\mathrm{Re}(A^*) = \mathrm{Re}(A)$.
- $(iii) \Rightarrow (iv)$ is immediate.
- $(iv) \Rightarrow (iii)$ Suppose that $r(T) > 1$. Thus, there exists $\gamma \in \mathbb{C}$ such that $|\gamma| = r(T) > 1$, and there is a sequence (h_n) of vectors $h_n \in H$ such that $\|h_n\| = 1$ and $\|(T - \gamma I)h_n\| \to 0$ as $n \to \infty$. Let $0 < \epsilon < |\gamma| - 1$ and set $g_n := (T - \gamma I)h_n$. Let also $\eta = \epsilon e^{it}$, for some t that will be chosen later on. Let $z := \frac{1+\eta}{\gamma}$. Then, $|z| < \frac{1+(|\gamma|-1)}{|\gamma|} = 1$. Furthermore, we have

$$(I - zT)h_n = (I - \frac{1}{\gamma}T)h_n - \frac{\eta}{\gamma}Th_n + \eta h_n - \eta h_n = -zg_n - \eta h_n.$$

Thus, we obtain

$$\begin{aligned}
&\mathrm{Re}(\langle[\frac{2}{\rho}(I - zT) + (1 - \frac{2}{\rho})(I - zT)^*(I - zT)]h_n, h_n\rangle) \geq 0 \\
\Rightarrow &\mathrm{Re}(\frac{2}{\rho}[-\eta.\|h_n\|^2 - \langle zg_n, h_n\rangle] + (1 - \frac{2}{\rho})\|(I - zT)h_n\|^2) \geq 0 \\
\Rightarrow &\mathrm{Re}(\frac{2}{\rho}[-\eta - \langle zg_n, h_n\rangle] + (1 - \frac{2}{\rho})[|\eta|^2 + 2\mathrm{Re}(\langle zg_n, h_n\rangle) + |z|^2\|g_n\|^2]) \geq 0
\end{aligned}$$

Hence, by taking the limit as $n \to +\infty$, we obtain

$$\mathrm{Re}(\frac{2}{\rho}(-\eta) + (1 - \frac{2}{\rho})|\eta|^2) = \mathrm{Re}(\frac{-2}{\rho}e^{it})\epsilon + \mathrm{Re}(1 - \frac{2}{\rho})\epsilon^2 \geq 0.$$

We can then choose $t \in \mathbb{R}$ depending on $arg(\rho)$ and $sgn(\text{Re}(1-\frac{2}{\rho}))$ to obtain

$$\text{either } \frac{-2}{|\rho|}\epsilon + |\text{Re}(1-\frac{2}{\rho})|\epsilon^2 \geq 0 \text{ or } \frac{2}{|\rho|}\epsilon - |\text{Re}(1-\frac{2}{\rho})|\epsilon^2 \geq 0.$$

But since $\frac{-2}{|\rho|} < 0$, there is some $\epsilon > 0$ such that $\frac{-2}{|\rho|} + |\text{Re}(1-\frac{2}{\rho})|\epsilon$ is strictly negative, which is impossible. This contradiction shows that $r(T) \leq 1$, which concludes the proof. □

Lemma 1.3.2. — *Let $\rho \neq 0$ and $\alpha > 0$ be two scalars. Let $T \in \mathcal{L}(H)$.*
The following assertions are equivalent:

(i) $w_{(\rho)}(T) \leq \alpha$;

(ii) $r(T) \leq \alpha$, $((\rho-1)zT - \rho\alpha I)$ *is invertible and* $\|(zT)((\rho-1)zT - \rho\alpha I)^{-1}\| \leq 1$, $\forall z \in \mathbb{D}$;

(iii) $r(T) \leq \alpha$, $((\rho-1)T - \rho w I)$ *is invertible and* $\|T((\rho-1)T - \rho w I)^{-1}\| \leq 1$, $\forall |w| > \alpha$.

Proof. $(i) \Rightarrow (ii)$ When replacing T with $\frac{T}{\alpha}$, all expressions in (i) and (ii) are reduced to the case $\alpha = 1$. Now, as $w_{(\rho_n)}(T) \leq \alpha = 1$, we use Lemma 1.3.1 to have $r(T) \leq 1$ and

$$\text{Re}((1-\frac{2}{\rho})I + \frac{2}{\rho}(I - zT)^{-1}) \geq 0, \ \forall z \in \mathbb{D}.$$

We denote $C_z := (1-\frac{2}{\rho})I + \frac{2}{\rho}(I - zT)^{-1}$, for $z \in \mathbb{D}$. We recall that since $\text{Re}(C_z) \geq 0$, we have $(C_z + I)$ invertible and

$$\|(C_z - I)(C_z + I)^{-1}\| \leq 1.$$

A computation gives

$$C_z - I = \frac{2}{\rho} zT(I - zT)^{-1} \text{ and } C_z + I = [2I + (\frac{2}{\rho} - 2)zT](I - zT)^{-1}.$$

Thus,

$$(C_z - I)(C_z + I)^{-1} = \frac{1}{\rho} zT[I + (\frac{1}{\rho} - 1)zT]^{-1} = zT[\rho I + (1-\rho)zT]^{-1} = -zT[-\rho I + (\rho - 1)zT]^{-1}.$$

This means that all the conditions of (ii) are fulfilled.
- $(ii) \Rightarrow (i)$ We again reduce to the case $\alpha = 1$. We denote $D_z = zT[\rho I - (\rho-1)zT]^{-1}$, for $z \in \mathbb{D}$. Since $\|D_z\| \leq 1$, we have $D_z \in C_{(1)}$ so $r(D_z) \leq 1$ and $\text{Re}((I + wD_z)(I - wD_z)^{-1}) \geq 0$, for all $w \in \mathbb{D}$. We obtain:

$$I + wD_z = [\rho I + (w + 1 - \rho)zT][\rho I - (\rho - 1)zT]^{-1}$$

and

$$I - wD_z = [\rho I + (-w + 1 - \rho)zT][\rho I - (\rho - 1)zT]^{-1}.$$

Thus,

$$(I + wD_z)(I - wD_z)^{-1} = [\rho I + (w + 1 - \rho)zT][\rho I + (-w + 1 - \rho)zT]^{-1}.$$

Since $r(T) \leq 1$, $(I - zT)$ is invertible so $[\rho I + (-w + 1 - \rho)zT]^{-1}$ converges to $\frac{1}{\rho}(I - zT)^{-1}$ when w tends to 1, by continuity of the inverse map. Thus,

$$\lim_{w \to 1,\, w \in \mathbb{D}} (I + wD_z)(I - wD_z)^{-1} = \frac{1}{\rho}(\rho I + (2-\rho)zT)(I - zT)^{-1} = C_z.$$

Hence, $\text{Re}(C_z) \geq 0$ for all $z \in \mathbb{D}$ and $r(T) \leq 1$, so $T \in C_{(\rho)}$.
- $(ii) \Leftrightarrow (iii)$ For $z \neq 0$, we take $w = \frac{\alpha}{z}$ to obtain the result. The converse gives the result for all $z \in \mathbb{D}$, $z \neq 0$, which extends to $\mathbb{D}$ by continuity. □

1.3.B Reducing to the case $\rho > 0$.

With this characterization of $C_{(\rho)}$ classes, we are now able to obtain the main relationship between (ρ)-radii and (τ)-radii, $\rho \in \mathbb{C}^*$, $\tau > 0$. This relationship extends the "symmetric" relationship

$$\tau w_{(\tau)}(T) = (2-\tau) w_{(\tau)}(T),\ 0 < \tau < 2,$$

that was already known (see [AN73, Thm.3]).

Proposition 1.3.3. — *Let $\rho \neq 0$ and $\alpha > 0$ be two scalars. Let $T \in \mathcal{L}(H)$. The following assertions are equivalent:*

(i) $w_{(\rho)}(T) \leq \alpha$;

(ii) $((\rho-1)zT - \rho\alpha I)$ *is invertible and* $\|(zT)((\rho-1)zT - \rho\alpha I)^{-1}\| \leq 1$, $\forall z \in \mathbb{D}$;

(iii) $((\rho-1)T - \rho w I)$ *is invertible and* $\|T((\rho-1)T - \rho w I)^{-1}\| \leq 1$, $\forall |w| > \alpha$;

(iv) $\|T(h)\| \leq \|(\rho-1)T(h) - \rho w h\|$, $\forall h \in H$, $\forall |w| > \alpha$.

Furthermore, we have:

$$|\rho| w_{(\rho)}(T) = (1 + |\rho-1|) w_{1+|\rho-1|}(T). \tag{1.3.1}$$

Hence, the map $\rho \in \mathbb{C}^* \mapsto |\rho| w_{(\rho)}(T)$ is constant on circles of center 1, is continuous on $\mathbb{C}^*$ and can be extended continuously to $2w_{(2)}(T)$ at 0.

Proof. Using the results of Lemma 1.3.2, we can see that conditions (ii) and (iii) are equivalent and that condition (i) implies condition (ii). We can also see that the implication $(iii) \Rightarrow (iv)$ is immediate.

- $(iv) \Rightarrow (iii)$ Let $w \in \mathbb{C}$ with $|w| > \alpha$. We need to show that $((\rho-1)T - \rho w I)$ is invertible. If $\rho = 1$ this operator is equal to $-\rho w I$ which is invertible. Suppose that $\rho \neq 1$. Let $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$. We then have $h_n \in H$ such that $\|h_n\| = 1$ and $T(h_n) - \lambda h_n \to_{n\to+\infty} 0$. Then,

$$\begin{aligned} 0 &\leq \big|\|(\rho-1)T(h_n) - \rho w h_n\| - |-\rho w + (\rho-1)\lambda|\,\big| \\ &= \big|\|(\rho-1)T(h_n) - \rho w h_n\| - \|-\rho w h_n + (\rho-1)\lambda h_n\|\big| \\ &\leq \|(\rho-1)T(h_n) - (\rho-1)\lambda h_n\| \to_{n\to+\infty} 0. \end{aligned}$$

Condition (iv) gives us $\|T(h_n)\| \leq \|(\rho-1)T(h_n) - \rho w h_n\|$. As the left-hand term converges to $|\lambda|$ and as the right-hand term converges to $|-\rho w + (\rho-1)\lambda|$, we obtain

$$r(T) = |\lambda| \leq |(\rho-1)\lambda - \rho w|,\ \forall w \in \mathbb{C},\ |w| > \alpha.$$

Thus, we cannot have $r(T) \geq \frac{|\rho|\alpha}{|\rho-1|} > 0$ as the previous inequality would imply

$$r(T) = |\lambda| \leq \inf\{|(\rho-1)\lambda - \rho w|,\ |w| > \alpha\} = 0,$$

which contradicts the fact that $r(T) > 0$. Hence, we have $r(T) < \frac{|\rho|\alpha}{|\rho-1|}$, so $((\rho-1)T - \rho w I)$ is invertible for every w with $|w| > \alpha$.

- $(ii) \Rightarrow (i)$ We only need to show that item (ii) implies $r(T) \leq \alpha$. We can reduce the proof to the case $\alpha = 1$ by considering $\frac{T}{\alpha}$ instead of T. We also recall that if $\rho > 0$, the result is valid (see [Sta82, Thm.1] or [Dav70] for a proof). Let $\rho \neq 0$. We denote $S = \frac{1+|\rho-1|}{|\rho|}T$. Suppose that $[(\rho-1)zT - \rho I]^{-1}$ exists and that $\|zT[(\rho-1)zT-\rho I]^{-1}\| \leq 1$, for all $z \in \mathbb{D}$. With $\rho - 1 = |\rho-1|e^{it}$, $\rho = |\rho|e^{is}$ and $w = z.e^{-is+it}$ we then have

$$\begin{aligned}
&\|zT[(\rho-1)zT-\rho I]^{-1}\| \leq 1 \\
\Leftrightarrow &\|zT[|\rho-1|e^{it}zT - |\rho|e^{is}I]^{-1}\| \leq 1 \\
\Leftrightarrow &\|ze^{-is}e^{it}e^{-it}T[|\rho-1|ze^{-is}e^{it}T - |\rho|I]^{-1}\| \leq 1. \\
\Leftrightarrow &|e^{-it}|\|wT[|\rho-1|wT - |\rho|I]^{-1}\| \leq 1 \\
\Leftrightarrow &\|wT[(1+|\rho-1|-1)wT - |\rho|I]^{-1}\| \leq 1 \\
\Leftrightarrow &\|w\frac{1+|\rho-1|}{|\rho|}T[(1+|\rho-1|-1)w\frac{1+|\rho-1|}{|\rho|}T - (1+|\rho-1|)I]^{-1}\| \leq 1 \\
\Leftrightarrow &\|wS[(1+|\rho-1|-1)wS - (1+|\rho-1|)I]^{-1}\| \leq 1.
\end{aligned}$$

Since w describes $\mathbb{D}$ when z does, this is true for all $w \in \mathbb{D}$. Therefore $w_{(1+|\rho-1|)}(S) \leq 1$ as $1+|\rho-1| > 0$ (see the beginning of the proof and Lemma 1.3.2). Thus, $r(S) \leq 1$, which implies $r(T) \leq \frac{|\rho|}{1+|\rho-1|} \leq 1$.

Now that we have showed that the condition about the spectral radius of T is not necessary, we can see that the equivalences in the previous computations give

$$w_{(\rho)}(T) \leq 1 \;\Leftrightarrow\; w_{(1+|\rho-1|)}\left(\frac{1+|\rho-1|}{|\rho|}T\right) \leq 1.$$

By homogeneity of the (ρ_n)-radii, this is equivalent to

$$|\rho|w_{(\rho)}(T) = (1+|\rho-1|)w_{(1+|\rho-1|)}(T).$$

- The properties of the map $\rho \in \mathbb{C}^* \mapsto |\rho|w_{(\rho)}(T)$ can now be obtained from its restriction to $[1, +\infty[$, which is known to be continuous (see [AN73, Cor.2] for example). ☐

Equation (1.3.1) gives a simple geometric understanding of a formula that was previously known only for real numbers ρ between 0 and 2. It also implies the following relationship between C_ρ classes.

Corollary 1.3.4. — *We have*

$$C_{(\rho)} = \frac{1+|\rho-1|}{|\rho|}C_{(1+|\rho-1|)}.$$

We conclude that complex (ρ)-radii of an operator T can be expressed in terms of the real positive ones.

Corollary 1.3.5. — *Let $\rho \neq 0$ and let $T \in \mathcal{L}(H)$. We have:*

(i) $w_{(\rho)}(I) = \frac{1+|\rho-1|}{|\rho|}$, $\forall \rho \neq 0$;

(ii) If T is normal, then $w_{(\rho)}(T) = \|T\|\frac{1+|\rho-1|}{|\rho|}$;

(iii) If $T^2 = 0$, then $w_{(\rho_n)}(T) = w_{(\rho_1)}(T) = \frac{2w(T)}{|\rho_1|} = \frac{\|T\|}{|\rho_1|}$;

(iv) *If* $T^2 = bI$, $b \in \mathbb{C}$, *then* $|\rho| w_{(\rho)}(T) = w(T) + \sqrt{w_2(T)^2 + |b|(|\rho - 1|^2 - 1)}$;

(v) *If* $T^2 = aT$, $a \in \mathbb{C}$, *then* $|\rho| w_\rho(T) = 2w(T) + |a|(|\rho - 1| - 1) = \|T\| + |a||\rho - 1|$.

Proof. - (i) It is known that $w_{(\rho)}(I) = 1$ when $1 \leq \rho$. The relationship of Proposition 1.3.3 gives the result.
- (ii) When T is normal, we have $w_{(\rho_n)}(T) = \|T\| w_{(\rho_n)}(I)$.
- (iii) If $T^2 = 0$, then $T \in C_{(\rho_n)}$ if and only if $I + \mathrm{Re}(\frac{2}{\rho_1} zT) \geq 0$ for all $z \in \mathbb{D}$. By Corollary 1.2.7, this is equivalent to $\frac{T}{|\rho_1|} \in C_{(1)}$, to $\frac{2T}{|\rho_1|} \in C_{(2)}$ and to $T \in C_{(\rho_1)}$. Thus, Lemma 1.2.12 and the following facts

$$w_{(2)}(T) = w(T) \text{ and } w_{(1)}(T) = \|T\|$$

imply that

$$w_{(\rho_n)}(T) = w_{(\rho_1)}(T) = \frac{2w(T)}{|\rho_1|} = \frac{\|T\|}{|\rho_1|}.$$

- (iv), (v) We can reduce these cases to $T^2 = I$ (respectively $T^2 = T$) by taking δ to be a square root of b (respectively a) and looking at $\frac{T}{\delta}$ (respectively $\frac{T}{\delta^2}$). Then, [AN73, Theorem 6] gives the result when $\rho > 0$, and we extend it to $\rho \in \mathbb{C}^*$ by using Proposition 1.3.3. □

1.3.C Computations and some applications.

For the next auxiliary result we need some notation. For an operator T acting on H and for $h \in H$, define

$$V_h := \overline{\mathrm{Span}(T^n(h),\, n \geq 0)} \text{ and } T_h := T|_{V_h} \in \mathcal{L}(V_h).$$

We recall here that for a family S of vectors, $\mathrm{Span}(S)$ is the smallest vector space containing S, so it may not be closed.

Lemma 1.3.6. — *Let* $T \in \mathcal{L}(H)$. *Let* $\rho \neq 0$. *Then, with the previous notation, we have*

$$w_{(\rho)}(T) = \sup_{h \in H}(w_{(\rho)}(T_h)).$$

If we also have $P(T) = 0$ *for some* $P \in C[X]$ *with* $deg(P) = n$, *then* T_h *can be identified as some matrix* $S \in M_n(\mathbb{C})$ *such that* $P(S) = 0$, *and the computation of* $w_{(\rho)}(T_h)$ *can be obtained from the computation of* $w_{(\rho)}(S)$.

Proof.
Let $h \in H$. We already proved in Corollary 1.2.16 that $w_{(\rho)}(T_h) \leq w_{(\rho)}(T)$. Conversely, for $\frac{1}{u} = \sup_{h \in H}(w_{(\rho)}(T_h))$, $(I - z\frac{T}{u})$ is invertible as $(I - z\frac{T_h}{u})$ is invertible for all $h \in H$ and we have $\mathrm{Re}(\langle (I + f_{(\rho)}(z\frac{T}{u}))g, g\rangle) \geq 0$ for all $g \in H$. Thus $\frac{T}{u} \in C_{(\rho)}$, which implies $\sup_{h \in H}(w_{(\rho)}(T_h)) \geq w_{(\rho)}(T)$ and concludes the proof. □

Lemma 1.3.7. *Let* $\rho \geq 1$. *For* $a, b, c \in \mathbb{C}$, *denote* $T = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$. *Then,*

(i) $w_\rho(T)$ *is the largest* $r > 0$ *such that there exists* $t \in [0, 2\pi[$ *for which we have*

$$|c|^2 r^2 = [(\rho - 2)|a|^2 - 2(\rho - 1)r Re(ae^{it}) + \rho r^2][(\rho - 2)|b|^2 - 2(\rho - 1)r Re(be^{it}) + \rho r^2].$$

(ii) for $\rho = 2$, we have

$$w_2(T) = \max_{t\in[0,2\pi[} \frac{Re((a+b)e^{it}) + \sqrt{Re((a-b)e^{it})^2 + |c|^2}}{2}.$$

Proof. (i) Let $\rho \geq 1$. We can see that the map $\|w \mapsto T[(\rho-1)T - \rho wI]^{-1}\|$ satisfies the maximum principle as a sub-harmonic map on $\{w\colon\ |w| > \frac{r(T)(\rho-1)}{\rho}\}$. Hence, using equivalence $(i) \Leftrightarrow (iii)$ of Proposition 1.3.3 for $\alpha = w_\rho(T)$ tells us that $w_\rho(T)$ is the modulus of the largest in modulus z such that $\|T[(\rho-1)T - \rho zI]^{-1}\| = 1$, with $|z| > \frac{r(T)(\rho-1)}{\rho}$.
Let $z \in \mathbb{C}$ such that $|z| > \frac{r(T)(\rho-1)}{\rho} \geq r(T)(\rho-1)$. Then $(\rho-1)T - zI$ is invertible. By denoting $S_z = [(\rho-1)T - \rho zI]^{-1}$, we have

$$S_z = \begin{pmatrix} \frac{1}{(\rho-1)a-\rho z} & \frac{-c(\rho-1)}{((\rho-1)a-\rho z)((\rho-1)b-\rho z)} \\ 0 & \frac{1}{(\rho-1)b-\rho z} \end{pmatrix}.$$

Since the matrix $(TS_z)(TS_z)^*$ is self-adjoint, its norm is one of its eigenvalues and we have $\|(TS_z)(TS_z)^*\| = \|TS_z\|^2$. Hence, we can see that

$$\|TS_z\| = 1 \;\Leftrightarrow\; \|(TS_z)(TS_z)^*\| = 1 \;\Leftrightarrow\; Tr((TS_z)(TS_z)^*) = 1 + |det(TS_z)|^2.$$

Writing the coefficients in the right-hand side equation, and multiplying each side by

$$\frac{1}{|det(S_z)|^2} = |((\rho-1)a - \rho z)((\rho-1)b - \rho z)|^2$$

gives the following equation:

$$|a|^2|(\rho-1)b - \rho z|^2 + |b|^2|(\rho-1)a - \rho z|^2 + |c|^2\rho^2|z|^2 = |((\rho-1)a - \rho z)((\rho-1)b - \rho z)|^2 + |ab|^2.$$

Thus $w_\rho(T)$ is the modulus of the largest in modulus z that satisfies this equation.
Write $z = re^{-it}$, with $r > 0$. We then have

$$\begin{aligned}
&|a|^2|(\rho-1)b - \rho z|^2 + |b|^2|(\rho-1)a - \rho z|^2 + |c|^2\rho^2|z|^2 = |((\rho-1)a - z)(\rho-1)b - z)|^2 + |ab|^2\\
\Leftrightarrow &|c|^2\rho^2r^2 = (|(\rho-1)b - \rho z|^2 - |b|^2)(|(\rho-1)a - \rho z|^2 - |a|^2)\\
\Leftrightarrow &|c|^2\rho^2r^2 = [|b|^2\rho(\rho-2) - 2\rho(\rho-1)r\mathrm{Re}(be^{it}) + \rho^2r^2][|a|^2\rho(\rho-2) - 2\rho(\rho-1)r\mathrm{Re}(ae^{it}) + \rho^2r^2]\\
\Leftrightarrow &|c|^2r^2 = [|b|^2(\rho-2) - 2(\rho-1)r\mathrm{Re}(be^{it}) + \rho r^2][|a|^2(\rho-2) - 2(\rho-1)r\mathrm{Re}(ae^{it}) + \rho r^2],
\end{aligned}$$

which gives the desired equation.
- (ii) Let $\rho = 2$, $r > 0$ and $t \in [0, 2\pi[$. We have

$$\begin{aligned}
&|c|^2r^2 = [|b|^2(\rho-2) - 2(\rho-1)r\mathrm{Re}(be^{it}) + \rho r^2][|a|^2(\rho-2) - 2(\rho-1)r\mathrm{Re}(ae^{it}) + \rho r^2]\\
\Leftrightarrow &|c|^2r^2 = (-2r\mathrm{Re}(be^{it}) + 2r^2)(-2r\mathrm{Re}(ae^{it}) + 2r^2)\\
\Leftrightarrow &|c|^2 = 4(-\mathrm{Re}(be^{it}) + r)(-\mathrm{Re}(ae^{it}) + r)\\
\Leftrightarrow &0 = 4r^2 - 4r\mathrm{Re}((a+b)e^{it}) + 4\mathrm{Re}(ae^{it})\mathrm{Re}(be^{it}) - |c|^2.
\end{aligned}$$

The discriminant of this polynomial in r is equal to

$$\begin{aligned}
\Delta &= 16(\mathrm{Re}(ae^{it}) + \mathrm{Re}(be^{it}))^2 + 4|c|^2 - 16\mathrm{Re}(ae^{it})\mathrm{Re}(be^{it})\\
&= 16(\mathrm{Re}((a-b)e^{it})^2 + |c|^2).
\end{aligned}$$

As this discriminant is always positive, the largest root of this polynomial is equal to

$$\begin{aligned} r_t &= \frac{4\mathrm{Re}((a+b)e^{it}) + \sqrt{16(\mathrm{Re}((a-b)e^{it})^2 + |c|^2)}}{8} \\ &= \frac{\mathrm{Re}((a+b)e^{it}) + \sqrt{\mathrm{Re}((a-b)e^{it})^2 + |c|^2}}{2}. \end{aligned}$$

Using item (i) tells us that $w_2(T)$ is then equal to the largest possible r_t, that is $w_2(T) = \sup_{t\in[0,2\pi[}(r_t)$, which concludes the proof. □

Remark 1.3.8. — The idea at the beginning of the proof of Lemma 1.3.7 comes from [AN73], who used it to compute $w_\rho(T)$ when $a = 0$ and when $b = -a$.
Similar inequalities appear in Theorem 3 and Corollary 7 of [CZ07]. These results give a condition on c in order to have $w_\rho(T) \leq 1$. Hence, by looking at every $u > 0$ such that $w_\rho(\frac{T}{u}) \leq 1$, we can use the inequality of Theorem 3 to obtain the equation in item (i) of Lemma 1.3.7. The inequalities in Corollary 7 allow us to obtain similar equations while removing the parameter t in the cases $|a| = |b|$ and $a\bar{b} \in \mathbb{R}$. In both cases $w_\rho(T)$ can still only be expressed implicitly as the largest root of a degree 4 polynomial.

In the following proposition, we compute the value of $w_\rho(T)$ depending on $w_2(T)$ in the case $a = b$.

Proposition 1.3.9. *Let H be a Hilbert space. Let $a \in \mathbb{C}$ and let $T \in \mathcal{L}(H)$ such that $(T-aI)^2 = 0$. Let $\rho \in \mathbb{C}^*$. Then,*

(i) For $\rho \in [1, +\infty[$, we have

$$\rho w_\rho(T) = w_2(T) + |a|(\rho-2) + \sqrt{(w_2(T) + |a|(\rho-2))^2 - |a|^2\rho(\rho-2)}.$$

(ii) If $T \neq 0$, then

$$\|T\| = w_2(T) - |a| + \sqrt{(w_2(T) - |a|)^2 + |a|^2} \text{ and } w_2(T) = |a| + \frac{\|T\|^2 - |a|^2}{2\|T\|}.$$

(iii) For $\rho \neq 0$, we have

$$|\rho| w_\rho(T) = w_2(T) + |a|(|\rho-1|-1) + \sqrt{(w_2(T) + |a|(|\rho-1|-1))^2 - |a|^2(|\rho-1|^2-1)}.$$

Proof. (i) Let $\rho \in [1, +\infty[$. Let $h \in H$ that is non-zero. Since T is an algebraic operator with a minimal polynomial of degree at most 2, the subspace $V_h = \overline{\mathrm{Span}(T^n(h),\, n \geq 0)}$ has a dimension of 1 or 2. Denote $T_h := T|_{V_h}$.
If $dim(V_h) = 1$ we then have $T(h) = ah$, so $T_h = aI$ and $w_\rho(T_h) = |a|$. Hence, we have

$$\begin{aligned} \rho w_\rho(T_h) &= |a|\rho = |a|(\rho-1) + |a| \\ &= |a|(\rho-1) + \sqrt{|a|^2(\rho-1)^2 - \rho(\rho-2)} \\ &= w_2(T_h) + |a|(\rho-2) + \sqrt{(w_2(T_h) + |a|(\rho-2))^2 - |a|^2\rho(\rho-2)}. \end{aligned}$$

If $dim(V_h) = 2$, then up to taking a suitable orthonormal basis of V_h, the operator T_h can be represented as the matrix $M = \begin{pmatrix} a & c \\ 0 & a \end{pmatrix}$, for some $c \in \mathbb{C}$ that depends on h.

Since we have $w_\rho(T_h) = w_\rho(M)$, we can use Lemma 1.3.7 to compute $w_\rho(T_h)$.
According to this lemma, $w_\rho(M)$ is the largest $r > 0$ such that there exists $t \in [0, 2\pi[$ for which we have

$$|c|^2 r^2 = [(\rho-2)|a|^2 - 2(\rho-1)r\mathrm{Re}(ae^{it}) + \rho r^2][(\rho-2)|a|^2 - 2(\rho-1)r\mathrm{Re}(be^{it}) + \rho r^2].$$

Let $r > 0$ and $t \in [0, 2\pi[$. We have

$$\begin{aligned} &|c|^2 r^2 = [(\rho-2)|a|^2 - 2(\rho-1)r\mathrm{Re}(ae^{it}) + \rho r^2]^2 \\ \Leftrightarrow &0 = [(\rho-2)|a|^2 - 2(\rho-1)r\mathrm{Re}(ae^{it}) + \rho r^2 - |c|r][(\rho-2)|a|^2 - 2(\rho-1)r\mathrm{Re}(ae^{it}) + \rho r^2 + |c|r] \\ \Leftrightarrow &0 = (\rho-2)|a|^2 - 2(\rho-1)r\mathrm{Re}(ae^{it}) + \rho r^2 \pm |c|r \end{aligned}$$

The discriminant of such a polynomial in r is

$$\Delta = (-2(\rho-1)\mathrm{Re}(ae^{it}) \pm |c|)^2 - 4|a|^2\rho(\rho-2).$$

Let t_0 such that $\mathrm{Re}(ae^{it_0}) = |a|$. Since we have $-|a| \leq \mathrm{Re}(ae^{it}) \leq |a|$, we can see that

$$\begin{aligned} \inf_t(-2(\rho-1)\mathrm{Re}(ae^{it}) - |c|) &= -(2(\rho-1)|a| + |c|), \\ \sup_t(-2(\rho-1)\mathrm{Re}(ae^{it}) - |c|)^2 &= (2(\rho-1)|a| + |c|)^2, \end{aligned}$$

and that these extrema are both attained when $t = t_0$.
When $t = t_0$ we have

$$\begin{aligned} \Delta &= (2(\rho-1)a| + |c|)^2 - 4|a|^2\rho(\rho-2) \\ &= |a|^2(4(\rho-1)^2 - 4\rho(\rho-2)) + 4(\rho-1)|a||c| + |c|^2 \\ &= 4|a|^2 + 4(\rho-1)|a||c| + |c|^2 \geq 0, \end{aligned}$$

so the polynomial we are considering possesses real roots in this situation.
As we also have

$$\begin{aligned} -2(\rho-1)\mathrm{Re}(ae^{it}) + |c| &\geq -2(\rho-1)|a| - |c|, \\ (-2(\rho-1)\mathrm{Re}(ae^{it}) + |c|)^2 &\leq (2(\rho-1)|a| + |c|)^2, \end{aligned}$$

we can see that the largest root for this family of polynomials is then

$$r' = \frac{2(\rho-1)|a| + |c| + \sqrt{(2(\rho-1)|a| + |c|)^2 - 4|a|^2\rho(\rho-2)}}{2\rho}.$$

Therefore, item (i) of Lemma 1.3.7 tells us that

$$\begin{aligned} w_\rho(T_h) &= \frac{2(\rho-1)a| + |c| + \sqrt{(2(\rho-1)a| + |c|)^2 - 4|a|^2\rho(\rho-2)}}{2\rho} \\ \Rightarrow \rho w_\rho(T_h) &= (\rho-1)|a| + \frac{|c|}{2} + \frac{\sqrt{(2(\rho-1)|a| + |c|)^2 - 4|a|^2\rho(\rho-2)}}{2}. \end{aligned}$$

For $\rho = 2$ we obtain

$$w_2(T_h) = \frac{2|a| + |c| + \sqrt{(|c| + 2|a|)^2}}{4} = |a| + \frac{|c|}{2}.$$

For $\rho = 1$ we obtain

$$\|T\| = \frac{|c| + \sqrt{|c|^2 + 4|a|^2}}{2}.$$

As we have $|c| = 2(w_2(T_h) - |a|)$, we can now give an expression $w_\rho(T_h)$ depending on $w_2(T_h)$:

$$\begin{aligned}
\rho w_\rho(T_h) &= (\rho-1)|a| + \frac{|c|}{2} + \frac{\sqrt{(2(\rho-1)|a| + |c|)^2 - 4|a|^2\rho(\rho-2)}}{2} \\
\Rightarrow \rho w_\rho(T_h) &= (\rho-1)|a| + w_2(T_h) - |a| + \frac{\sqrt{(2(\rho-1)|a| + 2(w_2(T_h) - |a|))^2 - 4|a|^2\rho(\rho-2)}}{2} \\
\Rightarrow \rho w_\rho(T_h) &= (\rho-2)|a| + w_2(T_h) + \frac{\sqrt{4((\rho-2)|a| + w_2(T_h))^2 - 4|a|^2\rho(\rho-2)}}{2} \\
\Rightarrow \rho w_\rho(T_h) &= (\rho-2)|a| + w_2(T_h) + \sqrt{((\rho-2)|a| + w_2(T_h))^2 - 4|a|^2\rho(\rho-2)}.
\end{aligned}$$

Hence, if we denote $f : x \mapsto (\rho-2)|a| + x + \sqrt{(x + |a|(\rho-1))^2 - |a|^2\rho(\rho-2)}$, we proved that for every $h \in H$ that is non-zero, we have $\rho w_\rho(T_h) = f(w_2(T_h))$.
For every $x \geq |a|$ we have

$$(x + |a|(\rho-1))^2 - |a|^2\rho(\rho-2) \geq |a|^2((\rho-1)^2 - \rho(\rho-2)) = |a|^2 \geq 0,$$

so we can see that f is well-defined on $[|a|, +\infty[$, and that f is continuous and increasing on $[|a|, +\infty[$. As we have $w_2(T_h) \geq |a|$ for every $h \in H \setminus \{0\}$, using Lemma 1.3.6 gives us

$$\rho w_\rho(T) = \sup_{h\in H} \rho w_\rho(T_h) = \sup_{h\in H} f(w_2(T_h)) = f(\sup_{h\in H} w_2(T_h)) = f(w_2(T)),$$

which gives the desired formula.
- (ii) We already obtained in item (i) that $\|T\| = w_2(T) - |a| + \sqrt{(w_2(T) - |a|)^2 + |a|^2}$. Suppose now that $T \neq 0$. We have

$$\begin{aligned}
\|T\| - w_2(T) + |a| &= \sqrt{(w_2(T) - |a|)^2 + |a|^2} \\
\Rightarrow (\|T\| - w_2(T) + |a|)^2 &= (w_2(T) - |a|)^2 + |a|^2 \\
\Rightarrow \|T\|^2 + 2\|T\|(|a| - w_2(T)) + (|a| - w_2(T))^2 &= (w_2(T) - |a|)^2 + |a|^2 \\
\Rightarrow \|T\|^2 + 2\|T\||a| - 2\|T\|w_2(T) &= |a|^2 \\
\Rightarrow w_2(T) &= \frac{\|T\|^2 + 2|a|\|T\| - |a|^2}{2\|T\|} = |a| + \frac{\|T\|^2 - |a|^2}{2\|T\|},
\end{aligned}$$

which gives the desired result.
- (iii) Let $\rho \neq 0$. Using Proposition 1.3.3 we obtain

$$\begin{aligned}
|\rho|w_\rho(T) &= (1 + |\rho-1|)w_{1+|\rho-1|}(T) \\
&= w_2(T) + |a|(|\rho-1| - 1) \\
&\quad + \sqrt{(w_2(T) + |a|(|\rho-1| - 1))^2 - |a|^2(|\rho-1| + 1)(|\rho-1| - 1)} \\
&= w_2(T) + |a|(|\rho-1| - 1) + \sqrt{(w_2(T) + |a|(|\rho-1| - 1))^2 - |a|^2(|\rho-1|^2 - 1)},
\end{aligned}$$

which concludes the proof. □

Remark 1.3.10. For H a Hilbert space, $\rho \in [1, +\infty[$, and $T \in \mathcal{L}(H)$ such that T is the zero of a polynomial P of degree 2, we do not know yet a general formula that gives an explicit expression of $w_\rho(T)$ depending on $w_2(T)$ and on the coefficients of P. We can see that the formulas we obtained in Corollary 1.3.5 and Proposition 1.3.9 give $w_\rho(T)$ as the zero of a degree 2 (or 1) polynomial depending on $w_2(T)$ and on the coefficients of P. Indeed:

- If $T^2 + cT + 0 = 0$, then $\rho w_\rho(T) = w_2(T) + |c|(\rho - 2) + \sqrt{w_2(T)^2 + 0}$;
- If $T^2 + 0 + bI = 0$, then $\rho w_{(\rho)}(T) = w_2(T) + 0 + \sqrt{w_2(T)^2 + |b|\rho(\rho - 2)}$;
- If $T^2 + 2aT + a^2 I = 0$, then $\rho w_\rho(T) = w_2(T) + |a|(\rho - 2) + \sqrt{(w_2(T) + |a|(\rho - 2))^2 - |a|^2 \rho(\rho - 2)}$.

Using Proposition 1.3.3, we can also generalize some results of [AL10] about characterizing unitary operators through their ρ-radii.

Proposition 1.3.11. — *Let $T \in \mathcal{L}(H)$ be invertible. Then*

(i) T is unitary if and only if there exists $\rho \in \mathbb{C}^*$ such that

$$w_{(\rho)}(T) \leq w_{(\rho)}(I) \text{ and } w_{(\rho)}(T^{-1}) \leq w_{(\rho)}(I).$$

(ii) $T = \|T\|U$ for U unitary if and only if there exists $\rho \in \mathbb{C}^*$ and $m > 0$ such that

$$\frac{w_{(\rho)}(T^{-m})}{w_{(\rho)}(I)} = \left(\frac{w_{(\rho)}(T)}{w_{(\rho)}(I)}\right)^{-m}.$$

Proof. - (i) The formula of Proposition 1.3.3 can be rewritten as $w_{(\rho)}(S) = w_{(\rho)}(I) w_{(1+|\rho-1|)}(S)$. It allows us to obtain the same relationship between T and I for $w_{(1+|\rho-1|)}$, and we can then apply [AL10, Theorem 2.3] to get the result.
- (ii) The formula of Proposition 1.3.3 allows us to obtain the same relationship for $w_{(1+|\rho-1|)}$, which simplifies into:

$$w_{1+|\rho-1|}(T^{-m}) = w_{1+|\rho-1|}(T)^{-m}.$$

We can now apply [AL10, Theorem 1.1], and the proof is complete. □

Proposition 1.3.12. — *Let $\rho \neq 0$ be a complex number. Then*

(i) The ρ-radius $w_{(\rho)}(\cdot)$ is a norm on $\mathcal{L}(H)$ if and only if $|\rho - 1| \leq 1$;

(ii) If $|\rho - 1| > 1$, then, for all operators T_1 and T_2 in $\mathcal{L}(H)$, we have

$$w_\rho(T_1 + T_2) \leq (1 + |\rho - 1|)\left(w_\rho(T_1) + w_\rho(T_2)\right).$$

Proof. For two operators T_1, T_2, we have $w_{(\rho)}(T_1 + T_2) \leq C\left(w_{(\rho)}(T_1) + w_{(\rho)}(T_2)\right)$ if and only if the same is true for $w_{(1+|\rho-1|)}$. It is known [SNBFK10, AO76] that for $\tau > 0$, $w_{(\tau)}$ is a norm if and only if $0 < \tau \leq 2$. We conclude that $w_{(\rho)}(\cdot)$ is a norm if and only if ρ lies in the closed circle of center 1 and radius 1. Moreover, when $\tau > 2$, $w_{(\tau)}$ is a quasi-norm with multiplicative constant (also called the modulus of concavity of the quasi-norm [Kal03]) lower or equal to τ. We thus obtain (ii). □

For the next proposition we recall that for $r > 0$ the disc algebra over the disc $\mathbb{D}(0,r)$, $\mathbb{A}(\mathbb{D}(0,r))$, is the set of holomorphic functions on $\mathbb{D}(0,r)$ that are continuous on $\overline{\mathbb{D}(0,r)}$.

Proposition 1.3.13. — *Let $\rho \neq 0$ be a complex number. Let $T \in C_{(\rho)}$. Then the functional calculus map $f \mapsto f(T)$ that sends a polynomial f into $f(T)$ can be extended continuously to the disk algebra $\mathbb{A}(\mathbb{D}(0, \frac{1}{w_{(\rho)}(I)}))$. It is a morphism of Banach algebras, and satisfies*

$$\|f(T)\| \leq (1 + |\rho - 1|)\|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))}.$$

Furthermore, for $f \in \mathbb{A}(\mathbb{D}(0, \frac{1}{w_{(\rho)}(I)}))$ such that $f(0) = 0$, we have

$$w_{(\rho)}(f(T)) \leq w_{(\rho)}(I)\|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))}.$$

If $f \in \mathbb{A}(\mathbb{D})$ with $f(0) = 0$, we also have

$$w_{(\rho)}(f(T)) \leq \|f\|_{L^\infty(\mathbb{D})}.$$

The constants in these three inequalities are optimal.

Proof. We notice first that $T \in C_{(\rho)}$ is equivalent to $w_{(\rho)}(T) \leq 1$, which is equivalent to

$$w_{(1+|\rho-1|)}(T) \leq \frac{|\rho|}{1 + |\rho - 1|} = \frac{1}{w_{(\rho)}(I)} \leq 1.$$

Hence, $w_{(\rho)}(I)T$ lies in $C_{(1+|\rho-1|)}$, so there exists a Hilbert space K and an unitary operator U over K such that

$$(w_{(\rho)}(I)T)^n = (1 + |\rho - 1|)P_H U^n|_H, \ \forall n \geq 1.$$

Therefore, if we denote $V := \frac{U}{w_{(\rho)}(I)}$, for any polynomial P we get

$$P(T) = P_H[(1 + |\rho - 1|)P(V) - |\rho - 1|P(0)I]|_H.$$

Since V is a normal operator with spectral radius $\frac{1}{w_{(\rho)}(I)}$, we then have

$$\|P(T)\| \leq \|(1 + |\rho - 1|)P(V) - |\rho - 1|P(0)I\| \leq \|(1 + |\rho - 1|)P - |\rho - 1|P(0)\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))}.$$

As the polynomials are dense in the algebra $\mathbb{A}(\mathbb{D}(0, \frac{1}{w_{(\rho)}(I)}))$, the morphism of algebras $P \mapsto P(T)$ extends continuously on $\mathbb{A}(\mathbb{D}(0, \frac{1}{w_{(\rho)}(I)}))$.

Let us estimate the norm of this map. For f in the algebra we denote $g(z) := f(\frac{z}{w_{(\rho)}(I)})$. Hence, $g \in \mathbb{A}(\mathbb{D})$, and we have $f(T) = g(w_{(\rho)}(I)T)$. Applying a reformulation of Theorem 2 in [AO75] by Ando and Okubo, we obtain

$$\|f(T)\| = \|g(w_{(\rho)}(I)T)\| \leq \max(1, 1 + |\rho - 1|)\|g\|_{L^\infty(\mathbb{D})} = (1 + |\rho - 1|)\|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))}.$$

We will now prove the two remaining inequalities. The fact that V is normal implies that $f \mapsto f(V)$ is well defined and bounded on $\mathbb{A}(\mathbb{D}(0, \frac{1}{w_{(\rho)}(I)}))$. Therefore

$$f(T) = P_H[(1 + |\rho - 1|)f(V) - |\rho - 1|f(0)I]|_H, \ \forall f \in \mathbb{A}(\mathbb{D}(0, \frac{1}{w_{(\rho)}(I)})).$$

We now suppose that f satisfies $f(0) = 0$. If $f \equiv 0$, then $f(T) = 0$ and the statements are true. Otherwise, up to dividing f by its norm, we may assume that $\|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))} = 1$. For a fixed $n \geq 1$, we get

$$f(T)^n = f^n(T) = (1 + |\rho - 1|)P_H f^n(V)|_H = (1 + |\rho - 1|)P_H f(V)^n|_H.$$

As we have $\|f(V)\| \leq \|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))} = 1$, the operator $f(V)$ lies in $C_{(1)}$ which in turn implies that $f(V)$ can be dilated on a larger Hilbert space as follows

$$f(V)^m = P_K W^m|_K, \, \forall m \geq 1,$$

with W a suitable unitary operator. Combining the two dilations, we obtain

$$f(T)^n = (1 + |\rho - 1|)P_H W^n|_H, \, \forall n \geq 1.$$

Therefore $f(T)$ lies in $C_{(1+|\rho-1|)}$, which is equivalent to $w_{(1+|\rho-1|)}(f(T)) \leq 1$. This inequality is in turn equivalent to $w_{(\rho)}(f(T)) \leq w_{(\rho)}(I)$, which proves the second inequality of this Proposition. Lastly, if $f \in \mathbb{A}(\mathbb{D})$ with $f(0) = 0$, we can use the Schwarz lemma to obtain

$$\|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))} \leq \frac{1}{w_{(\rho)}(I)} \|f\|_{L^\infty(\mathbb{D})},$$

which in turn gives $w_{(\rho)}(f(T)) \leq \|f\|_{L^\infty(\mathbb{D})}$.
For the optimality of these inequalities, let us take T such that $T^2 = 0$ and $\|T\| = |\rho|$, and $f(z) = z$. We then have

$$w_{(\rho)}(T) = \frac{\|T\|}{|\rho|} = 1 = w_{(\rho)}(I)\|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))} = \|f\|_{L^\infty(\mathbb{D})}$$

and

$$\|f(T)\| = |\rho| = (1 + |\rho - 1|)\|f\|_{L^\infty(\mathbb{D}(0,\frac{1}{w_{(\rho)}(I)}))}.$$

The proof is complete. □

When ρ does not lie in $[1, +\infty[$, the algebra where the functional calculus is defined strictly contains the disc algebra $\mathbb{A}(\mathbb{D})$. For $0 < \rho < 1$, the norm of this map is then $2 - \rho$. This result differs from [AO75, Theorem 2] as Ando and Okubo looked in [AO75] at the map $f \mapsto f(T)$ on $\mathbb{A}(\mathbb{D})$ and not on a larger algebra.

A generalization of classes $C_{(\rho_n)}$ to Banach spaces

The statements of Proposition 1.3.3 can be generalized to classes $C_{(\rho_n)}$ using the same ideas.

Proposition 1.3.14. *Let $T \in \mathcal{L}(H)$. Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. The following are equivalent:*

(i) $T \in C_{(\rho_n)}$;

(ii) $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$, $2I + f_{(\rho_n)}(zT)$ *is invertible for every* $z \in \mathbb{D}$, *and*

$$\|f_{(\rho_n)}(zT)(2I + f_{(\rho_n)}(zT))^{-1}\| \leq 1, \, \forall z \in \mathbb{D};$$

(iii) $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$, *$2I + f_{(\rho_n)}(zT)$ is invertible for every $z \in \mathbb{D}$, and*

$$\|f_{(\rho_n)}(zT)(h)\| \leq \|2h + f_{(\rho_n)}(zT)(h)\|, \forall h \in H, \forall z \in \mathbb{D};$$

(iv) $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$, *and*

$$\|f_{(\rho_n)}(zT)(h)\| \leq \|2h + f_{(\rho_n)}(zT)(h)\|, \forall h \in H, \forall z \in \mathbb{D};$$

Proof. $(i) \Rightarrow (ii)$ Let $T \in C_{(\rho_n)}$. We have $r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$ and $\mathrm{Re}(I + f_{(\rho_n)}(zT)) \geq 0$ for every $z \in \mathbb{D}$. Let $z \in \mathbb{D}$. Denote $C_z = I + f_{(\rho_n)}(zT)$. Since $\mathrm{Re}(C_z) \geq 0$, then $C_z + I = 2I + f_{(\rho_n)}(zT)$ is invertible and we have

$$\|(C_z - I)(C_z + I)^{-1}\| = \|f_{(\rho_n)}(zT)(2I + f_{(\rho_n)}(zT))^{-1}\| \leq 1.$$

- $(ii) \Rightarrow (i)$ Let $z \in \mathbb{D}$. Then $D_z = f_{(\rho_n)}(zT)(2I + f_{(\rho_n)}(zT))^{-1}$ is well-defined. As $\|D_z\| \leq 1$, for every $w \in \mathbb{D}$ we have $\mathrm{Re}((I + wD_z)(I - wD_z)^{-1}) \geq 0$. When w tends to 1, $(I + wD_z)(I - wD_z)^{-1}$ converges to $C_z = I + f_{(\rho_n)}(zT)$. Thus, we have $\mathrm{Re}(I + f_{(\rho_n)}(zT)) \geq 0$, which implies in turn that $T \in C_{(\rho_n)}$.
- $(ii) \Leftrightarrow (iii)$ is immediate as $2I + f_{(\rho_n)}(zT)$ is invertible. $(iii) \Rightarrow (iv)$ is also immediate.
- $(iv) \Rightarrow (i)$ Let $z \in \mathbb{D}$. Denote $S_z = f_{(\rho_n)}(zT)$ and let $\lambda \in \partial\sigma(S_z)$. Hence λ lies in the approximate spectrum of S_z: there exists $h_n \in H$ such that $\|h_n\| = 1$ and $g_n = S_z(h_n) - \lambda h_n \to 0$. We have

$$0 \leq |\|2h_n + S_z(h_n)\| - |2 + \lambda|| = |\|2h_n + S_z(h_n)\| - \|2h_n + \lambda h_n\|| \leq \|S_z(h_n) - \lambda h_n\| \to 0.$$

By hypothesis we have $\|S_z(h_n)\| \leq \|(2I + S_z)(h_n)\|$. The quantity on the left converges to $|\lambda|$ while the one on the right converges to $|2 + \lambda|$. Thus we have $|\lambda| \leq |2 + \lambda|$, which implies

$$\begin{aligned} &|\lambda| \leq |2 + \lambda| \\ \Rightarrow &|\lambda|^2 \leq |2 + \lambda|^2 \\ \Rightarrow &\mathrm{Re}(\lambda)^2 \leq (\mathrm{Re}(\lambda) + 2)^2 \\ \Rightarrow &-1 \leq \mathrm{Re}(\lambda). \end{aligned}$$

Therefore we have $\partial\sigma(S_z) \subset Re_{\geq -1}$, so $\sigma(S_z) \subset Re_{\geq -1}$. This is equivalent to $\mathrm{Re}(I + S_z) \geq 0$. Hence, we have $T \in C_{(\rho_n)}$. □

Remark 1.3.15. Following the ideas developed by Carrot in [Car05], we can use conditions (ii) or (iv) of Proposition 1.3.14 in order to define classes $C_{(\rho_n)}$ of operators on any Banach space X, as these conditions do not require any structure related to Hilbert spaces (unitary operators or self-adjoint operators). Using this we could generalize most of the results that Carrot generalized for classes C_ρ.
We do note however that conditions (ii) and (iv) of Proposition 1.3.14 for (ρ_n) are less manageable than conditions (ii) and (iv) of Proposition 1.3.3 for ρ due to two constraints: they always require a condition on the spectral radius of T in order for $f_{(\rho_n)}(zT)$ to be well-defined, and the expression in condition (ii) of Proposition 1.3.14 cannot be simplified any further due to the general form of $f_{(\rho_n)}(zT)$. These constraints would induce additional work in proofs regarding classes $C_{(\rho_n)}$ and (ρ_n)-radii on general Banach spaces, and some results in [Car05] may not generalize to this case due to them.

1.4 Inequalities and Parametrizations for (ρ_n)-Radii

1.4.A Operator radii of products and tensor products.

A useful tool, used to study the behaviour of a product or sum of double-commuting operators, is the following result, proved in [Hol68, Thm.4.2].

Proposition 1.4.1. — *Let $T_n, S_n \in \mathcal{L}(H)$, $n \in \mathbb{Z}$, be such that for all $0 \le r < 1$, $t \in \mathbb{R}$, the series $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} T_n$ and $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} S_n$ converge absolutely and have self-adjoint non-negative sums. If, moreover, we have $T_n S_m = S_m T_n$, $\forall\, m, n \in \mathbb{Z}$, then the series $\sum_{n=-\infty}^{\infty} r^{|n|} e^{int} T_n S_n$ converges absolutely and has a self-adjoint non-negative sum, for all $0 \le r < 1$, $t \in \mathbb{R}$.*

Using Proposition 1.4.1 we can easily obtain the following auxiliary result.

Lemma 1.4.2. — *Let $T, S \in \mathcal{L}(H)$ be two operators that are double-commuting (i.e., $TS = ST$, $TS^* = S^*T$). Let $(\rho_n)_n, (\tau_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$ and $\liminf_n(|\tau_n|^{\frac{1}{n}}) > 0$. Then, we have*

$$w_{(\rho_n\tau_n)}(ST) \le w_{(\rho_n)}(S) w_{(\tau_n)}(T).$$

Proof. If $S = 0$ or $T = 0$, then $ST = 0$ and both sides of the inequality are equal to zero. If $S \ne 0$ and $T \ne 0$, then, up to dividing S and T by their respective radius, we can consider that $w_{(\rho_n)}(S) = w_{(\tau_n)}(T) = 1$. Thus, we need to prove that $w_{(\rho_n\tau_n)}(ST) \le 1$. We define

$$S_m := \begin{cases} \frac{1}{\rho_m} S^m \text{ if } m \ge 1 \\ I \text{ if } m = 0 \\ \frac{1}{\rho_{|m|}} (S^*)^{|m|} \text{ if } m \le -1. \end{cases}, \quad T_m := \begin{cases} \frac{1}{\tau_m} T^m \text{ if } m \ge 1 \\ I \text{ if } m = 0 \\ \frac{1}{\tau_{|m|}} (T^*)^{|m|} \text{ if } m \le -1 \end{cases}$$

The condition $w_{(\rho_n)}(S) = w_{(\tau_n)}(T) = 1$, together with Lemma 1.2.12 and Proposition 1.2.6, ensure us that the conditions of Proposition 1.4.1 are fulfilled, since $I + \mathrm{Re}(f_{(\rho_n)}(re^{it}S)) = \sum_{m\in\mathbb{Z}} r^{|m|} e^{imt} S_m$, for all $0 \le r < 1$, $t \in \mathbb{R}$ and since the same is truc for $(T_m)_m$. Applying 1.4.1 gives us that $\sum_{m\in\mathbb{Z}} r^{|m|} e^{imt} S_m T_m$ converges absolutely, is self-adjoint, and has a positive sum, for all $0 \le r < 1$, $t \in \mathbb{R}$. This implies that the series

$$\sum_{n\ge 1} \frac{2}{\rho_n\tau_n} (re^{it}ST)^n = f_{(\rho_n\tau_n)}(re^{it}ST)$$

is absolutely convergent and that $I + \mathrm{Re}(f_{(\rho_n\tau_n)}(re^{it}ST)) \ge 0$ for all $0 \le r < 1$, $t \in \mathbb{R}$. Thus $ST \in C_{(\rho_n\tau_n)}$ and $w_{(\rho_n\tau_n)}(ST) \le 1$, which concludes the proof. □

Corollary 1.4.3. — *Let $T, S \in \mathcal{L}(H)$ and let $(\rho_n)_n, (\tau_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$ and $\liminf_n(|\tau_n|^{\frac{1}{n}}) > 0$.*

(i) If T and S double-commute, then

$$w_{(\rho_n)}(ST) \le w_{(1)}(S) w_{(\rho_n)}(T) \le |\tau_1| w_{(\tau_n)}(S) w_{(\rho_n)}(T).$$

This inequality is optimal when $\dim(H) \ge 4$.

(ii) We have

$$w_{(1)}(ST) \le w_{(1)}(S) w_{(1)}(T) \le |\tau_1||\rho_1| w_{(\tau_n)}(S) w_{(\rho_n)}(T).$$

This inequality is optimal when $\dim(H) \ge 2$.

(iii) For $R \in \mathcal{L}(H')$, we have

$$w_{(\rho_n \tau_n)}(T \otimes R) \leq w_{(\rho_n)}(T) w_{(\tau_n)}(R).$$

Proof. - (i) We use Lemma 1.4.2 for S, T and $(1)_n, (\rho_n)_n$ to get the left-hand side inequality. The right-hand side inequality comes from the fact $w_{(\tau_n)}(S) \geq \frac{\|S\|}{|\tau_1|}$ (cf. Lemma 1.2.12). By taking

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ such that } ST = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

some computation show that S and T double-commute, and that

$$\|S\| = \|T\| = \|ST\|, \; S^2 = T^2 = (ST)^2 = 0.$$

Corollary 1.3.5,(iii), shows that all three quantities are equal to $\frac{\|ST\|}{|\rho_1|}$.
- (ii) The inequality on the right-hand side follows again from Lemma 1.2.12, while the left-hand one is the operator norm multiplicative inequality. By taking

$$S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ such that } ST = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\|S\| = \|T\| = \|ST\| = 1, \; S^2 = T^2 = 0, \text{ and } ST \text{ is self-adjoint.}$$

Thus, $w_{(\tau_n)}(S) = w_{(\rho_n)}(T) = \frac{1}{\rho}$ and $w_{(1)}(ST) = 1$, so all quantities are equal to 1.
- (iii) As I_H, T double-commute and $I_{H'}, R$ double-commute too, we can apply Lemma 1.4.2 to $(T \otimes I_{H'})(I_H \otimes R) = T \otimes R$. We then apply item (iii) of Corollary 1.2.16. □

Although these inequalities are optimal for some operators, they tend to lose a good part of the information in the general case. For example, we have $w_{(3)}(I) = 1 \leq w_{(-1)}(I) w_{(-3)}(I) = 5$. Such a loss of information on the radius of the identity operator I also impacts almost every estimate of radii for other operators in $\mathcal{L}(H)$.

Corollary 1.4.4. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Then,*

$$\frac{\|T\|}{|\rho_1|} \leq w_{(\rho_n)}(T) \leq \|T\| w_{(\rho_n)}(I).$$

Furthermore, the coefficients in this equivalence of quasi-norms are optimal.

Proof. The left-hand side inequality $\frac{\|T\|}{|\rho_1|} \leq w_{(\rho_n)}(T)$ has been obtained in Lemma 1.2.12. The equality case is obtained for T such that $T^2 = 0$, as seen in Corollary 1.3.5. The right-hand side inequality comes from Lemma 1.4.2, with $S = I$ and $\tau_n = 1$. It is an improvement of the one that was obtained in Lemma 1.2.12. The equality case is obtained for any T normal of norm 1. □

1.4.B Operator radii as 1-parameter families.

To better understand the behaviour of the associated radii associated with classes of operators, it is useful to look at (ρ_n)-radii as 1-parameter families. This is obtained by studying the map $z \mapsto w_{(z\rho_n)}$. We will present results for the real parameter case ($r \in]0, +\infty[$) and for the complex one ($z \in \mathbb{C}^*$).

The two main ingredients we are using are the double-commuting inequality of Lemma 1.4.2 for T, I and $(\rho_n)_n$,$(1)_1$, and the fact that $f_{(z\rho_n)} = \frac{1}{z} f_{(\rho_n)}$.

Proposition 1.4.5. — *Let $T \in \mathcal{L}(H)$ and consider $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ with $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$.*

(i) For all $z \neq 0$, we have:

$$\frac{|z|}{1+|z-1|} w_{(z\rho_n)}(T) \leq w_{(\rho_n)}(T) \leq w_{(z\rho_n)}(T)(|z| + |z-1|).$$

(ii) The map $z \mapsto w_{(z\rho_n)}(T)$ is continuous on $\mathbb{C}^*$, and $r \mapsto w_{(r.e^{it}\rho_n)}(T)$ is continuous and decreasing on $]0, +\infty[$, for all $t \in]-\pi, \pi]$.

(iii) We have

$$\frac{1}{3} w_{(z\rho_n)}(T) \leq w_{(|z|\rho_n)}(T) \leq 3 w_{(z\rho_n)}(T),$$

and these inequalities are optimal.

Proof. - (i) We use Lemma 1.4.2 to obtain

$$w_{(z\rho_n)}(T) \leq w_{(z)}(I) w_{(\rho_n)}(T) \text{ and } w_{(\rho_n)}(T) \leq w_{(z^{-1})}(I) w_{(z\rho_n)}(T).$$

As $w_{(z)}(I) = \frac{1+|z-1|}{|z|}$ and $w_{(z^{-1})}(I) = |z| + |z-1|$, we obtain the desired inequalities.
- (ii) Up to changing $(\rho_n)_n$ by $(w\rho_n)_n$, the continuity must only be shown at the point $w = 1$, that is when $z \to 1$. As we have

$$w_{(\rho_n)}(T) \leq w_{(z\rho_n)}(T)(|z| + |z-1|) \leq w_{(\rho_n)}(T)(|z| + |z-1|)\frac{1+|z-1|}{|z|}$$

and as $(|z| + |z-1|), \frac{1+|z-1|}{|z|}$ both tend to 1 from above as $z \to 1$, we obtain

$$\lim_{z \to 1} w_{(z\rho_n)}(T) = w_{(\rho_n)}(T).$$

For any $t \in \mathbb{R}$ and $0 < r < R$, we have

$$w_{(Re^{it}\rho_n)}(T) \leq w_{(Rr^{-1})}(I) w_{(re^{it}\rho_n)}(T) = w_{(re^{it}\rho_n)}(T).$$

Thus, $r \mapsto w_{(re^{it}\rho_n)}(T)$ is decreasing on $]0, +\infty[$.
- (iii) We use item (i) as well as the fact that $w_{(e^{it})}(I) = 1 + |e^{it} - 1|$ has a maximum of 3 when $e^{it} = -1$ to get the desired inequalities. The equality case for the inequality on the left-hand side is attained at $T = I$, $\rho_n = 1$ and $z = -1$, whereas the equality case for the one on the right-hand side is attained at $T = I$, $\rho_n = -1$, $z = -1$. □

Since $r \mapsto w_{(r\rho_n)}(T)$ is decreasing, the classes $C_{(r\rho_n)}$ are increasing (for the usual order of inclusion of sets), for $r \in]0,+\infty[$. By using nilpotent operators of order 2, and item (iii) of Corollary 1.3.5, we can also immediately show that these inclusions are always strict.

For the following propositions, we recall that $\frac{1}{\liminf_n(|\rho_n|^{\frac{1}{n}})} = 0$ if $\liminf_n(|\rho_n|^{\frac{1}{n}}) = +\infty$.

Proposition 1.4.6. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ and $T \in \mathcal{L}(H)$ be such that $\liminf_n(|\rho_n|^{\frac{1}{n}}) > r(T) \geq 0$. Then, there is $r > 0$ such that for all z with $|z| = r$,*

$$\frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})} \leq w_{(z\rho_n)}(T) \leq 1.$$

Proof. Let $s > 1$ be such that $r(sT) < \liminf_n(|\rho_n|^{\frac{1}{n}})$. As

$$\limsup_{n\to\infty} \left(\frac{2s^n\|T^n\|}{|\rho_n|}\right)^{\frac{1}{n}} = \frac{r(sT)}{\liminf_n(|\rho_n|^{\frac{1}{n}})} < 1,$$

there is $B > 0$ such that $\frac{2s^n\|T^n\|}{|\rho_n|} \leq B$. Thus, for all $w \in \mathbb{D}$, we have

$$\|f_{(z\rho_n)}(wT)\| \leq \sum_{n\geq 1} \frac{2\|T^n\|}{|z||\rho_n|} \leq \sum_{n\geq 1} \frac{B}{|z|s^n} = \frac{1}{|z|}\frac{sB}{s-1} < +\infty.$$

By taking $|z|$ large enough, we have $\|f_{(z\rho_n)}(wT)\| < 1$, which implies that

$$I + \mathrm{Re}(f_{(z\rho_n)}(wT)) \geq 0, \ \forall w \in \mathbb{D}.$$

Thus $w_{(z\rho_n)}(T) \leq 1$. The left-hand side inequality comes from items (i) and (ii) of Proposition 1.2.18: we have

$$w_{(z\rho_n)}(T) \geq r(T)w_{(z\rho_n)}(I) \text{ and } w_{(z\rho_n)}(I) \geq \frac{1}{\liminf_n(|z\rho_n|^{\frac{1}{n}})}.$$

□

Proposition 1.4.7. — *Let $T \in \mathcal{L}(H)$ and let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Then*

$$\lim_{|z|\to+\infty}(w_{(z\rho_n)}(T)) = \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}.$$

Proof. According to Proposition 1.4.5 and Proposition 1.2.18, the map $r \mapsto w_{(re^{it}\rho_n)}(T)$ is decreasing on $]0,+\infty[$ and

$$w_{(\rho_n)}(T) \geq r(T)w_{(\rho_n)}(I) \geq \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}.$$

We will show that $w_{(z\rho_n)}(T)$ is as close to this lower bound as we want, when z is large enough. Let $\epsilon > 0$. If $r(T) = 0$, then $r(\frac{1}{\epsilon}T) = 0$, so Proposition 1.4.6 implies the existence of $r > 0$ such that $w_{(z\rho_n)}(\frac{T}{\epsilon}) \leq 1$ for all z with $|z| = r$. Thus, $w_{(z\rho_n)}(T) \leq \epsilon$.
If $r(T) \neq 0$, for $0 < R < \liminf_n(|\rho_n|^{\frac{1}{n}})$ we have

$$r\left(\frac{RT}{(1+\epsilon)r(T)}\right) < \liminf_n(|\rho_n|^{\frac{1}{n}}).$$

Thus, by Proposition 1.4.6, there exists $r > 0$ such that $w_{(z\rho_n)}\left(\frac{RT}{(1+\epsilon)r(T)}\right) \leq 1$ for all z with $|z| = r$. Hence,

$$\frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})} \leq w_{(z\rho_n)}(T) \leq \frac{(1+\epsilon)r(T)}{R}.$$

We then obtain the result by taking $R = \liminf_n(|\rho_n|^{\frac{1}{n}})(1-\epsilon)$ if $\liminf_n(|\rho_n|^{\frac{1}{n}})$ is finite, or $R = \frac{1}{\epsilon}$ if $\liminf_n(|\rho_n|^{\frac{1}{n}}) = +\infty$. □

Proposition 1.4.8. — *Let $T \in \mathcal{L}(H)$. Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. We have:*

(i) $z \mapsto w_{(z\rho_n)}(T)$ is uniformly continuous on $\mathbb{C} \setminus \mathbb{D}(0,\epsilon)$, for all $\epsilon > 0$. This maps tends to $+\infty$ as $|z| \to 0$, and to $\frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}$ as $|z| \to +\infty$;

(ii) For any $t \in \mathbb{R}$, the map $r \mapsto w_{(re^{it}\rho_n)}(T)$ is log-convex on $]0,+\infty[$.

Proof. - *(i)* On the closed set $\mathbb{C} \setminus \mathbb{D}(0,\epsilon)$, the function

$$z \mapsto w_{(z\rho_n)}(T)$$

is continuous, decreasing on every half-line of the form $e^{it}[\epsilon,+\infty[$, and converges to

$$\frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})}$$

as $|z| \to +\infty$. Thus, a standard argument (considering two cases, $\epsilon \leq |z| \leq R$ and $|z| \geq R$) shows that this map is uniformly continuous. One can also use the double-commuting inequality of Lemma 1.4.2 for T and I_H, as well as the uniform continuity of the map $z \mapsto w_{(z)}(I)$ on $\mathbb{C} \setminus \mathbb{D}(0,\eta)$, in order to prove the uniform continuity of $z \mapsto w_{(z\rho_n)}(T)$. The limit as $|z| \to +\infty$ has been obtained in Proposition 1.4.7, while the limit as $|z| \to 0$ comes from the fact that $w_{(z\rho_n)}(T) \geq \frac{\|T\|}{|z||\rho_1|}$, as remarked in Lemma 1.2.12.
- *(ii)* Let $t \in \mathbb{R}$. Denote $G'(z) := -e^{-it}f_{(\rho_n)}(zT)$. For any $\alpha > 0$, we have $w_{(re^{it}\rho_n)}(T) \leq \alpha$ if and only if $f_{(e^{it}\rho_n)}(z\frac{T}{\alpha})$ is analytic on $\mathbb{D}$ and $I + \mathrm{Re}(\frac{1}{r}f_{(e^{it}\rho_n)}(z\frac{T}{\alpha})) \geq 0$, for all $z \in \mathbb{D}$. By taking $w = \frac{z}{\alpha}$, this is equivalent to $G'(w)$ being analytic on $\mathbb{D}(0,\frac{1}{\alpha})$ and $\mathrm{Re}(G'(w)) \leq rI$, for all $w \in \mathbb{D}(0,\frac{1}{\alpha})$. The result is then obtained by mimicking the proof of [AN73, Theorem 1] by Ando and Nishio and replacing G with G'. □

Even though the expression of $f_{(z\rho_n)}$ is more complex than $f_{(z)}(w) = \frac{2}{z}\frac{w}{1-w}$, the main regularity properties remain valid due to its analyticity.

Proposition 1.4.9. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. If one of the following assertions is true*

(i) $\liminf_n(|\rho_n|^{\frac{1}{n}}) < 1$;

(ii) $|\rho_n| < 1$ for some $n \geq 1$;

(iii) $w_{(\rho_n)}(I) > 1$;

(iv) $\rho_n = M + x_n$, $(x_n)_n \in \ell_2(\mathbb{C})$,

then all operators in $C_{(\rho_n)}(H)$ *are similar to contractions.*
If, on the contrary, we have:

(i') $w_{(\rho_n)}(I) < 1$,

then $C_{(\rho_n)}(H)$ *contains operators that are not similar to contractions.*
Both statements remain true if the conditions are only fulfilled for the subsequence $(\rho_{kn})_n$*, for some fixed* $k \geq 1$.

Proof. - $(i), (ii), (iii)$ We can see that $(i) \Rightarrow (ii) \Rightarrow (iii)$. If (iii) is true, then for $T \in C_{(\rho_n)}$, we have

$$r(T) \leq \frac{w_{(\rho_n)}(T)}{w_{(\rho_n)}(I)} < 1,$$

so T is similar to a contraction.
- (iv) It has been shown in [Rác74, Ch.2] (see also [Bad03, Cor 5.2.1]) that when $\rho_n = M + x_n$, $(x_n)_n \in \ell_2(\mathbb{C})$, all operators in $C_{(\rho_n)}$ are similar to contractions.
- (i') On the contrary, when $w_{(\rho_n)}(I) < 1$, $\frac{1}{w_{(\rho_n)}(I)} I \in C_{(\rho_n)}$ and this operator is not similar to a contraction.
The last assertion of the theorem follows from two facts. The first one is that $T \in C_{(\rho_n)}$ implies $T^k \in C_{(\rho_{kn})}$. The second one is that T^k is similar to a contraction if and only if T is similar to a contraction: see [Hal70, Problem 6,(ii)] for a proof when $k = 2$ that extends to any k by taking $((f, g)) := \sum_{j=1}^{k-1} \langle A^j f, A^j g \rangle$. □

Proposition 1.4.10. — *Let* $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ *be such that* $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$.

(i) If $\liminf_n(|\rho_n|^{\frac{1}{n}}) = +\infty$, *then* $\bigcup_{r>0} C_{(r\rho_n)}(H) = \mathcal{L}(H)$.

(ii) If $\liminf_n(|\rho_n|^{\frac{1}{n}}) < +\infty$, *then we have*

$$\{T\colon r(T) < \liminf_n(|\rho_n|^{\frac{1}{n}})\} \subset \bigcup_{r>0} C_{(r\rho_n)}(H) \subset \{T\colon r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})\}.$$

(iii) Moreover, we have

$$\{T\colon r(T) < \liminf_n(|\rho_n|^{\frac{1}{n}})\} = \bigcup_{r>0} C_{(r\rho_n)}(H)$$

if and only if

$$w_{(r\rho_n)}(\liminf_n(|\rho_n|^{\frac{1}{n}})I) > 1, \forall r > 0.$$

Proof. - (i) By using Proposition 1.4.6, for any T there exists $r > 0$ such that $w_{(r\rho_n)}(T) \leq 1$.
- (ii) We use again Proposition 1.4.6 in order to obtain the left-hand side inclusion. The other inclusion follows from Proposition 1.2.6.
- (iii) Suppose that there is a number $r > 0$ and an operator T with $r(T) = \liminf_n(|\rho_n|^{\frac{1}{n}})$ such that $w_{(r\rho_n)}(T) \leq 1$. Then

$$1 \geq w_{(r\rho_n)}(T) \geq r(T) w_{(r\rho_n)}(I) \geq \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})} = 1.$$

Thus all inequalities are equalities, and $w_{(r\rho_n)}\left(\liminf_n(|\rho_n|^{\frac{1}{n}}).I\right) = 1$. Conversely, if for some $r > 0$ we have

$$w_{(r\rho_n)}\left(\liminf_n(|\rho_n|^{\frac{1}{n}}).I\right) = 1,$$

then the union of all $C_{(r\rho_n)}$ contains $\liminf_n(|\rho_n|^{\frac{1}{n}})I$, so this set is not equal to $\{T\colon r(T) < \liminf_n(|\rho_n|^{\frac{1}{n}})\}$. Using the left-hand inclusion of item *(ii)* concludes the proof. □

Remark 1.4.11. — Replacing $(\rho_n)_n$ by $(e^{it}\rho_n)_n$ leaves unchanged the quantity $\liminf_n(|\rho_n|^{\frac{1}{n}})$. However, the union of all classes $C_{(r\rho_n)}$ can become a different set.
With $\rho_n = \rho$, we have $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 1$ and $w_{(\rho)}(I) = 1$ if and only if $\rho \in [1, +\infty[$. Thus,

$$\bigcup_{r>0} C_{(re^{it})_n}(H) = \{T\colon r(T) < 1\} \text{ if } t \neq 0 \ [2\pi].$$

This is not an equality if $t = 0$ (look at the identity operator I). However, the set $\bigcup_{r>0} C_{(r)}(H)$ does not contain all operators with spectral radius one. Indeed, it has been proven in [Rác74, Ch.2] (see also [Bad03, Cor 5.2.1]) that all operators contained in this union are all similar to contractions. Furthermore, all operators similar to a contraction are not in this union. For a counterexample, any non-orthogonal projection T (that is $T^2 = T$ and $\|T\| > 1$) is not in this union since Corollary 1.4.3,(v), says that $w_{(\rho)}(T) = \frac{\|T\|+|\rho-1|}{|\rho|} > 1$.
- For a sequence $(\rho_n)_n$ that satisfies $\alpha = \liminf_n(|\rho_n|^{\frac{1}{n}}) \in]0, +\infty[$, we can go back to the case $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 1$ by considering the sequence $(\frac{\rho_n}{\alpha^n})_n$. As this normalization is equivalent to a dilation by a factor $\frac{1}{\alpha}$ on the class $C_{(\rho_n)}$, we can then try to see if in this case the class $C_{(\rho_n)}$ is always included in the set of operators that are similar to contractions. This question is motivated by Properties 1.4.10 and 1.4.9. The answer is true when $w_{(\rho_n)}(I) > 1$, but Corollary 1.4.18 will give a negative answer in many remaining cases, even if we consider the inclusion in the set of power-bounded operators.

At this point we would like to mention that, for every $k \geq 2$, there is ([Găv08]) a Hilbert space operator $T \notin \cup_{\rho>0} C_\rho$ but with T^k belonging to $C_{(\tau)}$ for every $\tau \geq 1$. Related results are given in the next proposition.

Proposition 1.4.12. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. Let H a Hilbert space of dimension at least 2.*

(i) For $T \in \mathcal{L}(H)$ with $T^2 = 0$ and $\|T\| > |\rho_1|$, T^k is in the class $C_{(\rho_n)}$ for every $k \geq 2$, but T is not.

(ii) If $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 1$, then $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$ for some $k \geq 2$ implies that $T \in \bigcup_{r>0} C_{(r\rho_n)}$.

(iii) If $\liminf_n(|\rho_n|^{\frac{1}{n}}) < 1$, then there exists $T \in \mathcal{L}(H)$ such that T^k lies in $\bigcup_{r>0} C_{(r\rho_n)}$ for every $k \geq 2$ whereas T does not.

(iv) If $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 1$ and $I \notin \bigcup_{r>0} C_{(r\rho_n)}$, then $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$ for some $k \geq 2$ implies that $T \in \bigcup_{r>0} C_{(r\rho_n)}$.

Proof. - (i) As we have $\|T\| > |\rho_1|$, T cannot lie in $C_{(\rho_n)}$, whereas $T^k = 0$ does.
- (ii) Let T be such that $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$ for some $k \geq 2$. Then, $r(T^k) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})$. Hence,

$$r(T) \leq \liminf_n(|\rho_n|^{\frac{1}{n}})^{\frac{1}{k}} < \liminf_n(|\rho_n|^{\frac{1}{n}}),$$

that is $T \in \bigcup_{r>0} C_{(r\rho_n)}$ according to Proposition 1.4.10, (i) and (ii).
- (iii) Take $r > 0$ such that

$$\liminf_n(|\rho_n|^{\frac{1}{n}}) < r < \liminf_n(|\rho_n|^{\frac{1}{n}})^{\frac{1}{2}},$$

and denote $T = rI$. Thus, using item (ii) of Proposition 1.4.10, we can see that since for every $k \geq 2$ we have

$$r(T^k) \leq r(T^2) < \liminf_n(|\rho_n|^{\frac{1}{n}}) < r(T),$$

T doesn't lie in $\bigcup_{r>0} C_{(r\rho_n)}$ whereas T^k does.
- (iv) If $T^k \in \bigcup_{r>0} C_{(r\rho_n)}$, then $r(T^k) < 1$ according to item (iii) of Proposition 1.4.10. This implies that $r(T) < 1$, which implies in turn that $T \in \bigcup_{r>0} C_{(r\rho_n)}$. □

Remark 1.4.13. — As the classes $C_{(r\rho_n)}$ are increasing for the inclusion of sets, the assertion $T \in \bigcup_{r>0} C_{(r\rho_n)}$ is equivalent to the existence of $R > 0$ such that $T \in C_{(r\rho_n)}$ for every $r \geq R$. When $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 1$ and $I \in \bigcup_{r>0} C_{(r\rho_n)}$, which is the case when $\rho_n = \rho > 0$, we do not know if the result of Găvruţa [Găv08] stays true, as the type of operators he used in his proof is not suited in this setting: since there are sequences (ρ_n) such that $\bigcup_{r>0} C_{(r\rho_n)}$ contains all power-bounded operators (see Corollary 1.4.18), taking a T such that $T^k = I$ will not work.

Example 1.4.14. — For $\rho_n = 2(n!)$, we have $I + f_{(\rho_n)}(zT) = exp(zT)$, and a quick computation gives $w_{(2(n!))}(I) = \frac{2}{\pi} < 1$ (see item (iv) of Example 1.4.20 for another proof). Therefore $\frac{\pi}{2}I \in C_{(2(n!))}(H)$ and this class contains an operator not similar to a contraction.

We can also try to obtain some relationships between the $(\gamma_n\rho_n)$-radii of an operator, for sequences $(\gamma_n)_n \in \partial\mathbb{D}^{\mathbb{N}^*}$, in order to see for which sequences $(\gamma_n)_n$ the maximal or minimal radii are attained. The following Lemma answers the question for the maximal radii when $T = I$.

Lemma 1.4.15. — *Let $(\rho_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\alpha = \liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$. If we have $\lim_{x\to\alpha^-} f_{(|\rho_n|)}(x) > 1$, then $f_{(|\rho_n|)}(x) = 1$ has a unique solution, r_1, on $]0, \alpha[$. Otherwise, denote $r_1 = \alpha$. We then have:*

(i) $w_{(-|\rho_n|)}(I) = \frac{1}{r_1}$;

(ii) $w_{(-|\rho_n|)}(I) \geq w_{(\gamma_n\rho_n)}(I) \geq \frac{1}{\alpha}$, *for any* $(\gamma_n)_n \in \partial\mathbb{D}^{\mathbb{N}^*}$;

(iii) The condition

$$w_{(r\gamma_n\rho_n)}(I) = \frac{1}{\alpha},\ \forall r \geq 1,\ \forall \gamma_n \in \partial\mathbb{D}$$

is equivalent to

$$\lim_{x\to\alpha^-} f_{(|\rho_n|)}(x) \leq 1$$

and to

$$w_{(-|\rho_n|)}(I) = \frac{1}{\alpha}.$$

Proof. - $(i),(ii)$ The right-hand side inequality of (ii) is the last inequality of Lemma 1.2.12. For any $z \in \mathbb{D}(0,\alpha)$ and $\gamma_n \in \partial\mathbb{D}$, we have

$$|f_{(\gamma_n\rho_n)}(z)| \leq \sum_{n\geq 1} \frac{2|z|^n}{|\rho_n|} = f_{(|\rho_n|)}(|z|).$$

Also, the map $x \mapsto f_{(|\rho_n|)}(x)$ is strictly increasing on $]0,\alpha[$, as $f_{(|\rho_n|)}$ is non-constant with positive Taylor coefficients, so if $\lim_{x\to\alpha^-} f_{(|\rho_n|)}(x) > 1$ the real number r_1 is indeed unique. Let $u > 0$ be such that $u \geq \frac{1}{r_1} \geq \frac{1}{\alpha}$. Then $\frac{1}{u} \leq r_1$ and

$$\lim_{x\to\frac{1}{u}^-} f_{|\rho_n|}(x) \leq 1.$$

Since we have

$$f_{(\gamma_n\rho_n)}(\mathbb{D}(0,\frac{1}{u})) \subset \mathbb{D}(0, \lim_{x\to\frac{1}{u}^-} f_{(|\rho_n|)}(x)) \subset \mathbb{D},$$

Proposition 1.2.18 implies that $w_{(\gamma_n\rho_n)}(I) \leq u$ As this is true for every $u \geq \frac{1}{r_1}$, we obtain $w_{(\gamma_n\rho_n)}(I) \leq \frac{1}{r_1}$. When $\gamma_n = -\frac{\overline{\rho_n}}{|\rho_n|}$, we have

$$f_{(\gamma_n\rho_n)}(x) = f_{(-|\rho_n|)}(x) = -f_{(|\rho_n|)}(x).$$

Thus, the negative number $\lim_{x\to\frac{1}{u}^-}(-f_{(|\rho_n|)}(x))$ lies in the closure of $f_{(-|\rho_n|)}(\mathbb{D}(0,\frac{1}{u}))$, and the smallest $u \geq \frac{1}{\alpha}$ such that

$$f_{(-|\rho_n|)}(\mathbb{D}(0,\frac{1}{u})) \subset Re_{\geq -1}$$

is $\frac{1}{r_1}$. Hence,

$$w_{(-|\rho_n|)}(I) = \frac{1}{r_1} \geq w_{(\gamma_n\rho_n)}(I).$$

- (iii) By (ii) and using that $r \mapsto w_{(r\gamma_n\rho_n)}(I)$ is decreasing, we have

$$w_{(r\gamma_n\rho_n)}(I) = \frac{1}{\alpha},\ \forall r \geq 1,\ \forall(\gamma_n)_n \in \partial\mathbb{D}^{\mathbb{N}^*}$$

if and only if

$$w_{(-|\rho_n|)}(I) = \frac{1}{\alpha}.$$

This equation is equivalent to $r_1 = \alpha$, that is $\lim_{x\to\alpha^-}(f_{(|\rho_n|)}(x)) \leq 1$. □

We do not know if the $(|\rho_n|)$-radius of I is always the minimal one.
The idea of the proof of Lemma 1.4.15 can be transported to any operator T if we add a summability condition to the sequence $(\rho_n)_n$.

Proposition 1.4.16. — *Let $a = (a_n)_n \in (\mathbb{C}^*)^{\mathbb{N}^*}$ be such that $\sum_{n\geq 1} \frac{1}{|a_n|} \leq 1$. Let $T \in \mathcal{L}(H)$ and define*

$$\rho_n := \begin{cases} 2a_n\|T^n\| & \text{if } T^n \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

(i) If $r(T) > 0$ or if T is nilpotent, then $T \in C_{(\rho_n)}$.

(ii) *If* $r(T) > 0$ *and* $\liminf_n(|a_n|^{\frac{1}{n}}) = 1$*, then* $w_{(z_n\rho_n)}(T) = 1$*, for all* z_n *such that* $|z_n| \geq 1$ *and* $\lim_n(|z_n|^{\frac{1}{n}}) = 1$*.*

Proof. - (i) Suppose first that $r(T) > 0$. Since $\sum_n \frac{1}{|a_n|} < +\infty$, we have $\liminf_n(|a_n|^{\frac{1}{n}}) \geq 1$, thus $\liminf_n(|\rho_n|^{\frac{1}{n}}) \geq r(T) > 0$. We also have:

$$\|f_{(\rho_n)}(zT)\| \leq \sum_{n\geq 1} \frac{2|z|^n\|T^n\|}{2|a_n|\|T^n\|} \leq \sum_{n\geq 1} \frac{1}{|a_n|} \leq 1.$$

Thus, $I+\mathrm{Re}(f_{(\rho_n)}(zT)) \geq (1-\|f_{(\rho_n)}(zT)\|)I \geq 0$ for all $z \in \mathbb{D}$, so $T \in C_{(\rho_n)}$. If T is nilpotent then $f_{(\rho_n)}(zT)$ becomes a finite sum and the same computation gives the result, as $\liminf_n(|\rho_n|^{\frac{1}{n}}) > 0$.
- (ii) When $r(T) > 0$ and $\liminf_n(|a_n|^{\frac{1}{n}}) = 1$, we have $r(T) = \liminf_n(|\rho_n|^{\frac{1}{n}})$, so

$$1 \geq w_{(\rho_n)}(T) \geq \frac{r(T)}{\liminf_n(|\rho_n|^{\frac{1}{n}})} = 1.$$

Thus $w_{(\rho_n)}(T) = 1$. If we multiply each a_n by a complex number z_n with $|z_n| \geq 1$ and $\lim_n(|z_n|^{\frac{1}{n}}) = 1$, the sum $\sum_{n\geq 1} \frac{1}{|z_n a_n|}$ decreases, while $\liminf_n(|z_n a_n|^{\frac{1}{n}}) = 1$. Thus, we can apply the previous result to $(z_n\rho_n)_n$ and obtain $w_{(z_n\rho_n)}(T) = 1$. □

Remark 1.4.17. — For any T with $r(T) > 0$, if we take a sequence $(\rho_n)_n$ as in item (ii) of the previous Proposition, then the result says that $z \mapsto w_{(z\rho_n)}(T)$ is constant and equal to 1 on $\mathbb{C} \setminus \mathbb{D}$.
- The choice of $(\rho_n)_n$ only depends on $\|T^n\|$. For example, with any T normal with $\|T\| = 1$, by taking $a_n = \frac{\pi^2}{6}n^2$, we have $w_{(2a_n z_n)}(T) = 1$ for any sequence $(z_n)_n$ such that $1 \leq |z_n|$ and $\sup|z_n| < +\infty$.
- If T is quasi-nilpotent but not nilpotent, we have $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 0$. However, the statement of item (i) holds true for such a T, with a very similar proof.

Using the ideas in the proof of Proposition 1.4.16, we can show that some sets $\bigcup_{r>0} C_{(r\rho_n)}$ largely differ from $\bigcup_{\rho>0} C_{(\rho)}$ or $\{T\colon\ r(T) < 1\}$ even if $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 1$.

Corollary 1.4.18. — *Let* $(\rho_n)_n$ *be such that* $\liminf_n(|\rho_n|^{\frac{1}{n}}) = 1$*. The following assertions are true:*

(i) *If* $(\frac{1}{\rho_n}) \in \ell^1$*, then* $\bigcup_{r>0} C_{(r\rho_n)}$ *contains all power-bounded operators;*

(ii) *If* $f_{(\rho_n)} \in H^\infty(\mathbb{D})$ *and* $f'_{(\rho_n)} \in H^\infty(\mathbb{D})$*, then* $\bigcup_{r>0} C_{(r\rho_n)}$ *contains an operator that is not power-bounded.*

(iii) *If* $n^{k+1+\epsilon} = O(|\rho_n|)$ *for* $k \in \mathbb{N}^*$ *and some* $\epsilon > 0$*, then* $\bigcup_{r>0} C_{(r\rho_n)}$ *contains all operators* T *such that* $\|T^n\| = O(n^k)$*.*

Proof. - (i) Let T be a power-bounded operator with $\|T^n\| \leq C$. Let $r > 0$ and $z \in \mathbb{D}$. We have

$$\|f_{(r\rho_n)}(zT)\| \leq \sum_{n\geq 1} \frac{2}{r|\rho_n|}|z|^n\|T^n\| \leq \frac{2C}{r}\sum_{n\geq 1} \frac{1}{|\rho_n|} < +\infty.$$

Hence, for r large enough, we have $\|f_{(r\rho_n)}(zT)\| \leq 1$ for every $z \in \mathbb{D}$. This implies that

$$I + \mathrm{Re}(f_{(r\rho_n)}(zT) \geq 0, \, \forall z \in \mathbb{D}.$$

This in turn implies that $T \in C_{(r\rho_n)}$ since we also know that $r(T) \leq 1 = \liminf_n(|r\rho_n|^{\frac{1}{n}})$.
- (ii) We first note that both the entire series $f_{(\frac{\rho_n}{n})}(z) = \sum_{n\geq 1} \frac{2n}{\rho_n} z^n$ and $f_{(\rho_n)}$ have radii of convergence 1, so their sum is analytic on $\mathbb{D}$. We also have $f_{(\frac{\rho_n}{n})}(z) = zf'_{(\rho_n)}(z)$. Let N be a nilpotent operator of order 2 and set $T = I + N$. Since $T^n = I + nN$, we have $\|T^n\| \simeq n\|N\|$ so T is not power-bounded. We will show that T belongs to a class $C_{(r\rho_n)}$ for large enough $r > 0$. Let $r > 0$ and $z \in \mathbb{D}$. We have:

$$\begin{aligned}\|f_{(r\rho_n)}(zT)\| = \|\sum_{n\geq 1} \frac{2}{r\rho_n} z^n (I + nN)\| &= \|\frac{1}{r} f_{(\rho_n)}(z) I + \frac{1}{r} z f'_{(\rho_n)}(z) N\| \\ &\leq \frac{1}{r}(\|f_{(\rho_n)}\|_{H^\infty} + \|f'_{(\rho_n)}\|_{H^\infty}\|N\|) < +\infty.\end{aligned}$$

Hence, for r large enough, we have $\|f_{(r\rho_n)}(zT)\| \leq 1$ for every $z \in \mathbb{D}$, which implies that

$$I + \mathrm{Re}(f_{(r\rho_n)}(zT) \geq 0, \, \forall z \in \mathbb{D}.$$

This in turn implies that $T \in C_{(r\rho_n)}$ since we also know that $r(T) = 1 = \liminf_n(|r\rho_n|^{\frac{1}{n}})$.
- (iii) Let T be such that $\|T^n\| = O(n^k)$ and let $z \in \mathbb{D}$. We have $\frac{\|T^n\|}{|\rho_n|} = O(\frac{1}{n^{1+\epsilon}})$, so this sequence is in ℓ^1. If T is nilpotent, then T is power-bounded and we can apply (i) to get a positive $r > 0$ such that $T \in C_{(r\rho_n)}$. Otherwise, we can consider the complex numbers

$$a_n := \frac{\rho_n}{\|T^n\|} \left\| \left(\frac{\|T^n\|}{|\rho_n|} \right)_n \right\|_{\ell^1}.$$

We have

$$\sum_{n\geq 1} \frac{1}{|a_n|} = \left\| \left(\frac{\|T^n\|}{|\rho_n|} \right)_n \right\|_{\ell^1}^{-1} \sum_{n\geq 1} \frac{\|T^n\|}{|\rho_n|} = 1.$$

Thus, for $\tau_n := 2a_n\|T^n\|$, we can use Proposition 1.4.16 to obtain $T \in C_{(\tau_n)}$. Since $\tau_n = 2\rho_n \left\| \left(\frac{\|T^n\|}{|\rho_n|} \right)_n \right\|_{\ell^1}$, we have $\tau_n = r\rho_n$ for some $r > 0$, which concludes the proof. □

The condition $f'_{(\rho_n)} \in H^\infty(\mathbb{D})$ implies that the sequence $(\frac{n}{\rho_n})_n$ is bounded, but it does not imply the condition $(\frac{1}{\rho_n}) \in \ell^1$ from (i). Thus, for a sequence (ρ_n) satisfying the conditions of item (ii), the set of the power-bounded operators may not be fully included in $\bigcup_{r>0} C_{(r\rho_n)}$.

1.4.C Some examples.

We conclude this chapter by providing a computation of $w_{(z\rho_n)}(I)$ in two examples, where sequences $(\rho_n)_n$ were chosen to match some common analytic maps. The difficulty lies in the computation of the boundary of $f_{(z\rho_n)}(\mathbb{D}(1, \frac{1}{u}))$, as some specific points on the boundary do not always have an explicit expression.

Example 1.4.19. — Let $R > 0$ and $-\pi < t \leq \pi$. We have:

(i) $I + f_{(Re^{it}n)}(zI) = I - \frac{2}{Re^{it}} \log(1 - zI)$;

(ii) $w_{(Re^{it}n)}(I) = 1$ if $t = 0$ and $R \geq 2\log(2)$;

(iii) $w_{(Re^{it}n)}(I) = \frac{1}{\exp(\frac{R}{2})-1} > 1$ if $t = 0$ and $0 < R < 2\log(2)$;

(iv) $w_{(Re^{it}n)}(I) = \frac{1}{1-\exp(\frac{-R}{2})} > 1$ if $t = \pi$;

(v) $w_{(Re^{it}n)}(I) = 1$ if $t = \pm\frac{\pi}{2}$ and $R \geq \pi$;

(vi) $w_{(Re^{it}n)}(I) = \frac{1}{\sin(\frac{R}{2})}$ if $t = \pm\frac{\pi}{2}$ and $0 < R < \pi$;

(vii) $w_{(Re^{it}n)}(I) = 1$ if $0 < |t| < \frac{\pi}{2}$ and $R \geq 2\cos(t)\log(2\cos(t)) + 2\sin(t)t$;

(viii) If we have $0 < |t| < \frac{\pi}{2}$ and $0 < R < 2\cos(t)\log(2\cos(t)) + 2\sin(t)t$, then

$$w_{(Re^{it}n)}(I) = \inf(\{u > 1\colon\ 1 - \frac{2}{R} g_t(u) \geq 0\} > 1$$

with $g_t(u) := \cos(t)\log(\frac{\sqrt{u^2-\sin(t)^2}+\cos(t)}{u}) + \arcsin(\frac{\sin(t)}{u})\sin(t)$.
The same holds if $\frac{\pi}{2} < |t| < \pi$.

Proof. Let $R > 0$, $t \in]-\pi, \pi]$. As $n \in \mathbb{R}$, we have $w_{(Re^{-it}n)}(I) = w_{(Re^{it}n)}(I)$, so we can restrict the study to $t \in [0, \pi]$. A direct computation gives:

$$f_{(Re^{it}n)}(zT) = -\frac{2}{Re^{it}} \log(1 - zT).$$

As $\liminf_n(|n|^{\frac{1}{n}}) = 1$, we have $w_{(Re^{it}n)}(I) \geq 1$. Thus, we consider those $u > 1$ such that $I + \mathrm{Re}(f_{(R.e^{it}n)}(\frac{zI}{u}))$ is positive for every $z \in \mathbb{D}$. It is equivalent to look at the positivity of

$$1 + \mathrm{Re}(f_{(R.e^{it}n)}(\frac{z}{u})) = 1 - \frac{2}{R}\mathrm{Re}(e^{-it}\log(1 - \frac{z}{u})).$$

We start off by studying the boundary of $\log(\mathbb{D}(1, \frac{1}{u}))$. By analyticity, we have $\partial\log(\mathbb{D}(1, \frac{1}{u})) \subset \log(\partial\mathbb{D}(1, \frac{1}{u}))$. As $\log(e^{is}\mathbb{R}\cap\mathbb{D}(1, \frac{1}{u}))$ is a horizontal interval that is non-empty if and only if $|s| \leq \arcsin(\frac{1}{u})$, the previous sets are equal and $\log(\mathbb{D}(1, \frac{1}{u}))$ is convex. Thus, the set $\log(\partial\mathbb{D}(1, \frac{1}{u}))$ can be parametrized by two arcs depending on the imaginary part of its elements:

$$s \mapsto \log\left(\cos(s) \pm \frac{1}{u}\sqrt{1 - \sin(s)^2u^2}\right) + is,\ s \in [-\arcsin(\frac{1}{u}); \arcsin(\frac{1}{u})].$$

We want to compute the minimum of $1 - \frac{2}{R}\mathrm{Re}(e^{-it}\log(1 - \frac{e^{is}}{u}))$ in order to find for which $u > 1$ this minimum is non-negative. For the cases $t = 0$, $t = \pi$, and $t = \frac{\pi}{2}$, computing this minimum amounts to finding the extrema of the real or imaginary part of the elements in $\log(\partial\mathbb{D}(1, \frac{1}{u}))$. As these extrema are $\log(1 \pm \frac{1}{u})$ for the real part and $\pm\arcsin(\frac{1}{u})$ for the imaginary part, an easy computation gives all the $u > 1$ such that $\inf_{w\in\mathbb{R}}(1 - \frac{2}{R}\mathrm{Re}(e^{-it}\log(1 - \frac{e^{iw}}{u}))) \geq 0$ in all three cases, which proves the items $(ii), (iii), (iv), (v), (vi)$.
For $0 < t < \frac{\pi}{2}$, computing this minimum leads to searching the lower bound of

$$f_1(s) := \cos(\pi - t)\log(\cos(s) - \frac{1}{u}\sqrt{1 - \sin(s)^2u^2}) - s\sin(\pi - t).$$

For $\frac{\pi}{2} < t < \pi$, computing this minimum leads to searching the lower bound of

$$f_2(s) := \cos(\pi - t)\log(\cos(s) + \frac{1}{u}\sqrt{1 - \sin(s)^2 u^2}) - s\sin(\pi - t).$$

The derivatives of these maps are:

$$f_1'(s) = \frac{\sin(s)u\cos(\pi - t)}{\sqrt{1 - \sin(s)^2 u^2}} - \sin(\pi - t),\ f_2'(s) = -\frac{\sin(s)u\cos(\pi - t)}{\sqrt{1 - \sin(s)^2 u^2}} - \sin(\pi - t).$$

Both of them only have one zero, at $s = -\arcsin(\frac{\sin(t)}{u})$. And in both cases the searched minimum for $1 - \frac{2}{R}\mathrm{Re}(e^{-it}\log(1 - \frac{e^{is}}{u}))$ is:

$$1 - \frac{2}{R}[\cos(t)\log(\frac{\sqrt{u^2 - sin(t)^2} + \cos(t)}{u}) + \arcsin(\frac{\sin(t)}{u})\sin(t)] = 1 - \frac{2}{R}g_t(u).$$

If $0 < t < \frac{\pi}{2}$, this minimum decreases towards $1 - \frac{2}{R}g_t(1) := 1 - \frac{2}{R}[\cos(t)\log(2\cos(t)) + t\sin(t)]$ when $u \to 1^+$. So $\frac{I}{u} \in C_{(Re^{it}n)}$ for every $u > 1$ if and only if $1 - \frac{2}{R}g_t(1) \geq 0$, that is $R \geq 2g_t(1)$. This proves item (vii) and half of item $(viii)$.
If $\frac{\pi}{2} < t < \pi$, this minimum decreases towards $-\infty$ as $u \to 1^+$, so the smallest u for which this minimum is non-negative satisfies $u > 1$ and $w_{(Re^{it}n)}(I) = u$. This gives the other half of item $(viii)$ and concludes the proof. □

Example 1.4.20. — Let $R > 0$ and $-\pi < t \leq \pi$. We have:

(i) $I + f_{(Re^{it}n!)}(zI) = I + \frac{2}{R.e^{it}}(\exp(zI) - I)$;

(ii) $w_{(Re^{it}n!)}(I) = \frac{1}{\log(\frac{R}{2}+1)}$ if $t = \pi$;

(iii) $w_{(Re^{it}n!)}(I) = \frac{1}{\log(\frac{2}{2-R})}$ if $t = 0$ and $0 < R \leq 2 - \frac{2}{e}$;

(iv) $w_{(Re^{it}n!)}(I) = \frac{1}{\frac{\pi}{2}-t}$ if $0 \leq |t| < \frac{\pi}{2}$ and $R = 2\cos(t)$;

(v) $w_{(Re^{it}n!)}(I) \leq \frac{1}{\log(\frac{R}{2}-\cos(t))}$ for $R > 2 + 2\cos(t)$;

(vi) $w_{(Re^{it}n!)}(I) \geq \frac{1}{\sqrt{\pi^2+\log(\frac{R}{2\cos(t)}-1)^2}}$ if $0 \leq |t| < \frac{\pi}{2}$ and $R > 4\cos(t)$;

(vii) In general, we have

$$w_{(Re^{it}.n!)_n}(I) = \inf(\{u > 0\colon \forall\theta \in [-\pi, \pi] \text{ with } \theta + \frac{\sin(\theta)}{u} = t + k\pi,$$
$$k \in \mathbb{Z}, \text{ we have } (-1)^k e^{\frac{\cos(\theta)}{u}}\cos(\theta) \geq \cos(t) - \frac{R}{2}\}).$$

For $R \geq 2e^{\pi/2} - 2$, we can restrict the infimum after u in $]0, \frac{2}{\pi}]$ and to the smallest $\theta \in]\frac{-\pi}{2}, 0]$ such that $\theta + \frac{\sin(\theta)}{u} = t + k\pi$.

Proof. Let $R>0$, $t\in[-\pi,\pi]$ and $u>0$. As $n\in\mathbb{R}$, we have $w_{(Re^{-it}n!)}(I)=w_{(Re^{it}n!)}(I)$, so we restrict the study to $t\in[0,\pi]$. A computation gives

$$I+f_{(Re^{it}n!)}(zI)=I+\frac{2}{R.e^{it}}(\exp(zI)-I).$$

We will first use Lemma 1.4.15 to compute $w_{(-Rn!)}(I)$ and rule out the case $t=\pi$. As $f_{(Rn!)}(x)=\frac{2}{R}(\exp(x)-1)$, we get

$$f_{(Rn!)}(x)=1\Leftrightarrow x=\log(\frac{R}{2}+1).$$

Hence, $w_{(-Rn!)}(I)=\frac{1}{\log(\frac{R}{2}+1)}$ and item (ii) is proved.

As $\liminf_n(|n!|^{\frac{1}{n}})=+\infty$, we have $\frac{I}{u}\in C_{(Re^{it}n!)}$ if and only if $u\geq w_{(Re^{it}n!)}(I)$, if and only if $I+\mathrm{Re}(f_{(Re^{it}n!)}(z\frac{I}{u}))$ for every $z\in\mathbb{D}$. Thus, we need to study the positivity of

$$1+\mathrm{Re}(f_{(Re^{it}n!)}(\frac{z}{u}))=1+\frac{2}{R}\mathrm{Re}(e^{-it}(\exp(\frac{z}{u})-1)),$$

for every $z\in\mathbb{D}$ and for any $u>0$. By analyticity, we only need to make the computations for $z\in\partial\mathbb{D}$. We have

$$\begin{aligned}&1+\frac{2}{R}\mathrm{Re}(\exp(\frac{z}{u}-it)-e^{-it})\geq 0\\ \Leftrightarrow\ &\exp(\mathrm{Re}(\frac{z}{u}))\cos\left(\frac{\mathrm{Im}(z)}{u}-t\right)\geq-\frac{R}{2}+\cos(t).\end{aligned}$$

Denote, for $s\in[-\pi,\pi]$,

$$g_u(s):=e^{\frac{\cos(s)}{u}}\cos(t-\frac{\sin(s)}{u}).$$

Thus, $\frac{I}{u}\in C_{(Re^{it}n!)}$ is equivalent to

$$\min_{s\in[-\pi,\pi]}(g_u(s))\geq-\frac{R}{2}+\cos(t).$$

Therefore, this inequality will be satisfied if and only if $u\geq w_{(Re^{it}n!)}(I)$. Also, since

$$\min_s(g_u(s))=\min_{|w|=\frac{1}{u}}(\mathrm{Re}(\exp(w-it)))=\min_{|w|<\frac{1}{u}}(\mathrm{Re}(\exp(w-it))),$$

we can see that $\min_s(g_u(s))$ is the minimum of a harmonic non-constant map over the disc $\mathbb{D}(0,\frac{1}{u})$. The maximum principle implies that the map $u\mapsto\min_s(g_u(s))$ is strictly increasing. Hence, $w_{(Re^{it}n!)}(I)$ is the only number $u>0$ such that $\min_{s\in[-\pi,\pi]}(g_u(s))=-\frac{R}{2}+\cos(t)$.

Let us focus now on the minimum of g_u. The derivative of g_u is

$$g_u'(s)=\frac{1}{u}e^{\frac{\cos(s)}{u}}\sin(t-\frac{\sin(s)}{u}-s).$$

Hence, the minimum of g_u will be reached for a s_0 such that $h_u(s_0):=t-s_0-\frac{\sin(s_0)}{u}=k\pi$, for some $k\in\mathbb{Z}$. For such a s_0, we will also have

$$g_u(s_0)=(-1)^k e^{\frac{\cos(s_0)}{u}}\cos(s_0).$$

If $u \geq 1$, the map h_u is strictly decreasing, with range $[t-\pi, t+\pi]$. Hence, there will only be 2 (resp. 3) values of s such that $h_u(s) = k\pi$ if $t \in]0, \pi[$ (resp. $t = 0$). If $t = 0$ and $u \geq 1$, these values of s will be $-\pi$, 0, π, and the minimum of g_u will be $g_u(\pi) = \exp(\frac{-1}{u})$. Thus, if $t = 0$ and $w_{(Rn!)}(I) \geq 1$, we will have

$$\exp\left(\frac{-1}{w_{(Rn!)}(I)}\right) = -\frac{R}{2} + 1,$$

which is equivalent to $0 < R \leq 2 - \frac{2}{e}$. Thus $w_{(Rn!)}(I) = \frac{1}{\log(\frac{2}{2-R})}$, proving item (iii).

When $t \in]0, \pi[$ and $u \geq 1$, we have however no explicit formula for the two values of s mentioned above.

For $t \in [0, \frac{\pi}{2}[$ and $R = 2\cos(t)$, we will have $\min_s(g_{w_{(Re^{it}n!)}(I)}(s)) = 0$. As $e^{\frac{\cos(s)}{u}}\cos(s) = 0$ if and only if $s = \pm\frac{\pi}{2}$, this minimum will be attained at $\frac{\pi}{2}$ or $\frac{-\pi}{2}$, and $w_{(Re^{it}n!)}(I)$ will be the largest $u > 0$ such that $g_u(\frac{\pi}{2}) = 0$ or $g_u(-\frac{\pi}{2}) = 0$. The latter condition is equivalent to $\frac{1}{u} \pm t = \frac{\pi}{2} + k\pi$, that is $\frac{1}{u} = \frac{\pi}{2} \pm t + k\pi$. Since we have $0 \leq t < \frac{\pi}{2}$, the integer k needs to be positive. By looking at the smallest possible value for $\frac{1}{u}$ we get $w_{(Re^{it}n!)}(I) = \frac{1}{\frac{\pi}{2} - t}$, proving item (iv).

In general, we can see that $\min_s(g_u(s)) \geq -e^{\frac{1}{u}}$. When $R > 2 + 2\cos(t)$, the inequality $-e^{\frac{1}{u}} \geq \cos(t) - \frac{R}{2}$ is equivalent to $u \geq \frac{1}{\log(\frac{R}{2} - \cos(t))}$, which proves item (v).

If $u\pi < 1$, we have $u\pi = \sin(\alpha)$ for some $\alpha > 0$, and $g_u(\alpha) = -\cos(t)e^{\frac{\sqrt{1-u^2\pi^2}}{u}}$. When $R > 4\cos(t)$, the inequality $g_u(\alpha) \leq \cos(t) - \frac{R}{2}$ is equivalent to $u \leq \frac{1}{\sqrt{\pi^2 + \log(\frac{R}{2\cos(t)} - 1)^2}}$. Item (vi) is now proved.

Taking $R \geq 2e^{\pi/2} - 2 = R_0$, we get $-\frac{R}{2} + 1 \leq 2 - e^{\pi/2} < -1$ and

$$w_{(Re^{it}n!)}(I) \leq w_{(-Rn!)}(I) \leq w_{(-R_0 n!)}(I) = \frac{2}{\pi},$$

for every $0 \leq t \leq \pi$, according to Lemma 1.4.15. We can then see that for $u = w_{(Re^{it}n!)}(I)$ and for a number s_0 such that $g_u(s_0) = \min_s(g_u(s))$ and $h_u(s_0) = k\pi$, the relationship

$$(-1)^k e^{\frac{\cos(s_0)}{u}} \cos(s_0) = g_u(s_0) = -\frac{R}{2} + \cos(t) < -1$$

implies that $\cos(s_0) > 0$ and that k is odd. In this case, $|s_0|$ will be the smallest real s in $[0, \frac{\pi}{2}[$ such that $h_u(s)$ or $h_u(-s)$ is equal to $k\pi$ with k odd. As we also have $h_u(\frac{-\pi}{2}) = t + \frac{\pi}{2} + \frac{1}{u} \geq \pi \geq t = h_u(0) \geq 0$, we can see that s_0 lies in $]\frac{-\pi}{2}, 0]$. This gives all the announced results. □

Chapter 2

Hilbert space operators with unitary skew-dilations: Classes C_A

Classes C_A have been defined by H. Langer ; see also [Sue98a]. A large part of the initial results can be found in [Sue98a], we recall them here as they form a family of useful tools required later on. A few improvements are given (see Propositions 2.2.12 and 2.3.1), as well as new results (from Lemma 2.3.3 through Proposition 2.3.9).

2.1 Definition and Suen's Results

Definition 2.1.1. [Class C_A] Let H be a Hilbert space. Let $A \in \mathcal{L}(H)$ be a self-adjoint positive operator that is invertible. We define the class $C_A(H)$ as

$$C_A(H) := \{T \in \mathcal{L}(H)\colon \exists K \text{ Hilbert and } U \in \mathcal{L}(K) \text{ unitary such that } H \subset K \text{ and } A^{-\frac{1}{2}}T^nA^{-\frac{1}{2}} = P_HU^n|_H,\ \forall n \geq 1\}$$

where P_H is the orthogonal projection onto H. When the underlying Hilbert space H is not ambiguous, classes $C_A(H)$ will be abbreviated as C_A.

Remark 2.1.2. For the rest of this chapter, we will use the notation $A > 0$ to design a self-adjoint positive operator that is invertible (i.e. $A \geq \lambda I$ for some $\lambda > 0$).
By taking $\rho > 0$ and $A = \rho I$, we obtain the classical definition of the class C_ρ from Sz. Nagy and Foias [SNF66]. Hence, $C_{\rho I} = C_\rho$.
We could also define a more general class $\widetilde{C}_B$ for any invertible operator B by using $B^{-1}T^n(B^*)^{-1}$ instead of $A^{-\frac{1}{2}}T^nA^{-\frac{1}{2}}$ in its definition. However we can mimic the proof of the equivalence $(i) \Leftrightarrow (iii)$ in Proposition 2.1.4 in order to obtain the equivalence:

- $T \in \widetilde{C}_B$:
- $r(T) \leq 1$ and $BB^* + \mathrm{Re}(2\sum_{n\geq 1}(zT)^n) \geq 0$, $\forall z \in \mathbb{D}$.

Using Proposition 2.1.4 then gives $\widetilde{C}_B = C_{B.B^*}$. Hence this construction does not bring any new class regarding the case $A > 0$.

Remark 2.1.3. As A is self-adjoint, we can see from its definition that the class C_A is stable for the adjoint map: $T \in C_A \Leftrightarrow T^* \in C_A$.
We can also see that for any $T \in C_A$, we have $T^k \in C_A$ for every $k \geq 1$, as

$$A^{-\frac{1}{2}}(T^k)^n A^{-\frac{1}{2}} = P_H.(U^k)^n|_H.$$

Proposition 2.1.4. *Let H be a Hilbert space. Let $A, T \in \mathcal{L}(H)$ be such that $A > 0$. The following are equivalent*

(i) $T \in C_A(H)$;

(ii) $r(T) \leq 1$ *and* $I + Re(2\sum_{n\geq 1} A^{-\frac{1}{2}}(zT)^n A^{-\frac{1}{2}}) \geq 0$, $\forall z \in \mathbb{D}$;

(iii) $r(T) \leq 1$ *and* $A + Re(2\sum_{n\geq 1}(zT)^n) \geq 0$, $\forall z \in \mathbb{D}$;

(iv) $r(T) \leq 1$ *and* $A - 2Re(z(A-I)T) + |z|^2 T^*(A-2I)T \geq 0$, $\forall z \in \mathbb{D}$;

(v) Let $S := Span(\{z \mapsto z^n, z \mapsto \bar{z}^n, n \geq 0\}) \subset C^0(\partial\mathbb{D})$. *Let* $\phi : S \to \mathcal{L}(H)$ *be the linear map satisfying* $\phi(1) = A$, $\phi(z \mapsto z^n) = T^n$, $\phi(z \mapsto \bar{z}^n) = (T^*)^n$ $\forall n \geq 1$.
Then ϕ is positive: For every trigonometric polynomial $P \in S$ such that $P(w) \geq 0$ for all $w \in \partial\mathbb{D}$, we have $\phi(P) \geq 0$.

Proof. $(i) \Leftrightarrow (ii)$ We can mimic the proof of Proposition 1.2.6, that states a similar result for classes $C_{(\rho_n)}$, in order to obtain the result. The proof of the implication $(ii) \Rightarrow (i)$ relies mainly on Theorem 1.2.4.
- $(ii) \Leftrightarrow (iii)$ We recall that for $R, S \in \mathcal{L}(H)$ with S invertible, we have

$$\mathrm{Re}(R) \geq 0 \Leftrightarrow \mathrm{Re}(SRS^*) \geq 0.$$

We can then obtain the equivalence $(ii) \Leftrightarrow (iii)$ by choosing

$$R = I + 2\sum_{n\geq 1}(zT)^n,\ S = A^{\frac{1}{2}},$$

and by rearranging the expressions.
- $(iii) \Leftrightarrow (iv)$ Since $r(T) \leq 1$, $(I - zT)$ is invertible for every $z \in \mathbb{D}$. By using the previous property and the fact that A is self-adjoint, we get

$$\begin{aligned}
&\mathrm{Re}(A + 2\sum_{n\geq 1}(zT)^n) \geq 0 \\
\Leftrightarrow &\mathrm{Re}(A - 2I + 2(I - zT)^{-1}) \geq 0 \\
\Leftrightarrow &\mathrm{Re}((I - zT)^*(A - 2I)(I - zT) + 2(I - zT)^*) \geq 0 \\
\Leftrightarrow &(I - zT)^*(A - 2I)(I - zT) + 2\mathrm{Re}((I - zT)^*) \geq 0 \\
\Leftrightarrow &(A - 2I) - (zT)^*(A - 2I) - (A - 2I)zT + |z|^2 T^*(A - 2I)T + 2\mathrm{Re}(I - zT) \geq 0 \\
\Leftrightarrow &A - 2I - 2\mathrm{Re}((A - 2I)zT) + |z|^2 T^*(A - 2I)T + 2I - 2\mathrm{Re}(zT) \geq 0 \\
\Leftrightarrow &A - 2\mathrm{Re}(z(A - I)T) + |z|^2 T^*(A - 2I)T \geq 0 \\
\Leftrightarrow &A - 2\mathrm{Re}(z(A - I)T) + |z|^2 T^*(A - 2I)T \geq 0,
\end{aligned}$$

which proves the equivalence.
- $(i) \Rightarrow (v)$ Let $f_P \in S$ be a trigonometric polynomial such that $f_P(w) \geq 0$ for every $w \in \partial\mathbb{D}$. According to the Féjer-Riesz theorem, there exists $Q \in \mathbb{C}[X]$ such that $f_P(w) = Q(w)\overline{Q(w)}$ for every $w \in \partial\mathbb{D}$. Since T lies in C_A, we have a Hilbert space K containing H and an unitary operator $U \in \mathcal{L}(K)$ such that

$$T^n = A^{\frac{1}{2}} P_H . U^n|_H A^{\frac{1}{2}}, \text{ for every } n \geq 1.$$

This also gives us $(T^*)^n = A^{\frac{1}{2}} P_H.(U^*)^n|_H A^{\frac{1}{2}} = A^{\frac{1}{2}} P_H.(U^{-n})|_H A^{\frac{1}{2}}$ for every $n \geq 1$. Since we also have $A = A^{\frac{1}{2}} P_H(U^0)|_H A^{\frac{1}{2}}$, we can see that for every $n \in \mathbb{Z}$ we obtain

$$\phi(z \mapsto z^n) = A^{\frac{1}{2}} P_H.(U^n)|_H A^{\frac{1}{2}}.$$

Thus, we get

$$\phi(f_P) = A^{\frac{1}{2}} P_H f_P(U)|_H A^{\frac{1}{2}} = A^{\frac{1}{2}} P_H Q(U) Q(U)^*|_H A^{\frac{1}{2}},$$

as U is unitary. Such an operator is self-adjoint. For every $h \in H$ we have

$$\langle P_H Q(U) Q(U)^*|_H h, h\rangle = ||Q(U)^* h||^2 \geq 0,$$

so the self-adjoint operator $P_H Q(U) Q(U)^*|_H$ is also positive. Hence $\phi(f_P)$ is self-adjoint positive, so ϕ is positive.
- $(v) \Rightarrow (iii)$ The set S is a subspace of $C^0(\overline{\mathbb{D}})$ that is stable for the adjoint map $f \mapsto \overline{f}$. As the linear map $\phi : S \to \mathcal{L}(H)$ is positive, we can apply Proposition 2.1 of [Pau02] in order to obtain the continuity of ϕ for $\| \cdot \|_{L^\infty(\overline{\mathbb{D}})}$. Since S is dense in $C^0(\overline{\mathbb{D}})$ according to the Stone-Weierstrass Theorem, ϕ extends itself as a continuous linear map to $C^0(\overline{\mathbb{D}})$. We also have

$$\|\phi\| \leq 2\|\phi(1)\| = 2\|A\|,$$

according to [Pau02, Prop. 2.1]. As we also have $\phi(\overline{g}) = \overline{\phi(g)}$ for every $g \in S$, we obtain

$$\phi(\mathrm{Re}(f)) = \mathrm{Re}(\phi(f)), \text{ for every } f \in C^0(\overline{\mathbb{D}}).$$

Take $\lambda \in \mathbb{C}$, $|\lambda| > 1$ and $f(z) := \frac{1}{\lambda - z} - \lambda^{-1}$. Thus $f \in C^0(\overline{\mathbb{D}})$ and the power series decomposition of f in 0 converges uniformly to f on $\overline{\mathbb{D}}$. Denote $(S_n)_n$ the sequence of partial sums of this series. We have $S_n(z) = \sum_{k=1}^n \lambda^{-n-1} z^n$. By continuity of ϕ, $\phi(S_n)$ converges in $\mathcal{L}(H)$ towards $\phi(f)$. As $\phi(S_n) = \sum_{k=1}^n \lambda^{-n} T^n$, we obtain

$$\|\lambda^{-n-1} T^n\| \to_n 0.$$

Since this fact is true for every $|\lambda| > 1$, this implies that $r(T) \leq 1$. Now, let $z \in \mathbb{D}$ and denote $f_z(w) := \frac{1+zw}{1-zw}$. For every $w \in \overline{\mathbb{D}}$ $f_z(w)$ belongs to the right half-plane $Re_{\geq 0}$, so the map $\mathrm{Re}(f_z)$ is positive on $\partial\mathbb{D}$. Thus the map $\mathrm{Re}(\phi(f_z)) = \phi(\mathrm{Re}(f_z))$ is self-adjoint positive in $\mathcal{L}(H)$. On another hand, we have

$$f_z(w) = \frac{1+zw}{1-zw} = \frac{2}{1-zw} - 1 = 2\sum_{n\geq 1}(zw)^n + 1.$$

Since $r(T) \leq 1$, $|z| < 1$, and ϕ is continuous, we get $\phi(f_z) = 2\sum_{n\geq 1}(zT)^n + A$. Combining both facts yields $\mathrm{Re}(2\sum_{n\geq 1}(zT)^n + A) \geq 0$, which gives item (iii) and concludes the proof. □

Remark 2.1.5.

(i) We can see from condition (iii) of Proposition 2.1.4 that 0 belongs to the class C_A. Hence all classes C_A are non-empty.

(ii) The condition (iv) of Proposition 2.1.4 also allow us to apply Theorem 2.4 in Paulsen's book [Pau02] about positive maps over continuous functions on a compact Hausdorff space, in order to obtain

$$\|\phi\| = \|\phi(1)\| = \|A\|.$$

(iii) The linear map ϕ allows us to build a functional calculus over $C^0(\overline{\mathbb{D}})$ for any $T \in C_A$, with

$$f(T) := \phi(f) + (I - A)f(0) = \phi(f - f(0)) + f(0)I.$$

The map $f \mapsto f(T)$ is then well-defined and continuous. It is a morphism of $\mathbb{C}^*$-algebras on S that sends every polynomial map P onto $P(T)$. Thus it is also a morphism of $\mathbb{C}^*$-algebras on $C^0(\overline{\mathbb{D}})$ by continuity and by density of S.
For any f we then have

$$\|f(T)\| \leq \|A\| \cdot \|f\| + \|A - I\||f(0)| \leq (\|A\| + \|A - I\|)\|f\|.$$

If $f(0) = 0$ this inequality becomes $\|f(T)\| \leq \|A\|\|f\|$.

(iv) On the specific case $A = \rho I$, $\rho > 0$, it has been shown by [AO75, Thm.2] that we have in fact $\|f(T)\| \leq \rho\|f\|$ for every $f \in C^0(\overline{\mathbb{D}})$. However this proof relies on a specific factorization of operators in classes C_ρ, and we neither have a more elementary way to obtain such an upper bound nor a method to obtain a similar factorization result for operators in classes C_A. Hence, the best upper bound we can currently obtain for the norm of the functional calculus on operators in classes C_A is

$$\|f \mapsto f(T)\| \leq \|A\| + \|A - I\|.$$

It is the best result obtainable using the theory of positive maps (see [Pau02, Ch.2]) as we did.

Corollary 2.1.6. *Let H be a Hilbert space and let $A, B \in \mathcal{L}(H)$ with $A \geq B > 0$. Then $C_B \subset C_A$.*

Proof. Let $T \in C_B$. According to equivalence $(i) \Leftrightarrow (iii)$ of Proposition 2.1.4, we have $r(T) \leq 1$ and

$$B + \mathrm{Re}(2\sum_{n\geq 1}(zT)^n) \geq 0,\ \forall z \in \mathbb{D}.$$

Since $A \geq B$, we immediately obtain $r(T) \leq 1$ and

$$A + \mathrm{Re}(2\sum_{n\geq 1}(zT)^n) \geq 0,\ \forall z \in \mathbb{D},$$

hence $T \in C_A$. □

Hence, any class C_A contains the class $C_{\frac{1}{\|A^{-1}\|}}$.

Proposition 2.1.7. *Let $A, T \in \mathcal{L}(H)$ with $A > 0$. If $T \in C_A$ then T is similar to a contraction.*

Proof. Since $A \leq \|A\|I$, Corollary 2.1.6 tells us that T lies in $C_{\|A\|}$. As every operator in any class C_ρ is similar to a contraction, we obtain the desired result. □

Lemma 2.1.8. *Let H_1, H_2 be Hilbert spaces. Let $A, T \in \mathcal{L}(H_1)$, $B, S \in \mathcal{L}(H_2)$, be such that $A > 0, B > 0$. If $T \in C_A(H_1)$ and $S \in C_B(H_2)$, then $T \otimes S \in C_{A\otimes B}(H_1 \otimes H_2)$.*

Proof.
Let K_1 be a Hilbert space containing H_1, K_2 be a Hilbert space containing H_2, U an unitary operator acting on K_1, and V an unitary operator acting on K_2 such that:

$$T^n = A^{\frac{1}{2}} P_{H_1} U^n|_{H_1} A^{\frac{1}{2}},\ S^n = B^{\frac{1}{2}} P_{H_2} V^n|_{H_2} B^{\frac{1}{2}},,\ \forall n \geq 1.$$

Denote $W = U \otimes V$. Then W is an unitary operator acting on $K_1 \otimes K_2$ with $W^n = U^n \otimes V^n$, $\forall n \geq 1$. For every $k_1 \otimes k_2 \in K_1 \otimes K_2$, we also have:

$$P_{H_1\otimes H_2}(k_1 \otimes k_2) = (P_{H_1}(k_1)) \otimes (P_{H_2}(k_2)).$$

Thus, for every $h_1 \in H_1$, $h_2 \in H_2$, $n \geq 1$, we obtain

$$\begin{aligned}
&(A \otimes B)^{\frac{1}{2}} P_{H_1\otimes H_2} W^n|_{H_1\otimes H_2} (A \otimes B)^{\frac{1}{2}} (h_1 \otimes h_2) = (A \otimes B)^{\frac{1}{2}} P_{H_1\otimes H_2}[U^n A^{\frac{1}{2}}(h_1) \otimes V^n B^{\frac{1}{2}}(h_2)]\\
=&(A^{\frac{1}{2}} P_{H_1} U^n A^{\frac{1}{2}}(h_1)) \otimes (B^{\frac{1}{2}} P_{H_2} V^n B^{\frac{1}{2}}(h_2))] = (T^n(h_1)) \otimes (S^n(h_2))\\
=&(T \otimes S)^n(h_1 \otimes h_2).
\end{aligned}$$

Therefore, we have

$$(A \otimes B)^{-\frac{1}{2}} (T \otimes S)^n (A \otimes B)^{-\frac{1}{2}} = P_{H_1\otimes H_2} W^n|_{H_1\otimes H_2},$$

which concludes the proof. □

Proposition 2.1.9. *Let H be a Hilbert space. Let $A, T \in \mathcal{L}(H)$ be such that $A > 0$. The following are equivalent*

(i) $T \in C_A(H)$;

(ii) $\langle Ah, h\rangle - 2Re(\langle z(A - I)Th, h\rangle) + |z|^2\langle (A - 2I)Th, Th\rangle \geq 0$, $\forall z \in \mathbb{D}$, $\forall h \in H$;

(iii) $\langle Ah, h\rangle - 2r|\langle (A - I)Th, h\rangle| + r^2\langle (A - 2I)Th, Th\rangle \geq 0$, $\forall r \in [0, 1[$, $\forall h \in H$;

(iv) $P(A, z, T, n) := \begin{pmatrix} A & zT & \ldots & (zT)^n \\ (zT)^* & \ldots & & \ldots \\ \ldots & & \ldots & zT \\ ((zT)^*)^n & \ldots & (zT)^* & A \end{pmatrix}_{(n+1)\times(n+1)} \geq 0$, $\forall z \in \mathbb{D}$, $\forall n \geq 1$;

Proof. $(i) \Rightarrow (ii)$ By using the equivalence $(i) \Leftrightarrow (iv)$ in Proposition 2.1.4 we have $A - 2\mathrm{Re}(z(A - I)T) + |z|^2 T * (A - 2I)T \geq 0$, $\forall z \in \mathbb{D}$, which gives condition (ii).
- $(ii) \Rightarrow (i)$ With the previous argument, we only need to show that $r(T) \leq 1$. Using the proof of $(iii) \Leftrightarrow (iv)$ in Proposition 2.1.4, we can see that $A - 2\mathrm{Re}(z(A - I)T) + |z|^2 T * (A - 2I)T \geq 0$

is equivalent to $(I-zT)^*(A-2I)(I-zT)+2\mathrm{Re}((I-zT))\geq 0$. Since we have $\|A\|I\geq A$, we obtain

$$(\|A\|-2)(I-zT)^*(I-zT)+2\mathrm{Re}(I-zT)=(I-zT)^*(\|A\|I-2I)(I-zT)+2\mathrm{Re}(I-zT)\geq 0,\ \forall z\in\mathbb{D}.$$

Hence, item (iv) of Lemma 1.3.1 is satisfied for $\rho=\|A\|$, therefore we have $T\in C_{\|A\|}$ so $r(T)\leq 1$.
- $(ii)\Leftrightarrow(iii)$ Let $h\in H$ and let $r\in[0,1[$. As we have

$$\sup_{|z|=r}2\mathrm{Re}(\langle z(I-A)Th,h\rangle)=2r\mathrm{Re}(\langle (I-A)Th,h\rangle),$$

we obtain

$$\begin{aligned}&\langle Ah,h\rangle-2\mathrm{Re}(\langle z(I-A)Th,h\rangle)+|z|^2\langle (A-2I)Th,Th\rangle\geq 0,\ \forall z\colon |z|=r\\ \Leftrightarrow&\langle Ah,h\rangle-2r\mathrm{Re}(\langle (I-A)Th,h\rangle)+r^2\langle (A-2I)Th,Th\rangle\geq 0,\end{aligned}$$

therefore the two conditions are equivalent.
- $(i)\Rightarrow(iv)$ Let $n\geq 1$ and denote $H_1=H$, $H_2=\mathbb{C}^{n+1}$, $B=I_{n+1}$. Define $S=S_{n+1}\in\mathcal{L}(H_2)$ the left shift. As $\|S_{n+1}\|=1$, we have $S_{n+1}\in C_1(H_2)=C_{I_{n+1}}(H_2)$. We can then apply Lemma 2.1.8 to obtain $(T\otimes S_{n+1})\in C_{A\otimes I_{n+1}}(H_1\otimes H_2)$. Since $(T\otimes S_{n+1})$ is nilpotent of order at most $n+1$, Proposition 2.1.4 gives us

$$A\otimes I_{n+1}+2\mathrm{Re}(\sum_{k=1}^{n}z^k(T\otimes S_{n+1})^k)\geq 0,\ \forall z\in\mathbb{D},$$

which gives condition (iv).
- $(iv)\Rightarrow(ii)$ Let $n\geq 1$. Since $(T\otimes S_{n+1})$ is nilpotent, we have $r(T\otimes S_{n+1})=0\leq 1$. Hence $(T\otimes S_{n+1})$ satisfies condition (iii) of Proposition 2.1.4, so $(T\otimes S_{n+1})\in C_{A\otimes I_{n+1}}$. Now, let $h\in H$ and denote $\tilde{h}=\frac{1}{n+1}(h,...,h)$. We can then use condition (ii) of this Proposition for $(T\otimes S_{n+1})$, $A\otimes I_{n+1}$ and $\tilde{h}$. We obtain

$$\begin{aligned}&\frac{n+1}{n+1}\langle Ah,h\rangle-\mathrm{Re}(\frac{2z}{n+1}n\langle (I-A)Th,h\rangle)+\frac{|z|^2}{n+1}n\langle (A-2I)Th,Th\rangle\geq 0,\ \forall z\in\mathbb{D}\\ \Leftrightarrow&\langle Ah,h\rangle-\frac{n}{n+1}\mathrm{Re}(2z\langle (I-A)Th,h\rangle)+\frac{n}{n+1}|z|^2\langle (A-2I)Th,Th\rangle\geq 0,\ \forall z\in\mathbb{D}\end{aligned}$$

Since this is true for every $n\geq 1$, taking the limit when $n\to+\infty$ gives

$$\langle Ah,h\rangle-\mathrm{Re}(2z\langle (I-A)Th,h\rangle)+\frac{n}{n+1}|z|^2\langle (A-2I)Th,Th\rangle\geq 0,\ \forall z\in\mathbb{D},$$

so condition (ii) is obtained. □

Remark 2.1.10. Condition (iv) of Proposition 2.1.9 needs A to be self-adjoint positive, hence the operator matrix $P(A,z,T,n)$ has no chance to be self-adjoint positive for every $z\in\mathbb{D}$. Hence, we cannot really mimic this kind of idea for C_ρ classes with $\rho\in\mathbb{C}\setminus\mathbb{R}^+$. We could obtain a similar condition by using the fact that $C_\rho=\frac{|\rho|}{1+|\rho-1|}C_{1+|\rho-1|}$ and using Proposition 2.1.9 for $A=(1+|\rho-1|)I$ but this would only rely on Proposition 1.3.3 and no additional information would be obtained.

As we obtained multiple characterizations of classes C_A as well as some results regarding the operators they contain, we can now introduce and study the A-radius of an operator to quantify its distance to the class C_A, similarly to how the ρ-radius behaves with respect the class C_ρ (or (ρ_n)-radius with respect to the class $C_{(\rho_n)}$).

2.2 The Operator Radii w_A

Similarly to classes $C_{(\rho_n)}$, each class C_A can be associated to a map, called a *A-radius*, that can measure how "far" an operator T is from (or inside) the class. We will show that a A-radius is, similarly to (ρ_n)-radii, a quasi-norm that is equivalent to the operator norm $\|\cdot\|$, whose closed unit ball is the class C_A. Each property of the classes C_A can be transposed to a property on the A-radii.

Definition 2.2.1. Let $A, T \in \mathcal{L}(H)$ be such that $A > 0$. We define

$$w_A(T) := \inf(\{r > 0\colon \frac{1}{r}T \in C_A(H)\}),$$

which is called the *A-radius* of T.

We can see that when $A = \rho I$, $\rho > 0$, the A-radius coincides with the ρ-radius, that is $w_{\rho I} = w_\rho$.

Lemma 2.2.2. *Let $A \in \mathcal{L}(H)$ with $A > 0$. Then, for every $T \in \mathcal{L}(H)$, $w_A(T)$ is finite. We also have $w_A(T) = 0$ if and only if $T = 0$, and*

$$\frac{\|T\|}{\|A\|} \leq w_{\|A\|}(T) \leq w_A(T) \leq w_{\|A^{-1}\|^{-1}}(T).$$

Proof. Let $T \in \mathcal{L}(H)$. Since A is self-adjoint, positive and invertible, we have $\|A\|I \geq A \geq \frac{1}{\|A^{-1}\|}I > 0$. If $T = 0$ then condition (iii) of Proposition 2.1.4 is met, so $T \in C_A$ and $w_A(T) \leq 1$. Since for every $r > 0$ we have $\frac{1}{r}T = 0 \in C_A$, we obtain $w_A(T) = 0$.
If $T \neq 0$, then $w_{\|A^{-1}\|^{-1}}(T)$ is non-zero and finite. From our previous remark and from results on classes C_ρ (see Lemma 1.2.12), we have

$$\frac{1}{w_{\|A^{-1}\|^{-1}}(T)}T \in C_{\|A^{-1}\|^{-1}I}.$$

Thus $\frac{1}{w_{\|A^{-1}\|^{-1}}(T)}T \in C_A$ according to Corollary 2.1.6. Hence, $w_A(T) \leq w_{\|A^{-1}\|^{-1}}(T)$ and this quantity is finite.
When $T \neq 0$ we have $w_{\|A\|}(T) \neq 0$. Let $0 < r < w_{\|A\|}(T)$. Then $\frac{1}{r}T \notin C_{\|A\|}$. Hence, $\frac{1}{r}T$ cannot lie in C_A according to Corollary 2.1.6. Therefore we have $0 < w_{\|A\|}(T) \leq w_A(T)$. The results about ρ-radii give us the leftmost inequality, which concludes the proof. □

Proposition 2.2.3. *Let $A, T \in \mathcal{L}(H)$ with $A > 0$. The following are equivalent*

(i) $T \in C_A(H)$;

(ii) $w_A(T) \leq 1$;

(iii) $w_{A\otimes I_{n+1}}(T \otimes S_{n+1}) \leq 1$, $\forall n \geq 1$;

(iv) $P(A, 1, T, n) \geq 0$, $\forall n \geq 1$.

Proof. $(i) \Rightarrow (ii)$ Since $\frac{1}{1}T \in C_A$, we have $w_A(T) \leq 1$.
- $(ii) \Rightarrow (i)$ Let $r_n \in \mathbb{R}_+^*$ be a decreasing sequence such that $\frac{T}{r_n} \in C_A$ and $\lim_n(r_n) = w_A(T) \leq 1$. If $T = 0$ then $T \in C_A$ and there is nothing to prove. If $T \neq 0$, then $w_A(T) \neq 0$ according to Lemma 2.2.2. Condition (iii) of Proposition 2.1.4 gives us, for all $n \geq 1$:

$$r(\frac{T}{r_n}) \leq 1 \text{ and } A + 2\mathrm{Re}(\sum_{m\geq 1}(\frac{zT}{r_n})^m) \geq 0, \forall z \in \mathbb{D}.$$

Therefore we have on one hand $r(T) \leq r_n$, so $r(T) \leq w_A(T) \leq 1$. On another hand, we have $A + \sum_{m\geq 1}(wT)^m \geq 0$ for every w in $\bigcup_{n\geq 1}\mathbb{D}(0, \frac{1}{r_n})$. The properties of the sequence $(r_n)_n$ imply that $\bigcup_{n\geq 0}\mathbb{D}(0, \frac{1}{r_n}) = \mathbb{D}(0, \frac{1}{w_A(T)})$. Since $\frac{1}{w_A(T)} \geq 1$, we have $A + \sum_{m\geq 1}(wT)^m \geq 0$ for every $w \in \mathbb{D}$. Therefore T lies in C_A according to Proposition 2.1.4.
- $(iii) \Leftrightarrow (i)$ With equivalence $(i) \Leftrightarrow (ii)$ of this Proposition and with the fact that $(T \otimes S_{n+1})$ is nilpotent of order at most $n+1$, we can see that we have $w_{A\otimes I_{n+1}}(T \otimes S_{n+1}) \leq 1$ if and only if

$$A + 2\mathrm{Re}(\sum_{k=1}^{n}(zT \otimes S_{n+1})^k) \geq 0, \forall z \in \mathbb{D}.$$

The latter condition is equivalent to $P(A, z, T, n) \geq 0$ for every $z \in \mathbb{D}$. Hence, the equivalence $(iv) \Leftrightarrow (i)$ of Proposition 2.1.9 gives the desired result.
- $(iv) \Leftrightarrow (i)$ Let $n \geq 1$. For every $x \in H^{n+1}$, the map $g_x : z \mapsto \langle P(A, z, T, n)(x), x\rangle$ is harmonic on $\mathbb{C}$ as the real part of an analytic map. Due to harmonicity, this map takes positive values on $\mathbb{D}$ if and only if it takes positive values on $\partial\mathbb{D}$. For $z \in \partial\mathbb{D}$, denote $B_z := diag(I, zI, ..., z^nI)$. A computation then gives $B_z^* P(A, 1, T, n)B_z = P(A, z, T, n)$, so

$$\langle P(A, z, T, n)(x), x\rangle = \langle P(A, 1, T, n)(B_z(x)), B_z(x)\rangle.$$

Hence, the map g_x takes positive values on $\partial\mathbb{D}$ for every $x \in H^{n+1}$ if and only if $g_x(1)$ is positive for every $x \in H^{n+1}$. Thus, the operators $P(A, z, T, n)$ are positive for every $z \in \mathbb{D}$ if and only if $P(A, 1, T, n) \geq 0$. Therefore, we have $P(A, 1, T, n) \geq 0$ for every $n \geq 1$ if and only if we have $P(A, z, T, n) \geq 0$ for every $z \in \mathbb{D}$ and every $n \geq 1$, which is in turn equivalent to $T \in C_A$. □

Proposition 2.2.4. *Let H be a Hilbert space. Let $A \in \mathcal{L}(H)$, with $A > 0$. Then w_A is a quasi-norm that is equivalent to the operator norm $\|\cdot\|$, and whose closed unit ball is the class C_A.*

Proof. Let $S, T \in \mathcal{L}(H)$. We obtained in Lemma 2.2.2 that $w_{\|A^{-1}\|^{-1}}(T) \leq w_A(T) \leq w_{\|A\|}(T)$. Since all ρ-radius are quasi-norms, this inequality gives us a quasi-triangular inequality for w_A. Indeed,

$$\begin{aligned}
w_A(S+T) &\leq w_{\|A\|}(S+T) \leq w_{\|A\|}(I)\|S+T\| \leq w_{\|A\|}(I)(\|S\| + \|T\|) \\
w_A(S+T) &\leq w_{\|A\|}(I)(w_{\|A^{-1}\|}(I)w_{\|A^{-1}\|^{-1}}(S) + w_{\|A^{-1}\|}(I)w_{\|A^{-1}\|^{-1}}(S)) \\
w_A(S+T) &\leq w_{\|A\|}(I)w_{\|A^{-1}\|}(I)(w_A(S) + w_A(T)).
\end{aligned}$$

Hence we only need to prove that $w_A(zT) = |z|w_A(T)$ for every $z \in \mathbb{C}$ in order for w_A to be a quasi-norm. Let $z \in \mathbb{C}$. Since we know that $w_A(0) = 0$, there is nothing to prove when $z = 0$ or $T = 0$. Suppose that $z \neq 0$ and $T \neq 0$. We then have $w_A(T) \neq 0$. By looking at the proof of implication $(ii) \Rightarrow (i)$ in Proposition 2.2.3, we can see that $\frac{1}{w_A(S)}S \in C_A$ for any non-zero operator S. Take $S = |z|T$. We then have $\frac{|z|}{w_A(S)}T = \frac{S}{w_A(S)} \in C_A$. Hence, the definition of $w_A(T)$ gives us

$$w_A(T) \leq \frac{w_A(|z|T)}{|z|}.$$

If we now take $w = 1/z$ and $T' = |z|T$, we have

$$w_A(|z|T) = w_A(T') \leq \frac{w_A(|w|T')}{|w|} = |z|w_A(T),$$

thus $w_A(|z|T) = |z|w_A(T)$. Let $\lambda \in \partial\mathbb{D}$ and $U \in \mathcal{L}(H)$. Since $r(\lambda S) = r(S)$ and since $w\lambda$ describes the whole unit disc when w describes it, we can see from condition (ii) of Proposition 2.1.4 that $U \in C_A$ if and only if $\lambda U \in C_A$. Therefore, we obtain

$$\{r > 0\colon \frac{1}{r}|z|T \in C_A\} = \{r > 0\colon \frac{1}{r}zT \in C_A\},$$

and looking at their infimum gives $w_A(zT) = w_A(|z|T) = |z|w_A(T)$. Therefore the A-radius w_A is a quasi-norm. Since ρ-radii are equivalent to the norm $\|\cdot\|$, so does w_A. Also, the equivalence $(i) \Leftrightarrow (ii)$ from Proposition 2.2.3 tells us that the closed unit ball for w_A is exactly the class C_A. □

Remark 2.2.5. Since the A-radius is homogeneous and since its closed unit ball is C_A, we can find similarly to the (ρ_n)-radii that the set $\{r > 0\colon \frac{1}{r}T \in C_A\}$ is equal to $[w_A(T), +\infty[$ when $T \neq 0$ and to $\mathbb{R}_+^*$ when $T = 0$.

As the A-radius is also equivalent as a quasi-norm to $\|\cdot\|$, its closed unit ball is then closed for the operator norm $\|\cdot\|$. Hence, all classes C_A are closed subsets of $\mathcal{L}(H)$ that only contain operators that are similar to contractions.

Proposition 2.2.6. *Let H be a Hilbert space. Let $A, B, T \in \mathcal{L}(H)$ with $A, B > 0$. We have*

(i) $\lim_{\|B^{-1}\| \to 0} (w_B(T)) = r(T)$;

(ii) $w_A(T^*) = w_A(T)$;

(iii) $w_A(T^k) \leq w_A(T)^k$, $\forall k \geq 0$;

(iv) For $B \geq A$, we have $w_B(T) \leq w_A(T)$.

Proof. (i) Let $\eta > 0$ and $B > 0$ be such that $\|B^{-1}\| \leq \eta$. On one hand, we have

$$w_B(T) \geq w_{\|B\|}(T) \geq r(T).$$

While on the other hand, we have

$$w_B(T) \leq w_{\|B^{-1}\|^{-1}}(T) \leq w_{\frac{1}{\eta}}(T).$$

Since $\lim_{\rho \to +\infty} w_\rho(T) = r(T)$, then $\lim_{\|B^{-1}\| \to 0} (w_B(T))$ exists and is equal to $r(T)$.

- (ii) If $T = 0$ then there is nothing to prove. If $T \neq 0$, we have $\frac{T}{w_A(T)} \in C_A$. Remark 2.1.3 tells

us then that $\frac{T^*}{w_A(T)} \in C_A$, that is $w_A(\frac{T^*}{w_A(T)}) \leq 1$, or $w_A(T^*) \leq w_A(T)$ by homogeneity of w_A. We obtain the converse inequality by considering $S = T^*$.
- (iii) Let $k \geq 1$. If $T = 0$ then there is nothing to prove. If $T \neq 0$, we have $\frac{T}{w_A(T)} \in C_A$. Remark 2.1.3 tells us then that $(\frac{T}{w_A(T)})^k \in C_A$, that is $w_A(\frac{T^k}{w_A(T)^k}) \leq 1$, or $w_A(T^k) \leq w_A(T)^k$, by homogeneity of w_A.
- (iv) If $T = 0$ then there is nothing to prove. If $T \neq 0$, we have $\frac{T}{w_A(T)} \in C_A \subset C_B$ according to Corollary 2.1.6. Hence, $w_B(\frac{T}{w_A(T)}) \leq 1$, so $w_B(T) \leq w_A(T)$. □

Proposition 2.2.7. *Let H be a Hilbert space and let $A, B, S, T \in \mathcal{L}(H)$ with $A > 0$, $B > 0$. If these operators double-commute with each other (e.g. $TS = ST$,$TS^* = S^*T$), then*

$$w_{AB}(ST) \leq w_A(S)w_B(T).$$

Proof. If $S = 0$ or $T = 0$, then $ST = 0$ and both sides of the inequality are equal to zero. If $S \neq 0$ and $T \neq 0$ then, up to dividing S and T by their respective radius we can consider that $w_A(S) = w_B(T) = 1$. Thus, we need to prove that $w_{AB}(ST) \leq 1$. We define

$$T_m := \begin{cases} T^m \text{ if } m \geq 1 \\ B \text{ if } m = 0 \\ (T^*)^{|m|} \text{ if } m \leq -1 \end{cases}, \quad S_m := \begin{cases} S^m \text{ if } m \geq 1 \\ A \text{ if } m = 0 \\ (S^*)^{|m|} \text{ if } m \leq -1. \end{cases}$$

As $w_A(S) = w_B(T) = 1$, we have $S \in C_A$ and $T \in C_B$. Item (iii) of Proposition 2.1.4 then ensures us that the conditions of Proposition 1.4.1 are fulfilled, since

$$A + \mathrm{Re}(\sum_{n\geq 1}(re^{it}S)^n) = \sum_{m\in\mathbb{Z}} r^{|m|}e^{imt}S_m, \text{ for all } 0 \leq r < 1,\ t \in \mathbb{R},$$

and since the same is true for $(T_m)_m$. Hence, 2.1.4 tells us that $\sum_{m\in\mathbb{Z}} r^{|m|}e^{imt}S_mT_m$ converges absolutely, is self-adjoint, and has a positive sum, for all $0 \leq r < 1$, $t \in \mathbb{R}$. This implies that the series $\sum_{n\geq 1}(re^{it}ST)^n$ is absolutely convergent and that $AB + \mathrm{Re}(\sum_{n\geq 1}(re^{it}ST)^n) \geq 0$ for all $0 \leq r < 1$, $t \in \mathbb{R}$. Thus $ST \in C_{AB}$ and $w_{AB}(ST) \leq 1$, which concludes the proof. □

Lemma 2.2.8. *Let H be a Hilbert space and let $T \in \mathcal{L}(H)$. For every $r > 0$, we have*

$$\inf(\{\rho > 0\colon\ w_\rho(T) \leq r\}) = \inf(\{\|A\|,\ A > 0\colon\ w_A(T) \leq r\}).$$

Proof. Let $r > 0$. Denote a_1 and a_2 these infima. Since $w_{\|A\|}(T) \leq w_A(T)$, if there is no $\rho > 0$ such that $w_\rho(T) \leq r$, then there is no $A > 0$ such that $w_A(T) \leq r$. Conversely, if there is no $A > 0$ such that $w_A(T) \leq r$, then we have $w_\rho(T) = w_{\rho I}(T) > r$ for every $\rho > 0$. Thus, if one of these sets is empty so is the other one. Suppose now that both sets are not empty, so $a_1, a_2 < +\infty$. We recall from Proposition 1.4.8 that the map $\rho \mapsto w_\rho(T)$ is continuous on $]0, +\infty[$, decreasing, and tends to $+\infty$ when ρ tends to 0. Thus we also have $a_1 > 0$.

Since the set on the left is contained in the set on the right, we can see that $a_2 \leq a_1$ and $a_1 > 0$. If we had $a_2 < a_1$, then we would have $A > 0$ such that $w_A(T) \leq r$ and $\|A\| < a_1$. But this would mean that $w_{\|A\|}(T) \leq w_A(T) \leq r$, which contradicts the minimality of a_1. Hence we get $a_1 = a_2$, which concludes the proof. □

Example 2.2.9. Let $\rho, \tau > 1$. For $T = S_2$, $S = T^*$, $A = \rho I_2$, $B = \tau I_2$, we have $TS = diag(1,0)$ and

$$w_{AB}(TS) = 1 > \frac{1}{\rho\tau} = w_A(S)w_B(T).$$

Hence the result of Proposition 2.2.7 does not remain true when T and S do not double-commute.

Remark 2.2.10. We can try to obtain different inequalities like 2.2.7 by weakening the double-commutativity condition between S and T to commutativity only, similarly to the results in [AO76]. However, as A and B are self-adjoint commutativity between T (or S) and A or B implies double-commutativity between these operators (if $TA = AT$ then $AT^* = T^*A$).

Lemma 2.2.11. *Let $m \geq 1$. Let $H_1, ...H_m$ be Hilbert spaces. Let $A_i, T_i \in \mathcal{L}(H_i)$, with $A_i > 0$, for every $1 \leq i \leq m$. Define $H = H_1 \oplus .. \oplus H_m$, $A = diag(A_1, ..., A_m)$ and $T = diag(T_1, ..., T_m)$. Then, $A > 0$ and we have $w_A(T) = \max_i(w_{A_i}(T_i))$.*

Proof. Since all A_i are self-adjoint, positive and invertible, so is $A = diag(A_1, ..., A_m)$. Let $u > 0$. We have $r(\frac{T}{u}) = \max_i(r(\frac{T_i}{u})$. Hence, we have $r(\frac{T}{u}) \leq 1$ if and only if $r(\frac{T_i}{u}) \leq 1$ for every $1 \leq i \leq m$. Take $u > 0$ such that $r(\frac{T}{u}) \leq 1$. We then have

$$\begin{aligned}
&\frac{T}{u} \in C_A \\
\Leftrightarrow & A + 2\mathrm{Re}(\sum_{n\geq 1}(z\frac{T}{u})^n) \geq 0, \ \forall z \in \mathbb{D} \\
\Leftrightarrow & diag(A_1, .., A_m) + diag\left(2\mathrm{Re}(\sum_{n\geq 1}(z\frac{T_1}{u})^n), .., 2\mathrm{Re}(\sum_{n\geq 1}(z\frac{T_m}{u})^n)\right) \geq 0, \ \forall z \in \mathbb{D} \\
\Leftrightarrow & diag\left(A_1 + 2\mathrm{Re}(\sum_{n\geq 1}(z\frac{T_1}{u})^n), .., A_m + 2\mathrm{Re}(\sum_{n\geq 1}(z\frac{T_m}{u})^n)\right) \geq 0, \ \forall z \in \mathbb{D} \\
\Leftrightarrow & A_i + 2\mathrm{Re}(\sum_{n\geq 1}(z\frac{T_i}{u})^n) \geq 0, \ \forall z \in \mathbb{D}, \ \forall 1 \leq i \leq m \\
\Leftrightarrow & \frac{T_i}{u} \in C_{A_i}, \ \forall 1 \leq i \leq m.
\end{aligned}$$

Therefore, by definition of $w_A(T)$, we have

$$w_A(T) = \sup_i(\inf\{u > 0\colon \ \frac{T_i}{u} \in C_{A_i}\}) = \max_i(w_{A_i}(T_i)).$$

□

As we obtained the main characterizations and results about classes C_A and A-radii w_A, we can use them to make some computations and obtain various inequalities.

Proposition 2.2.12. *Let H be a Hilbert space and $A, B, T \in \mathcal{L}(H)$ with $A, B > 0$. We have*

(i) $w_A(I) = w_{\frac{1}{\|A^{-1}\|}}(I) = \begin{cases} 1 \text{ if } A \geq I \\ 2\|A^{-1}\| - 1 \text{ else} \end{cases}$;

(ii) $w_{\frac{A}{\|A\|}}(I) = 2\|A\|\|A^{-1}\| - 1 = w_{\frac{A^{-1}}{\|A^{-1}\|}}(I)$;

(iii) *If* T, A, B *commute with each other, then* $w_A(T) \leq w_{AB^{-1}}(I)w_B(T)$.
If we also have $A \leq B$*, then* $w_B(T) \leq w_A(T) \leq (2\|AB^{-1}\| - 1)w_B(T)$.

(iv) *The inequalities* $w_{\frac{1}{\|A^{-1}\|}}(T) \geq w_A(T) \geq w_{\|A\|}(T)$ *are optimal for a fixed* A.

Proof. (i) The numerical range of A is $W(A) = [\frac{1}{\|A^{-1}\|}, \|A\|]$. Let $u > 0$. Using condition (ii) of Proposition 2.1.4 we can notice that $\frac{1}{u}I$ lies in C_A if and only if $\frac{1}{u}I$ lies in C_ρ for every $\rho \in W(A)$. Since we have $C_\rho \subset C_\tau$ when $\rho \geq \tau$, this condition is equivalent to $\frac{1}{u}I \in C_{\|A^{-1}\|^{-1}}$. Hence, we obtain $w_A(I) = w_{\|A^{-1}\|^{-1}}(I)$, and we can use Corollary 1.3.5 to compute it.
- (ii) The result comes from item (i).
- (iii) We can use Proposition 2.2.7 to get the result. When $A \leq B$, we have $AB^{-1} \leq I$, so

$$w_{AB^{-1}}(I) = (2\|AB^{-1}\| - 1).$$

- (iv) If $A = \rho I$, both inequalities are always attained. Else, we have $\frac{1}{\|A^{-1}\|} \neq \|A\|$. The left hand inequality is attained for $T = I$. For the right hand one, take $\eta > 0$ such that $\frac{1}{\|A^{-1}\|} < \|A\| - \eta$. Using the theory of characteristic projections for self-adjoint operators (see [Sim15, Thm.5.1.5, p.292-295]), take $T = \chi_{[\|A\|-\eta, \|A\|]}(A)$. T is then an orthogonal projection that is non-zero, that commutes with A, and such that

$$W(A|_{\mathrm{Ran}(T)}) = [\|A\| - \eta, \|A\|] \cap W(A) \supset [\|A\| - \eta, \|A\|[.$$

Thus, we can define $H_1 = T(H)$, $H_2 = (I - T)(H)$, $A_1 = A|_{H_1}$, $A_2 = A|_{H_1}$, $T_1 = T|_{H_1}$, $T_2 = T|_{H_2}$. With $H = H_1 \oplus H_2$ we can see that we have $A = diag(A_1, A_2)$ and $T = diag(T_1, T_2) = diag(I_{H_1}, 0_{H_2})$. Therefore, Lemma 2.2.11 gives us

$$w_A(T) = \max(w_{A_1}(I_{H_1}), w_{A_2}(0)) = w_{A_1}(I_{H_1}) = w_{\|A\|-\eta}(I) = w_{\|A\|-\eta}(T).$$

Since the map $\rho \mapsto w_\rho(I)$ is continuous and decreasing, for every $\epsilon > 0$ we can then find $\eta > 0$ such that $w_{\|A\|}(I) - w_{\|A\|-\eta}(I) \leq \epsilon$. Hence, we can find a $T \in \mathcal{L}(H)$ such that $w_{\|A\|}(T) - w_A(T) \leq \epsilon$. ☐

Proposition 2.2.13. *Let* H *be a Hilbert space. Let* $A \in \mathcal{L}(H)$ *with* $A > 0$.
If $A \leq 2I$ *then* $w_A(\cdot)$ *is a norm.*

Proof. (ii) Since the A-radius w_A is a quasi-norm, showing that it is a norm amounts to showing that its closed unit ball, C_A, is convex. Let $S, T \in C_A$ and $t \in [0, 1]$. We need to prove that $tS + (1 - t)T \in C_A$. Since $0 < A \leq 2I$, $2I - A$ possesses a square root. Condition (ii) of Proposition 2.1.9 applied to T gives us

$$\begin{aligned}
&\langle Ah, h\rangle - 2\mathrm{Re}(\langle z(A - I)Th, h\rangle) + |z|^2\langle (A - 2I)Th, Th\rangle \geq 0, \forall z \in \mathbb{D}, \forall h \in H \\
\Leftrightarrow &\|\sqrt{A}h\|^2 - 2\mathrm{Re}(\langle z(A - I)Th, h\rangle) \geq |z|^2\langle (2I - A)Th, Th\rangle, \forall z \in \mathbb{D}, \forall h \in H \\
\Leftrightarrow &\|\sqrt{A}h\|^2 - 2\mathrm{Re}(\langle z(A - I)Th, h\rangle) \geq |z|^2\|\sqrt{2I - A}Th\|^2, \forall z \in \mathbb{D}, \forall h \in H
\end{aligned}$$

Hence, by combining this inequality with the same one for S, we obtain, for every $z \in \mathbb{D}$ and $h \in H$:

$$\|\sqrt{A}h\|^2 - 2\mathrm{Re}(\langle z(A - I)(tS + (1 - t)T)h, h\rangle) \geq |z|^2[t\|\sqrt{2I - A}Sh\|^2 + (1 - t)\|\sqrt{2I - A}Th\|^2].$$

The map $\|\cdot\| : H \to \mathbb{R}_+$ is convex and the map $x \in \mathbb{R}_+ \mapsto x^2 \in \mathbb{R}^+$ is convex and increasing. Hence, the map $h \mapsto \|h\|^2$ is convex. Therefore, we get

$$\|\sqrt{A}h\|^2 - 2\mathrm{Re}(\langle z(A-I)(tS+(1-t)T)h, h\rangle) \geq |z|^2\|\sqrt{2I-A}(tS+(1-t)T)h\|^2, \ \forall z \in \mathbb{D}, \ \forall h \in H.$$

The previous computations show that this inequality can be reformulated into

$$\langle Ah, h\rangle - 2\mathrm{Re}(\langle z(A-I)(tS+(1-t)T)h, h\rangle) + |z|^2\langle (A-2I)(tS+(1-t)T)h, (tS+(1-t)T)h\rangle \geq 0,$$

for every $z \in \mathbb{D}$ and $h \in H$. Thus, the operator $(tS + (1-t)T)$ satisfies condition (ii) of Proposition 2.1.9, so it belongs to the class C_A. □

2.3 Computations and Some Applications

The following proposition gives some improvements over Proposition 3.2 of [Sue98a]. The other results in this subsection (from Lemma 2.3.3 through Proposition 2.3.9) are believed to be new.

Proposition 2.3.1. *Let H be a Hilbert space, and $A, T, V \in \mathcal{L}(H)$ with $A > 0$ and V an isometry. We have*

(i) $w_{VAV^*}(VTV^*) = w_A(T)$*;*

(ii) $w_{V^*AV}(T) \leq w_A(VTV^*)$.
If we also have $[V, A] := VA - AV = 0$*, then* $w_A(VTV^*) = w_A(T)$.

(iii) If $[VV^*, T] = 0$ *and* $[V, A] := VA - AV = 0$*, then for* $H_1 = Ker(VV^*)$*,* $H_2 = Ran(VV^*)$ *we have*

$$w_A(T) = \max(w_{A|_{H_1}}(T|_{H_1}), w_{A|_{H_2}}(T|_{H_2})) \text{ and } w_A(V^*TV) = w_{A|_{H_2}}(T|_{H_2}).$$

If we also have $T = VT'$ *for some* T'*, then*

$$w_A(V^*T) = w_A(TV^*) \text{ and } w_A(T'V) = w_A(T) = w_A(VT').$$

(iv) Let U be an unitary. Then $w_A(UTU^*) = w_{U^*AU}(T)$.
If we also have $[U, A] = 0$ *then* $w_A(UTU^*) = w_A(U^*TU) = w_A(T)$.

(v) If $[V, A] = [V, T] = 0$*, then* $w_A(VT) \leq w_A(T)$.

(vi) If $A \leq 2I$*, then* $w_A(Re(T)) \leq w_A(T)$ *and* $w_A(Ran(T)) \leq w_A(T)$.

Proof. (i) If $T = 0$ there is nothing to prove. Assume now that $T \neq 0$. We recall that for any $B, C \in \mathcal{L}(H)$, we have $\mathrm{Re}(C) \geq 0 \Rightarrow \mathrm{Re}(BCB^*) = B\mathrm{Re}(C)B^* \geq 0$. Since V is an isometry, for any $n \geq 0$ we have $(VTV^*)^n = VT^nV^*$. Hence, we get

$$\|T^n\| \leq \|V^*\|\|VT^nV^*\|\|V\| = \|(VTV^*)^n\| = \|VT^nV^*\| \leq \|T^n\|,$$

so we obtain $r(T) = r(VTV^*)$. Let $u > 0$ be such that $r(\frac{T}{u}) \leq 1$. We then have

$$\begin{aligned}
&\mathrm{Re}(A + 2\sum_{n\geq 1}(z\frac{T}{u})^n) \geq 0,\ \forall z \in \mathbb{D} \\
\Rightarrow &\mathrm{Re}(V(A + 2\sum_{n\geq 1}(z\frac{T}{u})^n)V^*) \geq 0,\ \forall z \in \mathbb{D} \\
\Leftrightarrow &\mathrm{Re}(VAV^* + 2\sum_{n\geq 1}(z\frac{VTV^*}{u})^n) \geq 0,\ \forall z \in \mathbb{D} \\
\Rightarrow &\mathrm{Re}(V(A + 2\sum_{n\geq 1}(z\frac{T}{u})^n)V^*) \geq 0,\ \forall z \in \mathbb{D} \\
\Rightarrow &\mathrm{Re}(V^*V(A + 2\sum_{n\geq 1}(z\frac{T}{u})^n)V^*V) \geq 0,\ \forall z \in \mathbb{D}
\end{aligned}$$

Thus we end up with $\frac{T}{u} \in C_A$ if and only if $\frac{VTV^*}{u} \in C_{VAV^*}$, which gives in turn $w_A(T) = w_{VAV^*}(VTV^*)$.

- (ii) Let $u > 0$ be such that $\frac{VTV^*}{u} \in C_A$. Let $n \geq 1$, and recall that S_{n+1} is the left shift operator on $\mathbb{C}^{n+1}$. Using Propositions 2.1.9 and 2.2.3 we then have

$$\begin{aligned}
&P(A, 1, \frac{VTV^*}{u}, n) = \mathrm{Re}(A \otimes I_{n+1} + 2\sum_{k=1}^{n}(\frac{VTV^*}{u} \otimes S_{n+1})^k) \geq 0 \\
\Leftrightarrow &\mathrm{Re}(A \otimes I_{n+1} + 2\sum_{k=1}^{n} V \otimes I_{n+1} \cdot (\frac{T}{u} \otimes S_{n+1})^k \cdot V^* \otimes I_{n+1}) \geq 0 \\
\Rightarrow &\mathrm{Re}((V^*AV) \otimes I_{n+1} + 2\sum_{k=1}^{n}(\frac{T}{u} \otimes S_{n+1})^k) \geq 0 \\
\Rightarrow &P(V^*AV, 1, \frac{T}{u}, n) \geq 0.
\end{aligned}$$

Therefore, $\frac{VTV^*}{u} \in C_A$ implies that $\frac{T}{u} \in C_{V^*AV}$. This gives

$$w_{V^*AV}(T) \leq w_A(VTV^*).$$

When A commutes with V, we have $VAV^* = AVV^*$. Since VV^* is an orthogonal projection we have $0 \leq VV^* \leq I$. Since A is self-adjoint it commutes with V^* so it commutes with VV^*. Hence, we have $0 \leq AVV^* \leq A$. Thus,

$$w_A(VTV^*) \leq w_{AVV^*}(VTV^*) = w_A(T) = w_{AV^*A}(T) = w_{V^*AV}(T) \leq w_A(VTV^*).$$

- (iii) Since V is an isometry, VV^* is an orthogonal projection. Hence $H_1 = \mathrm{Ker}(VV^*)$ and $H_2 = \mathrm{Ran}(VV^*)$ satisfy $H = H_1 \overset{\perp}{\oplus} H_2$. As T and A commute with VV^* they leave H_1 and H_2 stable, so we can use Lemma 2.2.11 to obtain $w_A(T) = \max(w_{A|_{H_1}}(T|_{H_1}), w_{A|_{H_2}}(T|_{H_2}))$.
By using item (ii) we get $w_A(V^*TV) = w_A(VV^*TVV^*) = w_A(T(VV^*)^2) = w_A(TVV^*)$. Since TVV^* also commutes with VV^* and since $TVV^*|_{\mathrm{Ker}(VV^*)} = 0$ and $TVV^*|_{\mathrm{Ran}(VV^*)} = T|_{\mathrm{Ran}(VV^*)}$, we end up with

$$w_A(V^*TV) = w_A(TVV^*) = w_{A|_{H_2}}(T|_{H_2}).$$

If we also suppose that there exists T' such that $T = VT'$, then $w_A(V^*T) = w_A(T') = w_A(VT'V^*) = w_A(TV^*)$. We also have

$$TVV^* = VV^*T = VV^*VT' = VT' = T.$$

Therefore the previous computation gives us

$$w_A(T'V) = w_A(V^*TV) = w_A(TVV^*) = w_A(T).$$

- (iv) Let U be an unitary, so $U^* = U^{-1}$. Denote $B = U^*AU$. Using item (i) we obtain

$$w_A(UTU^*) = w_{UBU^*}(UTU^*) = w_B(T) = w_{U^*AU}(T).$$

Suppose now that U commutes with A. We then have $U^* = U^{-1}$, which also commutes with A. We can then get the result by applying item (i) with U and U^*.
- (v) If $T = 0$ there is nothing to prove. Assume now that $T \neq 0$. As V commutes with A, we have $V^*AV = A$. As V commutes with T, we have $V^*T^kV^{k+l} = T^kV^k = (TV)^k$ for every $k \geq 1$. Up to dividing T by $w_A(T)$, suppose that $w_A(T) = 1$. We then have $P(A, 1, T, n) \geq 0$ for every $n \geq 1$. A computation gives

$$diag(I, V, ..., V^n)^*P(A, 1, T, n)diag(I, V, ..., V^n) = P(A, 1, TV, n) \geq 0.$$

Therefore, we have $w_A(TV) \leq 1 = w_A(T)$.
- (vi) This item comes from the fact that w_A is a norm. Hence, we have

$$w_A(\mathrm{Re}(T)) = w_A(\frac{T + T^*}{2}) \leq \frac{1}{2}(w_A(T) + w_A(T^*)) = w_A(T).$$

We can make a similar computation for $\mathrm{Im}(T) = \frac{T-T^*}{2}$. □

Remark 2.3.2. The statement of item (iv) in Proposition 2.3.1 is false if we do not suppose that U commutes with A. Proposition 2.3.5 and Remark 2.3.7 will provide a counter-example.

Lemma 2.3.3. *Let H be a Hilbert space. Let $A, C, T \in \mathcal{L}(H)$ with $A > 0$. Let $\rho > 0$.*

(i) If $0 < A \leq 2I$, C is invertible, and A commutes with C, then

$$w_A(C^*TC) \leq \|C\|^2 w_A(T).$$

The result remains true if C lies in $\overline{Com(A)^\times} = \overline{\{S \in \mathcal{L}(H),\ SA = AS \text{ and } S \text{ invertible.}\}}$.

(ii) If $0 < \rho \leq 2$, and C lies in $\overline{\mathcal{L}(H)^\times}$, then

$$w_\rho(C^*TC) \leq \|C\|^2 w_\rho(T).$$

Proof. (i) If $T = 0$ there is nothing to prove. Suppose that $T \neq 0$ and that C is invertible. Up to considering $\frac{T}{w_A(T)}$, we can suppose that $w_A(T) = 1$. Denote $S = C^*TC$. Then, for every $h \in H$ and $z \in \mathbb{D}$, with equivalence $(i) \Leftrightarrow (ii)$ of Proposition 2.1.9 we obtain

$$\langle A(h), h\rangle - 2\mathrm{Re}(z\langle (A - I)T(h), h\rangle) + |z|^2\langle (A - 2I)T(h), T(h)\rangle \geq 0.$$

For $g = C^{-1}(h)$, we have

$$\|\sqrt{A}C(g)\|^2 - 2\mathrm{Re}(z\langle C^*(A-I)TC(g), g\rangle) + |z|^2\langle (A-2I)TC(g), TC(g)\rangle \geq 0.$$

Since $2I - A \geq 0$, it admits a square root. As A and C commute, we get

$$\|C\|^2\|\sqrt{A}(g)\|^2 - 2\mathrm{Re}(z\langle (A-I)S(g), g\rangle) - |z|^2\|\sqrt{2I-A}TC(g)\|^2 \geq 0.$$

We also have $\|TC(x)\| = \|(C^*)^{-1}C^*TC(x)\| \geq \frac{1}{\|C\|}\|C^*TC(x)\|$, so

$$\|C\|^2\|\sqrt{A}(g)\|^2 - 2\mathrm{Re}(z\langle (A-I)S(g), g\rangle) - |z|^2\frac{1}{\|C\|^2}\|\sqrt{2I-A}S(g)\|^2 \geq 0.$$

By denoting $S' = \frac{S}{\|C\|^2}$, we end up with

$$\|\sqrt{A}(g)\|^2 - 2\mathrm{Re}(z\langle (A-I)S'(g), g\rangle) - |z|^2\|\sqrt{2I-A}S'(g)\|^2 \geq 0.$$

Rewriting the expression gives

$$\langle A(g), g\rangle - 2\mathrm{Re}(z\langle (A-I)S'(g), g\rangle) + |z|^2\langle (A-2I)S'(g), S'(g)\rangle \geq 0.$$

Therefore, we can use the equivalence $(ii) \Leftrightarrow (i)$ in Proposition 2.1.9 to obtain

$$w_A(S') = \frac{w_A(S)}{\|C\|^2} \leq 1 = w_A(T),$$

which gives the desired inequality.
Since both maps $C \mapsto w_A(C^*TC)$ and $C \mapsto \|C\|^2 w_A(T)$ are continuous, the result remains true when C is a limit of a sequence of invertible operators commuting with A.
- (ii) For $0 < \rho \leq 2$ we take $A = \rho I$. Then, for any C that is a limit of a sequence of invertible operators we can apply item (i) and get the desired result. □

Proposition 2.3.4. *Let H be a Hilbert space. Let $A, T \in \mathcal{L}(H)$ with $A > 0$ and $T^2 = 0$. Then, we have $w_A(T) = 2w_2(A^{-\frac{1}{2}}TA^{-\frac{1}{2}})$.*
Furthermore, if T and A commute we have

$$w_A(T) = |\rho|w_\rho(A^{-\frac{1}{2}}TA^{-\frac{1}{2}}) = \|A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\|, \ \forall \rho \neq 0.$$

This equality is generally false without commutativity.

Proof. Since $r(T) = 0$ and $T^2 = 0$, we have $w_A(T) \leq 1$ if and only if

$$I + 2\mathrm{Re}(zA^{-\frac{1}{2}}TA^{-\frac{1}{2}}) \geq 0, \ \forall z \in \mathbb{D}.$$

Since we have $(A^{-\frac{1}{2}}TA^{-\frac{1}{2}})^2 = 0$, this condition is equivalent to $w_2(2A^{-\frac{1}{2}}TA^{-\frac{1}{2}}) \leq 1$. The quasi-norm properties of w_A and w_2 then give the desired result.
We recall that for S with $S^2 = 0$, we have $w_\rho(S) = \frac{\|S\|}{|\rho|} = \frac{2}{|\rho|}w_2(S)$. When T and A commute, the operator $A^{-\frac{1}{2}}TA^{-\frac{1}{2}}$ is nilpotent of order at most 2, which gives the result.
If we take $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $A^{-\frac{1}{2}} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ with $a, b, c \in \mathbb{R}_+^*$, we have $A^{-\frac{1}{2}}TA^{-\frac{1}{2}} = \begin{pmatrix} ab & ac \\ bb & bc \end{pmatrix}$.
The characteristic polynomial of this matrix is $X^2 - (ab + bc)X$. Hence, according to item (v) of Corollary 1.3.5, for every $\rho \in \mathbb{C}^*$, we have

$$|\rho|w_\rho(A^{-\frac{1}{2}}TA^{-\frac{1}{2}}) = \|A^{-\frac{1}{2}}TA^{-\frac{1}{2}}\| + |ab + bc||\rho - 1|,$$

so this quantity is not even constant. □

Proposition 2.3.5. *Let $A, T \in \mathcal{L}(\mathbb{C}^2)$ with $A > 0$ and $T^2 = 0$. Suppose that $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and write $A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$. Then,*

(i) $w_A(T) = \frac{1}{\sqrt{ac} - |b|}$;

(ii) *For λ, μ the eigenvalues of A, with $\lambda \leq \mu$, we have $\frac{1}{\sqrt{\lambda\mu}} \leq w_A(T) \leq \frac{1}{\lambda}$. Equality for the left hand inequality is attained when $b = 0$, whereas equality for the right hand inequality is attained when $|b| = \frac{\mu - \lambda}{2}$ (or equivalently $a = \frac{\mu+\lambda}{2}$).*

(iii) *For any $a > b > 0$, we have*

$$w_{\begin{pmatrix} a & b \\ b & a \end{pmatrix}}(T) = w_{a-b}(T) = \frac{1}{a-b}.$$

Proof. (i) Let $u > 0$. Denote

$$E = \{\mathrm{Re}(\langle (A + 2z\frac{T}{u}h, h\rangle),\ h \in \mathbb{C}^2,\ z \in \mathbb{D}\}.$$

Since $r(T) = 0$ and $T^2 = 0$, we have

$$\begin{aligned} \frac{T}{u} \in C_A &\Leftrightarrow \mathrm{Re}(A + 2z\frac{T}{u}) \geq 0,\ \forall z \in \mathbb{D} \\ &\Leftrightarrow \inf(E) \geq 0. \end{aligned}$$

We can also see that $\inf(E) = \inf(\overline{E})$ and that $\inf(E) > -\infty$ since $A + 2z\frac{T}{u}$ is bounded. For $h \in \mathbb{C}^2$ write $h = h_1e_1 + h_2e_2$. Let $z \in \mathbb{D}$. We recall that since A is positive we have $a, c \in [0, +\infty[$. Hence,

$$\mathrm{Re}(\langle (A + 2z\frac{T}{u}h, h\rangle) = a|h_1|^2 + c|h_2|^2 + 2\mathrm{Re}(bh_2\overline{h_1}) + 2\mathrm{Re}(\frac{z}{u}h_2\overline{h_1}).$$

For $b = |b|e^{it}$ take $h_1' = |h_1|$ and $h_2' = |h_2|e^{-it}$. Denote $h' = h_1'e_1 + h_2'e_2$. For $h_2'\overline{h_1'} = |h_1||h_2|e^{it'}$ and $n \geq 1$, define $z_n' = e^{-it'}(\frac{|z|}{n} + (1 - \frac{1}{n}))$. As $|z_n'| \geq |z|$ for every $n \geq 1$, this implies

$$\mathrm{Re}(\langle (A + 2z\frac{T}{u}h, h\rangle) \geq a|h_1|^2 + c|h_2|^2 - 2|b||h_2||h_1| - 2\frac{|z_n'|}{u}|h_2||h_1|.$$

The choice of h_2', h_1', z_n' also gives us

$$\mathrm{Re}(\langle (A + 2z_n'\frac{T}{u}h', h'\rangle) = a|h_1|^2 + c|h_2|^2 - 2|b||h_2||h_1| - 2\frac{|z_n'|}{u}|h_2||h_1|.$$

Since the sequence $(z_n')_n$ lies in $\mathbb{D}$ and converges to $e^{-it'}$, we can see that

$$a|h_1|^2 + c|h_2|^2 - 2|b||h_2||h_1| - 2\frac{1}{u}|h_2||h_1| \in \overline{E},$$

and that

$$\inf(E) = \inf(\{a|h_1|^2 + c|h_2|^2 - 2(|b| + \frac{1}{u})|h_2||h_1|,\ h_1, h_2 \in \mathbb{C}\}).$$

Therefore, we have $\frac{T}{u} \in C_A$ if and only if $ax^2 + cy^2 - 2(|b| + \frac{1}{u})xy \geq 0$ for every $x, y \in \mathbb{R}_+$. The polynomial $ax^2 + cy^2 - 2(|b| + \frac{1}{u})xy$ has a discriminant with respect to x equal to

$$\Delta = 4(|b| + \frac{1}{u})^2 y^2 - 4acy^2 = 4y^2((|b| + \frac{1}{u})^2 - ac).$$

Such a quantity is negative for every $y \in \mathbb{R}$ if and only if $(|b| + \frac{1}{u})^2 - ac \leq 0$. We have

$$(|b| + \frac{1}{u})^2 - ac \leq 0 \Leftrightarrow u \geq \frac{1}{\sqrt{ac} - |b|}.$$

Hence we have $\frac{T}{u} \in C_A$ if and only if $u \geq \frac{1}{\sqrt{ac}-|b|}$, so the A-radius of T is equal to $\frac{1}{\sqrt{ac}-|b|}$.
- (ii) Denote $\lambda \leq \mu$ the eigenvalues of A. We proved that $w_A(T) = \frac{1}{\sqrt{ac}-|b|} = \frac{\sqrt{ac}+|b|}{det(A)}$. We will express the quantities ac and $|b|$ depending on a, λ and μ, and find their extrema depending on the values of a. We have $Tr(A) = a + c$, so $c = Tr(A) - a$. We have $det(A) = ac - |b|^2$, so

$$a(Tr(A) - a) = ac = det(A) + |b|^2 \geq det(A).$$

The polynomial map $x \mapsto x(Tr(A) - x)$ has a maximum of $\frac{Tr(A)^2}{4}$ attained when $x = \frac{Tr(A)}{2}$ and a computation gives

$$x(Tr(A) - x) \geq det(A) \Leftrightarrow \lambda \leq x \leq \mu.$$

Hence the quantity $ac = a(Tr(A)-a)$ has a minimum of $det(A)$ when $a = \lambda$ or μ and a maximum of $\frac{Tr(A)^2}{4}$ when $a = \frac{Tr(A)}{2}$.
As we have

$$|b|^2 = ac - det(A) = a(Tr(A) - a) - det(A),$$

the quantity $|b|^2$ has a minimum of 0 when $a = \lambda$ or μ and a maximum of $\frac{Tr(A)^2}{4} - det(A)$ when $a = \frac{Tr(A)}{2}$. Another computation gives $\frac{Tr(A)^2}{4} - det(A) = \frac{(\mu-\lambda)^2}{4}$. Therefore, we obtain the following inequalities

$$\frac{1}{\sqrt{\lambda\mu}} = \frac{\sqrt{\lambda\mu} + 0}{det(A)} \leq \frac{\sqrt{ac} + |b|}{det(A)} = w_A(T) \leq \frac{\frac{Tr(A)}{2} + \frac{\mu-\lambda}{2}}{det(A)} = \frac{\mu}{\lambda\mu} = \frac{1}{\lambda}.$$

With the previous computations we can also see that $w_A(T) = \frac{1}{\sqrt{\lambda\mu}}$ if and only $|b| = 0$ and that $w_A(T) = \frac{1}{\lambda}$ if and only if $|b| = \frac{\mu-\lambda}{2}$ (or equivalently $a = \frac{\mu+\lambda}{2}$).
- (iii) By considering $a = c$ and $b = |b|$, we have $\sqrt{ac} - |b| = a - b$, which concludes the proof. □

Remark 2.3.6. Let $\lambda \in \mathbb{R}_+^*$ and define

$$A_\lambda = \begin{pmatrix} \frac{1+\lambda}{2} & \frac{1-\lambda}{2} \\ \frac{1-\lambda}{2} & \frac{1+\lambda}{2} \end{pmatrix}, T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

A computation gives that $A_\lambda \leq A_\mu$ when $\lambda \leq \mu$ and item (iii) of Proposition 2.3.5 tells us that $w_{A_\lambda}(T) = 1$.
Hence we have a one-parameter family of positive operators that is non-decreasing and that tends in norm to $+\infty$, for which $w_{A_\lambda}(T)$ is constant.
However there are no subspaces H_1, H_2 of $\mathbb{C}^2$ and operators $T_i, A_i \in \mathcal{L}(H_i)$ for which we would have $T = diag(T_1, T_2)$ and $A_\lambda = diag(A_1, A_2)$. This means that the computation of $w_{A_\lambda}(T)$ does not come from the formula of Lemma 2.2.11 and that this example comes from other properties of the A-radii.

Remark 2.3.7. For $0 < \lambda \leq \mu$, all self-adjoint matrices in $\mathcal{L}(\mathbb{C}^2)$ whose eigenvalues are λ and μ are unitarily similar to $diag(\lambda, \mu)$. When $\lambda < \mu$, Proposition 2.3.5 gives us an operator $T \in \mathcal{L}(\mathbb{C}^2)$ such that $w_A(T)$ is not constant over the set of operators A with $A > 0$ and $\sigma(A) = \{\lambda, \mu\}$. Hence we have positive operators A, B and an unitary U such that $A = UBU^*$, but $w_A(T) \neq w_B(T) = w_{UAU^*}(T) = w_A(U^*TU)$.

Lemma 2.3.8. *Let H be a Hilbert space, and let $A, T \in \mathcal{L}(H)$ with $A > 0$. We have*

(i) $w_A(T) \geq w_{\|A\|}(I)r(T)$, and this inequality is sharp.

(ii) If T is invertible then $w_A(T^{-m}) \geq w_A(T^m)^{-1} \geq w_A(T)^{-m}$, $\forall m \geq 1$.

Proof. (i) With item (ii) of Proposition 1.2.18 we have $w_A(T) \geq w_{\|A\|}(T) \geq w_{\|A\|}(I)r(T)$. If $A = \rho I$ then for $T = I$ we obtain an equality. Else, the proof of item (iv) of Proposition 2.2.12 gives, for every $\epsilon > 0$, an orthogonal projection $T \in \mathcal{L}(H)$ such that $w_{\|A\|}(T) - w_A(T) \leq \epsilon$. Hence we end up with $w_A(T) - w_{\|A\|}(I)r(T) = w_{\|A\|}(T) - w_A(T) \leq \epsilon$.
- (ii) Using the previous item and item (iii) of Proposition 2.2.6, we obtain

$$w_A(T^{-m}) \geq w_{\|A\|}(I)r(T^{-m}) \geq r(T^{-m}) \geq \frac{1}{r(T^m)} \geq \frac{1}{w_A(T^m)} \geq \frac{1}{w_A(T)^m}.$$

□

Proposition 2.3.9. *Let H be a Hilbert space. Let $A, T \in \mathcal{L}(H)$ with $A > 0$ and T invertible. If $\|A\| \geq 1$ and if there exists $m \geq 1$ such that $w_A(T^{-m}) = w_A(T)^{-m}$, then $T = \|T\|U$ for some unitary operator U.*
Conversely, if $T = \|T\|U$ for some unitary U, then for every $A \geq I$ and $m \geq 1$ we have $w_A(T^{-m}) = w_A(T)^{-m}$.

Proof. Since T is non-zero, we can consider $\frac{T}{w_A(T)}$ in order to assume that $w_A(T) = 1$. Thus, Lemma 2.3.8 gives us

$$1 = w_A(T^{-m}) \geq w_A(T^m)^{-1} \geq w_A(T)^{-m} = 1.$$

This implies $r(T^{-m}) \leq 1$ and $r(T^m) \leq 1$, so $r(T^m) = 1 = r(T^{-m})$. Hence, we have

$$1 = w_A(T) \geq w_{\|A\|}(T) \geq r(T) = 1;$$
$$1 = w_A(T^{-m}) \geq w_{\|A\|}(T^{-m}) \geq r(T^{-m}) = 1.$$

We end up with $w_{\|A\|}(T^{-m}) = 1 = \frac{1}{w_{\|A\|}(T)}^m$. Since $\|A\| \geq 1$, we can apply Theorem 1.1 from [AL10] to get the desired result.
The converse result comes from the fact that for U an unitary, when $A \geq I$ we have

$$1 = w_1(U) \geq w_A(U) \geq w_{\|A\|}(U) = 1.$$

Therefore, for $T = \|T\|U$, we have $w_A(T) = \|T\|$ and $w_A(T^{-m}) = \frac{1}{\|T\|^m} = w_A(T)^{-m}$. □

Remark 2.3.10. We cannot have a similar equality for some $A > 0$ with $\|A\| < 1$ as the fact that $\rho \mapsto w_\rho(T)$ is decreasing on $\mathbb{R}_+^*$ (see Prop.1.4.5) and the equation in Proposition 1.3.3 imply that $\rho \mapsto w_\rho(T)$ is strictly decreasing on $]0, 1]$, so

$$w_A(T) \geq w_{\|A\|}(T) > \lim_{\rho \to +\infty}(w_\rho(T)) = r(T).$$

Chapter 3

Similarity of operators and classes of projections

This third chapter studies two different notions: algebraic operators and projections. This chapter acts as a transition between classes $C_{(\rho_n)}$ and L^p-projections, the two main characters of this thesis. In Section 3.1 we focus on the study of algebraic operators (operators that are annihilated by a non-zero polynomial) by studying similarity to a contraction as well as some weaker conditions like polynomially boundedness or power-boundedness. We prove that most of these conditions are equivalent for algebraic operators. These results provide useful criteria for similarity to a contraction for algebraic operators.

As an algebraic operator possesses a spectral decomposition through its characteristic projections, we continue in Section 3.2 with the study of some classes of projections, comparing the way they behave with respect to each other. The classes of projections range from the class of norm one projections to classes of L^p-projections. As the properties defining some of these classes are mainly about the norm of vectors in a direct sum of two subspaces, we make some generalizations for direct sums of a finite number of subspaces in Subsection 3.3.B. Working with direct sums brings additional questions regarding these new properties, with some notable differences between direct sums of two or more subspaces (see Lemma 3.2.11).

In Section 3.3 we look at projections that are either Hermitian or L^p-projections in the specific case where X is equal to $L^p(\Omega)$ or to some subspace of $L^p(\Omega)$. We study some variations of properties of these projections by replacing the conditions which are satisfied for every $z \in \mathbb{C}$ by the weaker conditions satisfied "for every $z \in \partial\mathbb{D}$" or "for $z = \pm 1$". Also, as L^p-projections are particular cases of Hermitian projections, we look at subspaces of L^p for which every Hermitian projection is an L^p-projections, or at conditions that ensure the contrary. In Subsection 3.3.B we study the case $p = 2n$, where the relationship $|f+g|^{2n} = (f+g)^n(\overline{f}+\overline{g})^n$ allows us to obtain a useful additional property.

3.1 Similarity to a Contraction for Algebraic Operators

It is known that for Hilbert space operators we have the following implications :

$$\text{similar to a contraction } \Rightarrow \text{polynomially bounded } \Rightarrow \text{power bounded .} \tag{3.1.1}$$

Here, for a Hilbert space operator $T \in \mathcal{L}(H)$, we say that T is *similar to a contraction* if there is an invertible operator $L \in \mathcal{L}(H)$ such that $\|L^{-1}TL\| \leq 1$. We say that T is *polynomially* *bounded* if the von Neumann inequality holds true up to a constant, that is, there exists $C \geq 1$ such that $\|p(T)\| \leq C\|p\|_{\infty,\overline{\mathbb{D}}}$ for every polynomial p. Finally, we say that T is *power bounded* if $\sup_{n\geq 1} \|T^n\| < +\infty$.

Both implications in (3.1.1) cannot be reversed : the first example of a power bounded operator which is not similar to a contraction, so not polynomially bounded was given by Foguel (a different proof has been given by Lebow), while the first example of a polynomially bounded operator which is not similar to a contraction was constructed by Pisier. We refer to [Pis96] for more details and for the history of these examples.

The case of Banach spaces is more complicated: in general the von Neumann inequality is not true (in fact, the von Neumann inequality characterizes Hilbert spaces, see [Pis96]). Some general conditions under which an arbitrary Banach space contraction is polynomially bounded are given in [Zar05].

An operator T is said to be *algebraic* if it is annihilated by some non-zero polynomial Q. This section focuses on the study of algebraic operators regarding similarity to a contraction, power-boundedness and polynomial boundedness. We are able to obtain equivalences between many conditions thanks to the specific behaviour of algebraic operators regarding their spectrum and characteristic spaces.

Polynomial boundedness

We start by recalling a kernel lemma and stating some basic properties of algebraic polynomials.

Lemma 3.1.1 (Kernel lemma)**.** *Let X be Banach space, let $T \in \mathcal{L}(X)$, and let $P \in \mathbb{C}[Z]$. Write P as $P(Z) = \Pi_{i=1}^r (Z-\lambda_i)^{a_i}$, with λ_i that are pairwise disjoint. Then, we have*

$$Ker(P(T)) = \bigoplus_{i=1}^r Ker((T-\lambda_i)^{a_i}).$$

Lemma 3.1.2. *Let X be Banach space and let $T \in \mathcal{L}(X)$. Suppose that there exists $P \in \mathbb{C}[Z] \setminus \{0\}$ such that $P(T) = 0$. Then*

(i) T possesses a minimal polynomial μ_T: the smallest non-zero monic polynomial R such that $R(T) = 0$.
We write $\mu_T(Z) = \Pi_{i=1}^r (Z-\lambda_i)^{a_i}$, with λ_i that are pairwise disjoint.

(ii) For $1 \leq i \leq r$, denote $X_i = Ker((T-\lambda_i)^{a_i})$.
We have $X = X_1 \oplus ... \oplus X_r$, and $T(X_i) \subset X_i$.

(iii) For every $1 \leq i \leq r$, we have $T_i|_{X_i} = \lambda_i I_{X_i} + N_i$, with N_i a nilpotent operator of order a_i.

(iv) For $1 \leq i \leq r$, denote $P_i : X \to X_i$ the projection on X_i parallel to $\bigoplus_{j\neq i} X_j$, $J_i : X_i \to X$ the canonical inclusion, and $T_i = J_i T_i|_{X_i} P_i$.
Then $T = T_1 + ... + T_r$.

(v) For $1 \leq i, j \leq r$, we have $T_i T_j = 0$ if $i \neq j$.
For every polynomial $Q \in \mathbb{C}[Z]$ we have $Q(T) = Q(T_1) + .. + Q(T_r)$.

Proof. (i) As T is algebraic, the kernel of the ring morphism $\psi : Q \in \mathbb{C}[Z] \mapsto Q(T)$ is not reduced to $\{0\}$. Since it is an ideal of $\mathbb{C}[Z]$ and since $\mathbb{C}[Z]$ is a principal ring, there exists an

monic polynomial R such that $Ker(\psi) = (R)$. This polynomial satisfies the condition of item (i). We denote $\mu_T = R$.
- (ii) Applying the kernel lemma 3.1.1 to μ_T and T tells us that

$$X = Ker(\mu_T(T)) = \bigoplus_{i=1}^{r} \mathrm{Ker}((T-\lambda_i)^{a_i}) = \bigoplus_{i=1}^{r} X_i.$$

And since each subspace X_i is the kernel of a polynomial in T, they are stable under T.
- (iii) Let $1 \leq i \leq r$. Since we have $X_i = \mathrm{Ker}((T-\lambda_i)^{a_i})$, this implies that $(T_i|_{X_i} - \lambda_i I_{X_i})^{a_i} = 0$. Hence, $N_i = T_i|_{X_i} - \lambda_i I_{X_i}$ is a nilpotent operator.
Suppose that we have $m < a_i$ such that $N_i^m = 0$. This would imply that $\mathrm{Ker}((T-\lambda_i)^m) = \mathrm{Ker}((T-\lambda_i)^{a_i})$. Using the kernel lemma to the polynomial $Q(Z) = \Pi_{j\neq i}(Z-\lambda_j)^{a_j} \cdot (Z-\lambda_i)^m$, we would in turn obtain that $Q(T) = 0$, which contradicts the minimality of μ_T. Therefore N_i is nilpotent of order a_i.
- (iv) Let $x \in X$. Write $x = x_1 + ... + x_r$, with $x_i \in X_i$. For every $1 \leq i \leq r$, the subspace X_i is stable under T. Hence, we have $T(x_i) \in X_i$, that is $T(x_i) = J_i T_i|_{X_i} P_i(x_i) = T_i(x_i)$. For any $j \neq i$, we have $T_j(x_i) = J_j T_j|_{X_j} P_j(x_i) = 0$. Thus, we obtain

$$T(x) = T(x_1) + .. + T(x_r) = T_1(x_i) + ... + T_r(x_r) = T_1(x) + .. + T_r(x).$$

- (v) Let $1 \leq i \leq r$. As $T_i = J_i T_i|_{X_i} P_i$, we have $\bigoplus_{j\neq i} X_j = \mathrm{Ker}(P_i) \subset \mathrm{Ker}(T_i)$ and $\mathrm{Ran}(T_i) \subset \mathrm{Ran}(J_i) = X_i$. Therefore, for any $j \neq i$ we have $\mathrm{Ran}(T_j) \subset \mathrm{Ker}(T_i)$, so $T_i T_j = 0$.
Hence, we can now prove by induction on $n \geq 1$ that $T^n = T_1^n + .. + T_r^n$. Indeed, this result is true when $n = 1$, and if $T^n = T_1^n + .. + T_r^n$ for some $n \geq 1$ then we have

$$\begin{aligned} T^{n+1} &= T^n T = (T_1^n + .. + T_r^n)(T_1 + .. + T_r) \\ &= \sum_{k=1}^{r} T_k^n (T_1 + .. + T_r) = \sum_{k=1}^{r} T_k^{n+1}, \end{aligned}$$

which concludes the induction proof. Thus, for any polynomial $Q \in \mathbb{C}[Z]$, we have $Q(T) = Q(T_1) + .. + Q(T_r)$. The proof is complete. □

With the results of Lemma 3.1.2, we can now go into our main results for this section.

Proposition 3.1.3. *Let X be Banach space and let $T \in \mathcal{L}(X)$. Suppose that there exists $P \in \mathbb{C}[Z] \setminus \{0\}$ such that $P(T) = 0$. Let $\mu_T(Z)$ be the minimal polynomial of T. Write $\mu_T(Z) = \Pi_{i=1}^{r}(Z-\lambda_i)^{a_i}$, with λ_i that are pairwise disjoint. Then, the following assertions are equivalent*

(i) $|\lambda_i| \leq 1$ *for every* $1 \leq i \leq r$, *and if* $|\lambda_i| = 1$ *then* $a_i = 1$;

(ii) $\frac{\|T^n\|}{n} \to_n 0$;

(iii) T *is* power bounded: $\sup_n(\|T^n\|) < +\infty$;

(iv) T *is* polynomially bounded: *There exists* $M > 0$ *such that* $\|P(T)\| \leq M\|P\|_{\infty,\overline{\mathbb{D}}}$, *for every* $P \in \mathbb{C}[Z]$;

(v) There exists $M > 0$ *and a polygon* $\mathcal{P}$ *contained in* $\overline{\mathbb{D}}$ *such that* $\|P(T)\| \leq M\|P\|_{\infty,\mathcal{P}}$, $\forall P \in \mathbb{C}[Z]$.

Proof. The implications $(v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii)$ are immediate.
We will use the results and notations of Lemma 3.1.2.
Together with these results, we can easily prove that

- If $a_i = 1$ then $T_i|_{X_i} = \lambda_i I_{X_i}$, $T_i = \lambda_i P_i$, and for every $Q \in \mathbb{C}[Z]$ we have
$$Q(T_i) = \begin{cases} Q(0)I_X + \frac{Q(\lambda_i)-Q(0)}{\lambda_i}T_i & \text{if } \lambda_i \neq 0 \\ Q(0)I_X + 0 & \text{if } \lambda_i = 0. \end{cases}$$
In this case, we obtain
$$\|Q(T_i)\| \leq \begin{cases} (1 + 2\frac{\|T_i\|}{|\lambda_i|}) \max(|Q(0)|, |Q(\lambda_i)|) & \text{if } \lambda_i \neq 0 \\ |Q(0)| & \text{if } \lambda_i = 0. \end{cases}$$

- If $a_i > 1$ then $T_i|_{X_i} = \lambda_i I_{X_i} + N_i$ with N_i nilpotent of order a_i. And for $x \in \operatorname{Ker}(N_i^2) \setminus \operatorname{Ker}(N_i) \neq \emptyset$ we get
$$T^n(x) = T_i^n(x) = T_i|_{X_i}^n(x) = \lambda_i^n x + \lambda_i^{n-1} n N_i(x).$$
Thus, $\frac{\|T^n(x)\|}{n} \leq |\lambda_i|^{n-1}\|N_i(x)\| + |\lambda_i|^n \frac{\|x\|}{n}$, $\forall n \geq 1$.

We can now focus on the remaining implications.
- $(ii) \Rightarrow (i)$ As all $\operatorname{Ker}(T - \lambda_i)$ are non-empty by minimality of μ_T, for every $1 \leq i \leq r$ we have $x \in X$ such that $T(x) = \lambda_i x$ and $x \neq 0$. As we have $\frac{\|T^n(x)\|}{n} \to_n 0$, this implies that $|\lambda_i| \leq 1$ for every $1 \leq i \leq r$. Now suppose that there exists i with $|\lambda_i| = 1$ and $a_i > 1$. From the previous discussion, we have a $x \in X_i$ such that
$$T^n(x) = \lambda_i^n x + \lambda_i^{n-1} n N_i(x), \text{ with } N_i(x) \neq 0.$$
Thus, we obtain $\frac{\|T^n(x)\|}{n} \geq \|N_i(x)\| - \frac{\|x\|}{n} > 0$, which contradicts the fact that $\frac{\|T^n\|}{n} \to_n 0$. Therefore if $|\lambda_i| = 1$ we must have $a_i = 1$.
- $(i) \Rightarrow (v)$ Let $1 \leq i \leq r$ be such that $|\lambda_i| = 1$. As we have $a_i = 1$, a previous item gives us
$$\|Q(T_i)\| \leq M_i \sup_{\{0,\lambda_i\}} (|Q(z)|), \forall Q \in \mathbb{C}[Z],$$
with $M_i = 1 + 2\frac{\|T_i\|}{|\lambda_i|}$ if $\lambda_i \neq 0$ and $M_i = 1$ if $\lambda_i = 0$. We now want to show that there exists a closed disk D contained in $\mathbb{D}$ such that for every $1 \leq j \leq r$ with $|\lambda_j| < 1$ and for every polynomial $Q \in \mathbb{C}[Z]$, $\|Q(T_j)\| \leq M_j \|Q\|_{\infty,D}$ for some constant M_j.
Let $t = \max(\{|\lambda_j|\colon |\lambda_j| < 1\})$. Take $\epsilon > 0$ such that $t + \epsilon < 1$. Take $1 \leq j \leq r$ such that $|\lambda_j| < 1$. Let $Q \in \mathbb{C}[Z]$. We have
$$Q(T_j|_{X_j}) = Q(\lambda_j I_{X_j} + N_j) = \sum_{k=0}^{a_j-1} \frac{Q^{(k)}(\lambda_j)}{k!} N_j^k.$$
Take $t_j = t + \epsilon - |\lambda_j|$. The Taylor formula applied to Q_k on $\mathbb{D}(\lambda_j, t_j)$, for $k \geq 0$, gives us
$$\frac{Q^{(k)}(\lambda_j)}{k!} = \frac{1}{2i\pi} \int_{\partial\mathbb{D}(\lambda_j,t_j)} \frac{Q(w)dw}{(w-\lambda_j)^{k+1}}$$
$$\Rightarrow \frac{|Q^{(k)|}(\lambda_j)}{k!} \leq \frac{2\pi}{2\pi} \sup_{|z-\lambda_j|=t_j} \frac{|Q(z)|}{t_j^k}.$$

Since $|\lambda_j| + t_j = t + \epsilon$, we can apply the maximum principle to $\mathbb{D}(0, t+\epsilon)$ to obtain

$$\frac{|Q^{(k)}(\lambda_j)|}{k!} \leq \sup_{|z|=t+\epsilon} \frac{|Q(z)|}{t_j^k}, \text{ for every } 0 \leq k \leq a_j - 1.$$

With this inequality we can now estimate $\|Q(T_j)\|$. Indeed,

$$\begin{aligned}
&\|Q(T_j)\| = \|Q(J_j T_j|_{X_j} P_j)\| = \|J_j Q(T_j|_{X_j}) P_j\| \\
\Rightarrow &\|Q(T_j)\| \leq \|J_j\| \|P_j\| \| \sum_{k=0}^{a_i-1} \frac{Q^{(k)}(\lambda_j)}{k!} N_j^k \| \\
\Rightarrow &\|Q(T_j)\| \leq \|P_j\| (\sum_{k=0}^{a_i-1} \frac{|Q^{(k)}(\lambda_j)|}{k!} \|N_j^k\|) \\
\Rightarrow &\|Q(T_j)\| \leq \sup_{|z|=t+\epsilon} |Q(z)| \|P_j\| (\sum_{k=0}^{a_i-1} \frac{\|N_j^k\|}{k!}) \\
\Rightarrow &\|Q(T_j)\| \leq \sup_{|z|=t+\epsilon} |Q(z)| M_j.
\end{aligned}$$

Now, let $\mathcal{P}$ be a polygon contained in $\overline{\mathbb{D}}$ and containing all λ_i as well as $\overline{\mathbb{D}}(0, r+\epsilon)$. The previous computations shows that

$$\|Q(T)\| = \|Q(T_1) + ... + Q(T_r)\| \leq \sum_j \|Q(T_j)\| \leq \sup_{z \in \mathcal{P}} (|Q(z)|)(M_1 + .. + M_r).$$

The item (v) now follows, which concludes the proof. □

Similarity to a contraction

Recall that on a Hilbert space, an operator that is polynomially bounded is not always similar to a contraction. Thus, regarding Proposition 3.1.3, we wonder if an algebraic operator satisfying condition (i) is similar to a contraction or not.

Here, we can also ask the same question for more general Banach spaces and also focus on different similarity relationships depending on the Banach spaces over which the desired contraction can be defined.

Remark 3.1.4 (SQ^p spaces). For any $1 \leq p < +\infty$, the collection of subspaces of quotients of L^p spaces, also called SQ^p, is stable under quotients, subspaces, finite sums, and ultraproducts. These stability properties make this collection of Banach spaces an interesting setting for questions and results regarding similarity of operators.

Characterizing Banach spaces that are isomorphic to subspaces of quotients of L^p-spaces led to the notion of p-complete boundedness. Many details and results around this notion can be found in Pisier's book [Pis96, Ch.8]. Some aspects of it are related to the similarity of an operator to a contraction on a SQ^p space. Although this notion is not used in this section, we will look at the similarity of an algebraic operator to a contraction on a SQ^p space (see Proposition 3.1.9). Such elements are also a part of the shifting point from questions about similarity to a contraction in certain classes (like $C_{(\rho_n)}$) to questions about properties and characterizations of specific classes of projections, which is the main topic of Chapter 4.

For our next Proposition, we will need a few results about $\ell^p(X)$.

Remark 3.1.5. Let X be a Banach space, and $1 \leq p < +\infty$. We recall that $\ell^p(X)$ is the Banach space defined as

$$\ell^p(X) := \{(x_n)_n \in X^{\mathbb{N}} \colon \|(x_n)_n\| := \|(\|x_n\|)_n\|_{\ell^p} < +\infty\}.$$

We also recall that for a Banach space Y, a quotient of a subspace of Y is isometrically isomorphic to a subspace of some quotient of Y. Indeed, for F a closed subspace of Y and G a closed subspace of F, the quotient F/G of the subspace F by G can be identified as a closed subspace of the quotient of Y/G.

Lemma 3.1.6. *Let X be a Banach space, $1 \leq p < +\infty$, and $n \geq 2$. Let $X_1, ..., X_n$ be subspaces of quotients of $\ell^p(X)$. Then $X_1 \overset{\ell^p}{\oplus} ... \overset{\ell^p}{\oplus} X_n$ is isometrically isomorphic to a subspace of a quotient of $\ell^p(X)$.*

Proof. We will prove this fact by induction on $n \geq 2$. Let X_1, X_2 be subspaces of quotients of $\ell^p(X)$. Write $X_1 = F_1/G_1$, $X_2 = F_2/G_2$, with F_1, G_1, F_2, G_2 closed subspaces of $\ell^p(X)$ such that $G_1 \subset F_1$,$G_2 \subset F_2$. Define the map $\phi : \ell^p(X) \overset{\ell^p}{\oplus} \ell^p(X) \to \ell^p(X)$ that sends a pair of sequences $((x_n)_n, (y_n)_n)$ to the sequence $(z_n)_n$ such that $z_{2n} = x_n$ and $z_{2n+1} = y_n$. As we have

$$\|((x_n)_n, (y_n)_n)\|^p = \sum_{n\geq 0} \|x_n\|^p + \sum_{n\geq 0} \|y_n\|^p = \sum_{m\geq 0} \|z_m\|^p = \|(z_m)_m\|^p,$$

the map ϕ is well-defined and is an isometry which is also isomorphic. As the space $F_1/G_1 \overset{\ell^p}{\oplus} F_2/G_2$ is isometrically isomorphic to $(F_1 \overset{\ell^p}{\oplus} F_2)/(G_1 \overset{\ell^p}{\oplus} G_2)$, we end up with the following isomorphism

$$F_1/G_1 \overset{\ell^p}{\oplus} F_2/G_2 \simeq \phi(F_1 \overset{\ell^p}{\oplus} F_2)/\phi(G_1 \overset{\ell^p}{\oplus} G_2).$$

Hence, $X_1 \overset{\ell^p}{\oplus} X_2$ is isometrically isomorphic to a subspace of a quotient of $\ell^p(X)$.
Now, let $n \geq 2$ and suppose that the statement is true for n. Let $X_1, ..., X_{n+1}$ be subspaces of quotients of $\ell^p(X)$. Then, $X_1 \overset{\ell^p}{\oplus} ... \overset{\ell^p}{\oplus} X_n$ is isometrically isomorphic to a subspace of a quotient of $\ell^p(X)$. As we have

$$X_1 \overset{\ell^p}{\oplus} ... \overset{\ell^p}{\oplus} X_n \overset{\ell^p}{\oplus} X_{n+1} = (X_1 \overset{\ell^p}{\oplus} ... \overset{\ell^p}{\oplus} X_n) \overset{\ell^p}{\oplus} X_{n+1},$$

we can use the first part of the proof to see that $X_1 \overset{\ell^p}{\oplus} ... \overset{\ell^p}{\oplus} X_{n+1}$ is isometrically isomorphic to a subspace of a quotient of $\ell^p(X)$, which concludes the induction. □

Remark 3.1.7. As isometric isomorphisms do not change the structure of a given Banach space, the results of Lemma 3.1.6 stay true if the spaces $X_1, ..., X_n$ are isometrically isomorphic to subspaces of quotients of $\ell^p(X)$.

Now that we have a characterization of ℓ^p direct sums of subspaces of quotients of $\ell^p(X)$, we can prove the main result of this subsection.

Proposition 3.1.8. *Let X be a Banach space and $1 < p < +\infty$. Let $T \in \mathcal{L}(X)$ be an algebraic operator. The following are equivalent* *(i)* $\frac{\|T^n\|}{n} \to 0$;

(ii) *There exists* $M > 0$ *such that* $\|P(T)\| \leq M\|P\|_{\infty,\overline{\mathbb{D}}}$ *for every* $P \in \mathbb{C}[Z]$;

(iii) *There exists a Banach space* Y *and an isomorphism* $S : X \to Y$ *such that* $\|STS^{-1}\| \leq 1$;

(iv) *There exists* Y *a subspace of a quotient of* $\ell^p(X)$ *and an isomorphism* $S : X \to Y$ *such that* $\|STS^{-1}\| \leq 1$.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ comes from Proposition 3.1.3. The implication $(iv) \Rightarrow (iii)$ is immediate.
- $(iii) \Rightarrow (i)$ If condition (iii) is true then T is similar to a contraction, hence $sup_n\|T^n\|$ is finite, so $\frac{\|T^n\|}{n} \to 0$.
- $(i) \Rightarrow (iv)$ Using Proposition 3.1.3, we have a direct sum decomposition of X, $X = X_1 \oplus .. \oplus X_r$ such that each X_i is stable under T and such that we either have $T|_{X_i} = \lambda_i I_{X_i}$ for some $\lambda_i \in \partial\mathbb{D}$ or $r(T|_{X_i}) < 1$. Up to reordering, we can take $1 \leq m \leq r$ such that $T|_{X_i} = \lambda_i I_{X_i}$ for $1 \leq i \leq r$ and $r(T|_{X_i}) < 1$ for $m < i \leq r$. Denote $X_0 := \bigoplus_{i>m}^{r} X_i$. We then have $X = X_0 \oplus X_1 \oplus .. \oplus X_m$. As X_0 is a direct sum of characteristic spaces for T related to eigenvalues lying in $\mathbb{D}$, we have $r(T|_{X_0}) < 1$.
We can then use Corollary 5.1 of [Bad03] (a Banach space Rota theorem) to find a Banach space Y_0 that is a quotient of $\ell^p(X_0)$ and an isomorphism $S_0 : X_0 \to Y_0$ such that $\|S_0 T|_{X_0} S_0^{-1}\| \leq 1$.
For $1 \leq i \leq m$, denote $Y_i = X_i$. As X is isometric to the quotient of $\ell^p(X)$ by $\{(x_n)_n \in \ell^p(X)\colon\ x_0 = 0\}$, we can see that $Y_1, ..., Y_m$ are isomorphically isometric to subspaces of quotients of $\ell^p(X)$. According to a previous remark, we can also see that Y_0 is isomorphically isometric to a subspace of a quotient of $\ell^p(X)$.
Denote $Y = Y_0 \overset{\ell^p}{\oplus} Y_1 \overset{\ell^p}{\oplus} .. \overset{\ell^p}{\oplus} Y_m$. Lemma 3.1.6 then tells us that Y is isometrically isomorphic to a subspace of a quotient of $\ell^p(X)$.
Denote $S : x_0 + x_1 + .. + x_m \in X \mapsto S_0(x_0) + x_1 + ... + x_m \in Y$. For $0 \leq i \leq m$, the projection P_i on X_i parallel to $\oplus_{j\neq i} X_j$ is bounded. Hence, for any $x \in X$ we obtain

$$\begin{aligned}\|S(x)\|_Y^p &= \|S_0(x_0)\|^p + \|x_1\|^p + ... + \|x_m\|^p \\ &= \|S_0(P_0(x))\|^p + \|P_1(x)\|^p + ... + \|P_m(x)\|^p \\ &\leq \max(\|S_0 P_0\|^p, \|P_1\|^p, ..., \|P_m\|^p)(m+1)\|x\|^p,\end{aligned}$$

so S is bounded. Thus S is an isomorphism between X and Y. For every $1 \leq i \leq m$, Y_i is stable for STS^{-1} and $STS^{-1}|_{Y_i} = \lambda_i I_{Y_i}$. With our construction, for any $y \in Y$, we have

$$\begin{aligned}\|STS^{-1}(y)\|_Y^p &= \|STS^{-1}(y_0) + STS^{-1}(y_1) + ... + STS^{-1}(y_m)\|_Y^p \\ &= \|STS^{-1}(y_0) + \lambda_1 y_1 + ... + \lambda_m y_m\|_Y^p \\ &= \|STS^{-1}(y_0)\|^p + \|\lambda_1 y_1\|^p + ... + \|\lambda_m y_m\|^p \\ &= \|S_0 T|_{X_0} S_0^{-1}(y_0)\|^p + \|y_1\|^p + ... + \|y_m\|^p \\ &\leq \|y_0\|^p + \|y_1\|^p + ... + \|y_m\|^p = \|y\|^p.\end{aligned}$$

Therefore we end up with $\|STS^{-1}\| \leq 1$, which concludes the proof. $\square$

Corollary 3.1.9. *Let* X *be a Banach space and* $1 < p < +\infty$. *Let* $T \in \mathcal{L}(X)$ *be an algebraic operator. Suppose that* X *is isomorphic to a* SQ^p *space. The following are equivalent* *(i)* $\frac{\|T^n\|}{n} \to 0$;

(ii) *There exists* $M > 0$ *such that* $\|P(T)\| \leq M\|P\|_{\infty,\overline{\mathbb{D}}}$ *for every* $P \in \mathbb{C}[Z]$;

(iii) *There exists* Y *a* SQ^p *space and an isomorphism* $S : X \to Y$ *such that* $\|STS^{-1}\| \leq 1$.

Proof. By hypothesis, we have a SQ^p-space X' and an isomorphism $R : X \to X'$. The operator $T' = RTR^{-1}$ is then an algebraic operator on the SQ^p-space X', and for any polynomial $P \in \mathbb{C}[Z]$ we have

$$\|P(T')\| = \|SP(T)S^{-1}\| \leq \|S\|\|S^{-1}\|\|P(T)\|.$$

Hence up to considering X' and T' we can suppose that X is a SQ^p-space.
As the class of SQ^p-spaces is stable under countable ℓ^p-sum, if X is a SQ^p-space then $\ell^p(X) = \bigoplus_{n\geq 0}^{\ell^p} X$ is a SQ^p-space. Hence any subspace of quotient of $\ell^p(X)$ is a SQ^p-space. We can then apply Proposition 3.1.8 to obtain the equivalence. □

When $p = 2$, the class of SQ^p-spaces coincides with the class of Hilbert spaces. Thus we can use Corollary 3.1.9 to see that any algebraic operator on a Hilbert space is power-bounded if and only if it is similar to a contraction on a Hilbert space. A similar equivalence can be obtained using a result of Delaubenfels [dL98].

Corollary 3.1.10. Let H be a Hilbert space and let $T \in \mathcal{L}(H)$ be an algebraic operator. The following are equivalent

(i) $\frac{\|T^n\|}{n} \to_n 0$;

(ii) T is *power bounded*: $\sup_n(\|T^n\|) < +\infty$;

(iii) T is *polynomially bounded*: There exists $M > 0$ such that $\|P(T)\| \leq M\|P\|_{\infty,\overline{\mathbb{D}}}$, for every $P \in \mathbb{C}[Z]$;

(iv) T is *similar to a contraction* on H.

Proof. We have $(iv) \Rightarrow (ii)$, and Proposition 3.1.3 gives us $(iii) \Leftrightarrow (ii) \Leftrightarrow (i)$.
- $(iii) \Rightarrow (iv)$ According to Proposition 3.1.3, there exists $M' > 0$ and a polygon $\mathcal{P}$ contained in $\overline{\mathbb{D}}$ such that $\|P(T)\| \leq M'\|P\|_{\infty,\mathcal{P}}$, for every $P \in \mathbb{C}[Z]$. We can then use Theorem 4.4 of [dL98] to find that T is similar to a contraction on H, whose numerical range is contained in $\mathcal{P}$.
The implication $(ii) \Rightarrow (iv)$ has also been proven by Mlak in 1974 [Mla74, Cor.3]. □

Remark 3.1.11. Fundamentally, an algebraic operator T satisfying condition (i) of Proposition 3.1.3 has the form $T = T_1 + .. + T_r$ with $T_1, ..., T_r$ such that $T_iT_j = T_jT_i = 0$ for every $i \neq j$, and T_i similar to a contraction for every $1 \leq i \leq r$. These operators T_i are more specific than that, but our point of view shifted more onto the subspaces tied to them (that are in direct sum) rather than their specific properties (aside from the similarity to a contraction).
We can for example study complementary closed subspaces X_1, X_2 for which the operator $T = diag(C_1, C_2) \in \mathcal{L}(X_1 + X_2)$ is similar to a contraction (on some class of Banach spaces) for every contraction $C_i \in \mathcal{L}(X_i)$ (resp. for any operator similar to a contraction). Such a property is for example true when $X = X_1 \overset{N}{\oplus} X_2$, for N a norm on $\mathbb{R}^2$ (i.e. $\|x_1 + x_2\| = N(\|x_1\|, \|x_2\|)$), as we have $\|T\| = \max(\|C_1\|, \|C_2\|) \leq 1$. Such a question motivates the next section.

3.2 Classes of Projections on a Banach Space

As an algebraic operator possesses a spectral decomposition through its characteristic projections, the study shifts in this Section to classes of projections. We will define multiple classes of projections, with many classical ones, and study how these classes behave with respect to each other.
As the properties defining some of these classes are mainly about the norm of vectors in a direct sum of two subspaces, we make generalizations of some classes for direct sums of a finite number of subspaces in Subsection 3.3.B. Working with direct sums adds another parameter, namely the choice of the direct sum or of a smaller direct sum. This raises new questions regarding the classes we will be studying.

3.2.A Various classes of projections

Definition 3.2.1. Let X be a Banach space. Let X_1, X_2 be closed subspaces of X that are in direct sum. As we have $X = X_1 \oplus X_2$ and each of these spaces is closed, the projection P over X onto X_1 parallel to X_2 is bounded. We define the following properties

(P_1) P is a norm-one projection : $\|P\| = 1$;

(P_2) P and $I - P$ are norm-one projections :$\|P\| = \|I - P\| = 1$;

(P_3) P is *generalized bicircular*: There is $\lambda \in \partial\mathbb{D} \setminus \{1\}$ such that $\lambda P + (I - P)$ is a surjective isometry

(P_4) P is *Hermitian*: For every $\alpha \in \mathbb{R}$, $e^{i\alpha P}$ is a surjective isometry.

(P_5) For every $\lambda_i \in \mathbb{C}$, every $x_i \in X_i$, $1 \leq i \leq 2$, we have

$$\|\lambda_1 x_1 + \lambda_2 x_2\| \leq \max_i(|\lambda_i|)\|x_1 + x_2\|.$$

(P'_5) For every $\lambda \in \partial\mathbb{D}$, $x_i \in X_i$, $1 \leq i \leq 2$, we have

$$\|x_1 + \lambda x_2\| = \|x_1 + x_2\|.$$

(P_6) For every $x_i, y_i \in X_i$, $1 \leq i \leq 2$, with $\|x_2\| = \|y_2\|$, we have

$$\|x_1 + x_2\| = \|x_1 + y_2\|.$$

(P_7) For every closed subspaces $V_i \subset X_i$, every $C_i \in \mathcal{L}(X_i)$, and every $x_i \in V_i$, $1 \leq i \leq 2$, we have

$$\|C_1(x_1) + C_2(x_2)\| \leq \max_{i=1,2}\Big(\sup_{y_i \in V_i,\ \|y_i\|=1}(\|C_i(y_i)\|)\Big)\|x_1 + x_2\|.$$

(P_8) For every $x_i \in X_i \setminus \{0\}$, every $C_i \in \mathcal{L}(X_i)$, $1 \leq i \leq 2$, we have

$$\|C_1(x_1) + C_2(x_2)\| \leq \max_i\Big(\frac{\|C_i(x_i)\|}{\|x_i\|}\Big)\|x_1 + x_2\|.$$

(P_8') For every $x_i, y_i \in X_i$ with x_i non-zero, $1 \leq i \leq 2$, we have

$$\|y_1 + y_2\| \leq \max_i(\frac{\|y_i\|}{\|x_i\|})\|x_1 + x_2\|.$$

(P_9) For every $x_i, y_i \in X_i$ with $\|x_i\| = \|y_i\|$, $1 \leq i \leq 2$, we have

$$\|y_1 + y_2\| = \|x_1 + x_2\|.$$

(P_{10}) There exists $1 \leq p \leq +\infty$ such that P is a L^p*-projection*: For every $x_i \in X_i$ we have

$$\|x_1 + x_2\| = \|(\|x_1\|, \|x_2\|)\|_{\ell^p}.$$

Lemma 3.2.2. *Let X be a Banach space. Let X_1, X_2 be closed subspaces of X that are in direct sum. Let $P \in \mathcal{L}(X)$ be the projection onto X_1 parallel to X_2. We have*

$$(P_{10}) \Rightarrow (P_9) \Leftrightarrow (P_8') \Leftrightarrow (P_8) \Leftrightarrow (P_7) \Rightarrow (P_6) \Rightarrow (P_5) \Leftrightarrow (P_5') \Leftrightarrow (P_4) \Rightarrow (P_3) \Rightarrow (P_2) \Rightarrow (P_1).$$

Furthermore,

- $(P_1) \nRightarrow (P_2) \nRightarrow (P_3) \nRightarrow (P_4)$;
- $(P_5') \nRightarrow (P_6)$;
  If $dim(X_2) = 1$, then $(P_5') \Leftrightarrow (P_6)$;
- $(P_6) \nRightarrow (P_7)$;
  If $dim(X_1) = 1$, then $(P_7) \Leftrightarrow (P_6)$;
- $(P_9) \nRightarrow (P_{10})$.

Proof. The implication $(P_{10}) \Rightarrow (P_9)$ is immediate. The implications $(P_8') \Rightarrow (P_8) \Rightarrow (P_7)$ are immediate. The implications $(P_8') \Rightarrow (P_9)$ and $(P_7) \Rightarrow (P_6)$ are immediate, as well is the implication $(P_6) \Rightarrow (P_5')$. The implication $(P_2) \Rightarrow (P_1)$ is immediate.
For any $\alpha \in \mathbb{R}$, we have $e^{i\alpha P} = I + (e^{i\alpha} - 1)P = e^{i\alpha}P + (I - P)$. Thus we obtain $(P_4) \Rightarrow (P_3)$. We can also see that $e^{i\alpha}P + (I - P)$ is always surjective. Hence, we have $(P_5') \Rightarrow (P_4)$.
- $(P_9) \Rightarrow (P_8')$ We use the fact that for every $w_i \in X_i$, the map $z \mapsto \|w_1 + zw_2\|$ is sub-harmonic on $\mathbb{C}$. Thus, for every $z \in \mathbb{D}$, we have

$$\|w_1 + zw_2\| \leq \sup_{|\lambda|=1}(\|w_1 + \lambda w_2\|) = \|w_1 + w_2\|.$$

If we suppose by symmetry that $\frac{\|y_1\|}{\|x_1\|} = \max_i(\frac{\|y_i\|}{\|x_i\|})$, we then obtain

$$\begin{aligned}\|y_1 + y_2\| &= \|x_1\frac{\|y_1\|}{\|x_1\|} + x_2\frac{\|y_2\|}{\|x_2\|}\| = \frac{\|y_1\|}{\|x_1\|}\|x_1 + \frac{\|y_2\|\|x_1\|}{\|x_2\|\|y_1\|}x_2\| \\ &\leq \frac{\|y_1\|}{\|x_1\|}\|x_1 + x_2\| = \max_i(\frac{\|y_i\|}{\|x_i\|})\|x_1 + x_2\|.\end{aligned}$$

- $(P_8) \Rightarrow (P_8')$ We use the Hahn-Banach Theorem on X_1 and X_2 to get linear forms f_1, f_2 such that $f_i(x_i) \neq 0$ and $\|f_i\| = \frac{\|f_i(x_i)\|}{\|x_i\|}$. We can then build rank 1 linear maps C_i on X_i such that

C_i sends x_i onto y_i, which gives us (P_8').
- $(P_7) \Rightarrow (P_8)$ Let $x_i \in X_i \setminus \{0\}$, and $C_i \in \mathcal{L}(X_i)$, $1 \leq i \leq 2$. Denote $Y_i = \text{Span}(x_i)$. We then have $sup_{y_i \in Y_i,\ \|y_i\|=1} \|C_i(y_i)\| = \|C_i(\frac{x_i}{\|x_i\|})\|$, which allows us to obtain (P_8).
- $(P_5) \Rightarrow (P_5')$ We can notice that for every $\lambda \in \partial\mathbb{D}$ the operator $T(x_1 + x_2) := x_1 + \lambda x_2$ is invertible and satisfies $\|T\| \leq 1$, $\|T^{-1}\| \leq 1$. Hence T is an invertible isometry, so (P_5') is true.
- $(P_5') \Rightarrow (P_5)$ Let $x_1, x_2, \lambda_1, \lambda_2$ and suppose that $|\lambda_1| = \max_i(|\lambda_i|)$. Since $z \mapsto \|\lambda_1 x_1 + z x_2\|$ is sub-harmonic, Property (P_5'), gives us

$$\|\lambda_1 x_1 + \lambda_2 x_2\| \leq \sup_{|z|=|\lambda_1|} \|\lambda_1 x_1 + z x_2\| = \|\lambda_1 x_1 + \lambda_1 x_2\| = |\lambda_1| \|x_1 + x_2\|.$$

Similarly, if $|\lambda_2| = \max_i(|\lambda_i|)$ we have

$$\begin{aligned} \|\lambda_1 x_1 + \lambda_2 x_2\| &\leq \sup_{|z|=|\lambda_2|} \|z x_1 + \lambda_2 x_2\| = \||\lambda_2| e^{it} x_1 + \lambda_2 x_2\| \\ &\leq \||\lambda_2| e^{it} x_1 + |\lambda_2| e^{it} x_2\| = |\lambda_2| \|x_1 + x_2\|. \end{aligned}$$

- $(P_4) \Rightarrow (P_5')$ Since $e^{i\alpha P} = e^{i\alpha}(P + e^{-i\alpha}(I - P))$ is invertible, it is a surjective isometry for every $\alpha \in \mathbb{R}$ if and only if $P + e^{-i\alpha}(I - P)$ is an isometry for every $\alpha \in \mathbb{R}$. This gives us the equivalence between (P_4) and (P_5').
- $(P_3) \Rightarrow (P_2)$ The map $z \mapsto \|P + z(I - P)\|$ is subharmonic and equal to 1 on the unit circle. Hence, the mean inequality gives us $\|P + 0\| \leq 1$. Since P is a non-zero projection, we must have $\|P\| = 1$. By considering $w \mapsto \|wP + (I - P)\|$ we obtain that $\|I - P\| = 1$ in a similar way.
- $(P_9) \not\Rightarrow (P_{10})$ Take N a norm on $\mathbb{R}^2$ that is not an L^p norm. Let X_1, X_2 be non-trivial Banach spaces and $X = X_1 \oplus X_2$, with $\|x_1 + x_2\|_X = N(\|x_1\|_{X_1}, \|x_2\|_{X_2})$. Then (P_9) is satisfied but not (P_{10}).
- $(P_1) \not\Rightarrow (P_2)$ There are projections P such that $\|P\| = 1$ and $\|I - P\| = 2$, for example the projection on $\text{Span}(e_1)$ parallel to $\text{Span}(e_2)$ in $\mathbb{C}^2$, for $\|xe_1 + ye_2\| = |y| + |x - y|$.
- $(P_2) \not\Rightarrow (P_3)$ Take $X_1 = \mathbb{C}$ and $X_2 = \mathbb{C}$. We will build a norm on $\mathbb{C}^2$ such that (P_2) is satisfied but not (P_3). For this, denote

$$K = \text{Conv}(\{\lambda e_1, \gamma e_2, \delta(e_1 + e_2),\ \lambda, \gamma, \delta \in \overline{\mathbb{D}}\}.$$

The set K is convex, compact, contains 0 in its interior, and is stable for $x \mapsto \lambda x$, $\forall \lambda \in \overline{\mathbb{D}}$. Furthermore, K is contained in the polydisc $\overline{\mathbb{D}}^2$ and the vectors $e_1, e_2, e_1 + e_2$ are in its boundary, but $e_1 + \lambda e_2$ do not lie in K for every $\lambda \in \partial\mathbb{D}$ such that $\lambda \neq 1$.
Hence, there exists a norm N on $\mathbb{C}^2$ whose closed unit ball is K. As we have $N(e_1) = N(e_2) = 1$ and $K \subset \overline{\mathbb{D}}^2$, we obtain $\|P\| = \|I - P\| = 1$ so (P_2) is true. However, we have

$$N(e_1 + e_2) = 1 \neq N(e_1 + \lambda e_2),\ \forall \lambda \in \partial\mathbb{D} \setminus \{1\},$$

so (P_3) is false.
- $(P_5') \not\Rightarrow (P_6)$. Take $X_1 = \mathbb{C}$, $X_2 = \mathbb{C}^2$. We will build a norm on $\mathbb{C}^3$ such that (P_5') is satisfied but not (P_6). For this, denote

$$K = \text{Conv}(\{\lambda_1 e_1 + \lambda_2 e_2, \lambda_3 e_3, ,\ \lambda_i \in \overline{\mathbb{D}}\}).$$

The set K is convex, compact, contains 0 in its interior, and is stable for $x \mapsto \lambda x$ and $ae_1 + be_2 + ce_3 \mapsto ae_1 + \lambda e_2 + \lambda e_3$, $\forall \lambda \in \overline{\mathbb{D}}$. Furthermore, K is contained in the polydisc $\overline{\mathbb{D}}^3$ and the vectors

e_1, e_2, e_3, e_1+e_2 are in its boundary, but it does not contain e_1+e_3. Lastly, the boundary of K is stable for $ae_1+be_2+ce_3 \mapsto ae_1+\lambda be_2+\lambda ce_3$, $\forall \lambda \in \partial\mathbb{D}$.
Hence, there exists a norm N on $\mathbb{C}^3$ whose closed unit ball is K. As we have

$$N(ae_1+be_2+ce_3) = N(ae_1+\lambda be_2+\lambda ce_3),\ \forall \lambda \in \partial\mathbb{D},$$

we have Property (P_5'). However, as we also have $N(e_1) = N(e_2) = N(e_3) = 1$ and $N(e_1+e_2) = 1 < N(e_1+e_3)$, (P_6) is not satisfied.
- When $dim(X_2) = 1$, for every $x_2, y_2 \in X_2$ we have $\|x_2\| = \|y_2\|$ if and only if $y_2 = \lambda x_2$ for a $\lambda \in \partial\mathbb{D}$. Hence, Properties (P_5') and (P_6) coincide in this case.
- $(P_6) \not\Rightarrow (P_7)$ Take $X_1 = \mathbb{C}^2$, $X_2 = \mathbb{C}$. We will build a norm on $\mathbb{C}^3$ such that (P_6) is satisfied but not (P_7). For this, denote

$$K = \text{Conv}(\{\lambda_2 e_2 + \lambda_3 e_3, \lambda_1 e_1, ,\ \lambda_i \in \overline{\mathbb{D}}\}.$$

The set K is convex, compact, contains 0 in its interior, and is stable for $x \mapsto \lambda x$ and $ae_1 + be_2 + ce_3 \mapsto \lambda ae_1 + \lambda be_2 + ce_3$, $\forall \lambda \in \overline{\mathbb{D}}$. Furthermore, K is contained in the polydisc $\overline{\mathbb{D}}^3$ and the vectors e_1, e_2, e_3, e_2+e_3 are in its boundary, but it does not contain e_1+e_3. Lastly, the boundary of K is stable for the transformation $ae_1+be_2+ce_3 \mapsto \lambda ae_1+\lambda be_2+ce_3$, $\forall \lambda \in \partial\mathbb{D}$.
Hence, there exists a norm N on $\mathbb{C}^2$ whose closed unit ball is K. As we have

$$N(ae_1+be_2+ce_3) = N(\lambda ae_1+\lambda be_2+ce_3),\ \forall \lambda \in \partial\mathbb{D},$$

and as $dim(X_2) = 1$, (P_6) is satisfied. However, as we also have $N(e_1) = N(e_2) = N(e_3) = 1$ and $N(e_2+e_3) = 1 \neq N(e_1+e_3)$, (P_7) is not satisfied.
- When $dim(X_1) = 1$, for every $x_1, y_1 \in X_1$ we have $\|x_1\| = \|y_1\|$ if and only if $y_1 = \lambda x_1$ for some $\lambda \in \partial\mathbb{D}$. Hence, Properties (P_6) and (P_7) coincide in this case.
- $(P_3) \not\Rightarrow (P_4)$. If (P_3) is satisfied for a λ that is not a root of unity, then (P_4) is true. However there exist projections P that only satisfy (P_3) for $\lambda = -1$ or for some root of unity. Such counter-examples appear in [Lin08, Thm 4] (some projections on $L^p(\Omega, X)$, $p \neq 2$, (Ω, μ) of finite measure, X with an L^p-trivial structure, only satisfy (P_3) for $\lambda = -1$) and in [Kin13, Thm 2.2] (any projection on $H^p(\partial\mathbb{D}^2)$, $p \neq 2$, can only satisfy (P_3) for $\lambda = -1$). □

Lemma 3.2.3. *Let X be a Banach space. Let X_1, X_2 be closed subspaces of X that are in direct sum. Let $P \in \mathcal{L}(X)$ be the projection on X_1 parallel to X_2.*

(i) P satisfies Property (P_8) if and only if for every $C_i \in \mathcal{L}(X_i)$ we have

$$\|x_1+x_2 \mapsto C_1(x_1)+C_2(x_2)\| \leq \max_i(\|C_i\|).$$

In particular, if C_i are contractions, then $(x_1+x_2 \mapsto C_1(x_1)+C_2(x_2))$ is a contraction.

(ii) If P satisfies Property (P_8), and if for $C_i \in \mathcal{L}(X_i)$ we have $a_i > 0$ such that $a_i\|x_i\| \leq \|C_i(x_i)\|$, $\forall x_i \in X_i$, then the operator $T := x_1+x_2 \mapsto C_1(x_1)+C_2(x_2)$ satisfies

$$\min_i(a_i)\|x_1+x_2\| \leq \|T(x_1+x_2)\| \leq \max_i(\|C_i\|)\|x_1+x_2\|.$$

(iii) P satisfies (P_5) if and only if for $C_i = \lambda_i I_{X_i}$ with $\lambda_i \in \overline{\mathbb{D}}$, we have

$$\|x_1+x_2 \mapsto C_1(x_1)+C_2(x_2)\| = \max_i(|\lambda_i|) \leq 1.$$

Proof. (i) The direct implication is a consequence of (P_8).
Conversely, we will show that (P_9) is true, which is equivalent to (P_8). Let $x_i, y_i \in X_i$, be such that $\|x_i\| = \|y_i\|$, $1 \leq i \leq 2$. By using Hahn-Banach's Theorem on X_i, we obtain a linear form f_i such that

$$f_i(x_i) = 1 \text{ and } \|f_i\| = \|f_i(\frac{x_i}{\|x_i\|})\| = \frac{1}{\|x_i\|}.$$

We can then define $C_i \in \mathcal{L}(X_i)$ by $C_i(z_i) = f_i(z_i)y_i$. These are rank one operators, with

$$\|C_i\| = \|y_i\|\|f_i\| = \frac{\|y_i\|}{\|x_i\|} = 1.$$

Thus, we have $\|y_1 + y_2\| = \|C_1(x_1) + C_2(x_2)\| \leq \|x_1 + x_2\|$. Since this inequality is true for every $x_i, y_i \in X_i$, such that $\|x_i\| = \|y_i\|$, we obtain an equality by symmetry. Therefore, (P_9) is satisfied.
- (ii) In this context the operators $C_i : X_i \to C_i(X_i)$ are isomorphisms. Hence, for $C_i^{-1} : C_i(X_i) \to X_i$ we have $\|C_i^{-1}\| \leq \frac{1}{a_i}$. Take $x_i \in X_i$ non-zero and denote $y_i = C_i(x_i)$. Since y_i are non-zero, we can use (P_8') to obtain

$$\begin{aligned}
&\|C_1^{-1}(y_1) + C_2^{-1}(y_2)\| \leq \max_i(\|C_i^{-1}(\frac{y_i}{\|y_i\|})\|)\|y_1 + y_2\| \\
\Rightarrow &\|C_1^{-1}(y_1) + C_2^{-1}(y_2)\| \leq \max_i(\|C_i^{-1}\|)\|y_1 + y_2\| \\
\Rightarrow &\|x_1 + x_2\| \leq \max_i(\frac{1}{a_i})\|T(x_1 + x_2)\| \\
\Rightarrow &\min_i(a_i)\|x_1 + x_2\| \leq \|T(x_1 + x_2)\|
\end{aligned}$$

- (iii) The equivalence between these two conditions is immediate. □

Lemma 3.2.4. *Let X be a Banach spaces. Let X_1, X_2 be closed subspaces of X. Let $Y_1 \subset X_1$, $Y_2 \subset X_2$ be closed subspaces. Then*

(i) If (P_{10}) is true for $X_1 \oplus X_2$, then (P_{10}) is true for $Y_1 \oplus Y_2$.

(ii) If (P_9) is true for $X_1 \oplus X_2$, then (P_9) is true for $Y_1 \oplus Y_2$.

(iii) If (P_6) is true for $X_1 \oplus X_2$, then (P_6) is true for $Y_1 \oplus Y_2$.

(iv) If (P_5) is true for $X_1 \oplus X_2$, then (P_5) is true for $Y_1 \oplus Y_2$.

Proof. By taking $x_1, y_1 \in Y_1$, $x_2, y_2 \in Y_2$ such that $\|x_i\| = \|y_i\|$ (or only $\|x_2\| = \|y_2\|$), and $\lambda_1, \lambda_2 \in \mathbb{C}$, a quick computation gives each implication. □

Lemma 3.2.5. *Let X be a Banach space. Let $X_1, .., X_r$ be closed subspaces of X such that $X = X_1 \oplus X_2 \oplus ... \oplus X_r$. If Property (P_9) is true for $X_i \oplus (X_{i+1} \oplus ... \oplus X_r)$ for every $1 \leq i \leq r-1$, then Property (P_9) is true for $X_i \oplus X_j$, for every $1 \leq i < j \leq r$.*
The converse is false in general.

Proof. We can use Lemma 3.2.4 in order to obtain the implication.
We will exhibit a counter-example for the converse. Take $X_1 = \mathbb{C}$, $X_2 = \mathbb{C}$, $X_3 = \mathbb{C}$. We will

build a norm on $\mathbb{C}^3$ such that Property (P_9) is satisfied for $X_1 \oplus X_2$, $X_1 \oplus X_3$, $X_2 \oplus X_3$, but not for $X_1 \oplus (X_2 \oplus X_3)$. For this, denote

$$K = \text{Conv}(\{\lambda_1 e_1 + \lambda_2 e_2, \lambda_3 e_1 + \lambda_4 e_3, \lambda_5 e_2 + \lambda_6 e_3, \lambda_7(e_1 + e_2 + e_3), \lambda_i \in \overline{\mathbb{D}}\}.$$

The set K is then convex, closed, contains 0 in its interior, and is invariant under $x \mapsto \lambda x$, $ae_1 + be_2 \mapsto ae_1 + \lambda be_2$, $ae_1 + ce_3 \mapsto ae_1 + \lambda ce_3$ and $be_2 + ce_3 \mapsto be_2 + \lambda ce_3$, for every $\lambda \in \overline{\mathbb{D}}$. Furthermore, K is included in the polydisc $\overline{\mathbb{D}}^3$ and $e_1, e_2, e_3, e_1 + \lambda e_2, e_1 + \lambda e_3, e_2 + \lambda e_3, e_1 + e_2 + e_3$ are on its frontier, while $e_1 + e_2 - e_3$ is not in K. Hence, there exists a norm N on X whose closed unit ball is K. As $dim(X_i) = 1$, for every $x_i, y_i \in X_i$ we have $\|x_i\| = \|y_i\|$ if and only if $y_i = \lambda x_i$ for some $\lambda \in \partial\mathbb{D}$. As we have

$$N(ae_1 + be_2) = N(ae_1 + \lambda be_2),\ N(ae_1 + ce_3) = N(ae_1 + \lambda ce_3),$$
$$N(be_2 + ce_3) = N(be_2 + \lambda ce_3),\ \forall \lambda \in \partial\mathbb{D},$$

the spaces $X_1 \oplus X_2$, $X_1 \oplus X_3$, $X_2 \oplus X_3$ satisfy Property (P_9). However, as

$$N(e_1) = N(e_2 + e_3) = N(e_2 - e_3) = 1,\ N(e_1 + e_2 + e_3) = 1 \neq N(e_1 + e_2 - e_3),$$

the space $X_1 \oplus (X_2 \oplus X_3)$ does not satisfy Property (P_9). □

Remark 3.2.6. We will see in the next chapter that the implication of Lemma 3.2.5 holds for Property (P_{10}) (L^p-projections). We will also see, in this context, that the converse is true if we add some properties to the underlying Banach space X (properties extracted from L^p spaces).

Lemma 3.2.7. *Let X be a Banach space. Let X_1, X_2 be closed subspaces of X that are in direct sum. The following are equivalent*

(i) For any $\lambda \in \partial\mathbb{D}$, the operator V_λ defined by $(x_1 + x_2 \mapsto x_1 + \lambda x_2)$ is similar to a contraction in $\mathcal{L}(X)$, i.e., there exists an invertible operator $S \in \mathcal{L}(X)$ such that $\|S^{-1} V_\lambda S\| \leq 1$.

(ii) There exists Y_1, Y_2 closed subspaces of X with $X = Y_1 \oplus Y_2$, $Y_i \simeq X_i$, such that Property (P_5') is satisfied for $Y_1 \oplus Y_2$.

Proof. $(i) \Rightarrow (ii)$ Take $\lambda = e^i$, $C_1 = I_{X_1}$ and $C_2 = \lambda I_{X_2}$. Let $S \in \mathcal{L}(X)$ be an invertible operator and denote $Y_i = S^{-1}(X_i)$, $1 \leq i \leq 2$. We have $X = Y_1 \oplus Y_2$ and

$$S^{-1} C_1 S(y_1) + S^{-1} C_2 S(y_2) = y_1 + \lambda y_2,\ \forall y_i \in Y_i.$$

Thus, we have

$$\|y_1 + \lambda y_2\| \leq \|y_1 + y_2\|,\ \forall y_i \in Y_i.$$

With an induction on $n \geq 1$, we then obtain

$$\|y_1 + \lambda^n y_2\| \leq \|y_1 + y_2\|,\ \forall y_i \in Y_i,\ \forall n \geq 1.$$

By density of $\{e^{in},\ n \geq 0\}$ in $\partial\mathbb{D}$ and by continuity of $\|\cdot\|$, we get

$$\|y_1 + \gamma y_2\| \leq \|y_1 + y_2\|,\ \forall y_i \in Y_i,\ \forall \gamma \in \partial\mathbb{D}.$$

Since this inequality is true for γ and $\overline{\gamma}$, for all $\gamma \in \partial\mathbb{D}$, we must have an equality. Therefore, Property (P_5') is true for $X = Y_1 \oplus Y_2$.
- $(ii) \Rightarrow (i)$ Conversely, we have two invertible operators, $S_1 : X_1 \to Y_1$ and $S_2 : X_2 \to Y_2$. As $X = X_1 \oplus X_2$, the linear map $S : x_1 + x_2 \in X \mapsto S_1(x_1) + S_2(x_2) \in X$ is bounded and invertible. Then, for any $\lambda \in \partial\mathbb{D}$ and for $T_\lambda(x_1 + x_2) := x_1 + \lambda x_2$, we have $STS^{-1}(y_1 + y_2) = y_1 + \lambda y_2$, which is a contraction due to Property (P_5'). □

Lemma 3.2.8. *Let X be a Banach space and let $X_1, ..., X_r$ be closed subspaces such that $X = X_1 \oplus X_2 \oplus ... \oplus X_r$. The following are equivalent*

(i) There exists closed subspaces $Y_1, .., Y_r$ such that $X = Y_1 \oplus ... \oplus Y_r$, and such that for every $N_i \in \mathcal{L}(X_i)$ nilpotent, there is $S \in \mathcal{L}(X)$ invertible with $S^{-1}(X_i) = Y_i$, such that $\|S^{-1}VS\| \leq 1$, where V denotes the operator

$$x_1 + .. + x_r \mapsto N_1(x_1) + .. + N_r(x_r).$$

(ii) There exists closed subspaces $Y_1, .., Y_r$ such that $X = Y_1 \oplus ... \oplus Y_r$, $Y_i \simeq X_i$, and such that for every $N_i \in \mathcal{L}(Y_i)$ nilpotent, we have

$$\inf_{\{S_i \textit{invertible},\ S_i \in \mathcal{L}(Y_i)\}} (\|S_i^{-1} N_i S_i\|) = 0.$$

Proof. $(i) \Rightarrow (ii)$ Let $1 \leq i \leq r$ fixed. Let $N_i \in \mathcal{L}(Y_i)$ be nilpotent and non-zero. Condition (i) implies the existence of $S \in \mathcal{L}(X)$ invertible such that $S^{-1}(X_i) = Y_i$. For every $n \geq 1$, we define $T_n(x_1 + .. + x_r) := nSN_iS^{-1}(x_r)$. Then, we have $T_n = nT_1$. Since $nSN_iS^{-1} \in \mathcal{L}(X_i)$ and is nilpotent, for every $n \geq 1$ we have $U_n \in \mathcal{L}(X)$ invertible such that

$$U_n^{-1}(X_i) = Y_i, \text{ and } \|U_n^{-1} T_n U_n\| \leq 1.$$

Define $R_i(y_1 + .. + y_r) := N_i(y_i)$. We then have $SR_iS^{-1} = T_1 = \frac{1}{n} T_n$, so

$$\|(U_n^{-1} S) R_i (S^{-1} U_n)\| \leq \frac{1}{n},\ \forall n \geq 1,$$

which gives condition (ii).
- $(ii) \Rightarrow (i)$ Let $T_i : X_i \to Y_i$ be bounded and invertible. The linear map $T(x_1 + ... + x_r) := T_1(x_1) + ... + T_r(x_r)$ is then bounded and invertible. Let $N_i \in \mathcal{L}(X_i)$ be nilpotent, for $1 \leq i \leq r$. Define $V(x_1 + ... + x_r) := N_1(x_1) + ... + N_r(x_r)$. We then have

$$TVT^{-1}(y_1 + ... + y_r) = T_1 N_1 T_1^{-1}(y_1) + ... + T_r N_r T_r^{-1}(y_r).$$

Let $\epsilon > 0$. For every $1 \leq i \leq r$, condition (ii) gives us $S_i \in \mathcal{L}(Y_i)$ invertible such that $\|S_i T_i N_i T_i^{-1} S_i^{-1}\| \leq \epsilon$. Define $S(y_1 + ... + y_r) := S_1(y_1) + ... + S_r(y_r)$. Then S is bounded and invertible. Denote $P_i \in \mathcal{L}(X)$ the projection on Y_i parallel to $\bigoplus_{j \neq i} Y_j$. Let $y = y_1 + .. + y_r \in X$. We have

$$\begin{aligned}\|STVT^{-1}S^{-1}(y)\| &= \|\sum_{i=1}^r S_i T_i N_i T_i^{-1} S_i^{-1}(y_i)\| = \|\sum_{i=1}^r S_i T_i N_i T_i^{-1} S_i^{-1} P_i(y)\| \\ &\leq \sum_{i=1}^r \|S_i T_i Y_i T_i^{-1} S_i^{-1}\| \|P_i\| \|y\| \leq \epsilon \sum_{i=1}^r \|P_i\| \|y\|.\end{aligned}$$

Therefore, with ϵ small enough we obtain condition (i). □

The properties defined in Definition 3.2.1 focused on a single projection, that is a direct sum decomposition of X in two subspaces. We will now study the more general case of direct sums of more than two subspaces.

3.2.B Generalizations to finite direct sums of subspaces

In this Subsection, we generalize some classes of projections (identified to a direct sum of two subspaces) to classes of finite direct sums satisfying a given condition. We will look at relationships between these classes in a similar way as in Subsection 3.2.A and also raise new questions that naturally appear.

Definition 3.2.9. Let X be a Banach space, $r \geq 2$, and let $X_1, .., X_r$ be closed subspaces such that $X = X_1 \oplus X_2 \oplus ... \oplus X_r$. We define the following properties

$(P_{8,r})$ For every $x_i \in X_i$ and $C_i \in \mathcal{L}(X_i)$, $1 \leq i \leq r$, we have

$$\|C_1(x_1) + ... + C_r(x_r)\| \leq \max_i(a(C_i, x_i))\|x_1 + ... + x_r\|,$$

where $a(C_i, x_i) := \begin{cases} \frac{\|C_i(x_i)\|}{\|x_i\|} & \text{if } x_i \neq 0 \\ 0 & \text{else} \end{cases}$

$(P_{9,r})$ For every $x_i, y_i \in X_i$ such that $\|x_i\| = \|y_i\|$, $1 \leq i \leq r$, we have

$$\|x_1 + ... + x_r\| = \|y_1 + ... + y_r\|.$$

$(P_{5,r})$ For every $x_i \in X_i$ and $\lambda_i \in \mathbb{C}$, $1 \leq i \leq r$, we have

$$\|\lambda_1 x_i + ... + \lambda_r x_r\| \leq \max_i(|\lambda_i|)\|x_1 + ... + x_r\|.$$

$(P'_{5,r})$ For every $x_i \in X_i$ and $\lambda_i \in \partial\mathbb{D}$, $1 \leq i \leq r$, we have

$$\|\lambda_1 x_i + ... + \lambda_r x_r\| = \|x_1 + ... + x_r\|.$$

Lemma 3.2.10. *Let X be a Banach space, $r \geq 2$, and let $X_1, .., X_r$ be closed subspaces such that $X = X_1 \oplus X_2 \oplus ... \oplus X_r$. We have*

$$(P_{8,r}) \Leftrightarrow (P_{9,r}) \Rightarrow (P'_{5,r}) \Leftrightarrow (P_{5,r}).$$

Proof. The implications $(P_{9,r}) \Rightarrow (P'_{5,r}) \Rightarrow (P_{5,r})$ are immediate.
- $(P_{5,r}) \Rightarrow (P'_{5,r})$. Take $\lambda_i \in \partial\mathbb{D}$, $1 \leq i \leq r$, and define $T(x_1 + ... + x_r) := \lambda_1 x_1 + ... + \lambda_r x_r$. Then T is an invertible operator on X, and $(P_{5,r})$ implies that $\|T\| \leq 1$ and $\|T^{-1}\| \leq 1$, as $\max(|\lambda_i|) = 1 = \max(|\lambda_i^{-1}|)$. Hence T is an (invertible) isometry, thus

$$\|\lambda_1 x_i + ... + \lambda_r x_r\| = \|x_1 + ... + x_r\|, \forall x_1 + ... + x_r \in X.$$

- $(P_{9,r}) \Rightarrow (P_{8,r})$ We recall that for every $w_i \in X_i$, $1 \leq i \leq r$, the map $(z_1, .., z_r) \mapsto \|z_1 w_1 + ... + z_r w_r\|$ is pluri-subharmonic. Hence, for every $(z_1, .., z_r) \in \overline{\mathbb{D}}^r$, we obtain

$$\|z_1 w_1 + ... + z_r w_r\| \leq \sup_{\partial\mathbb{D}^r}(\|\lambda_1 w_1 + ... + \lambda_r w_r\|) = \|w_1 + ... + w_r\|.$$

Let $x_i \in X_i$ and $C_i \in \mathcal{L}(X_i)$, for $1 \le i \le r$. Denote $a(C_i, x_i) := \begin{cases} \frac{\|C_i(x_i)\|}{\|x_i\|} \text{ si } x_i \neq 0 \\ 0 \text{ else} \end{cases}$ and take $b = \max_i(a(C_i, x_i))$. If all $C_i(x_i)$ are zero, then $b = 0$ and there is nothing to prove. Suppose that at least one $C_i(x_i)$ is non-zero. Then, $x_i \neq 0$ and $b \neq 0$. This gives us

$$\|C_i(x_i)\| = \|x_i.a(C_i, x_i)\| \text{ and } \frac{a(C_i, x_i)}{b} \le 1, \forall 1 \le i \le r.$$

Hence,

$$\begin{aligned} \|C_1(x_1) + ... + C_r(x_r)\| &= \|x_1 a(C_1, x_1) + ... + x_r a(C_r, x_r)\| \\ &= b\|x_1 \frac{a(C_1, x_1)}{b} + ... + x_r \frac{a(C_r, x_r)}{b}\| \le b\|x_1 + ... + x_r\|, \end{aligned}$$

which gives $(P_{8,r})$.

- $(P_{8,r}) \Rightarrow (P_{9,r})$. Take $x_i, y_i \in X_i$ such that $\|x_i\| = \|y_i\|$, $1 \le i \le r$. For every i such that $x_i = 0$ we have $y_i = 0$, so we take $C_i = 0$. For every j such that $x_j \neq 0$, we can use the Hahn-Banach Theorem on X_j to find a linear form f_j such that

$$f_j(x_j) \neq 0 \text{ and } \|f_j\| = \frac{\|f_j(x_j)\|}{\|x_j\|}.$$

By defining $C_j(w) := \frac{f_j(w)}{f_j(x_j)} y_j$, C_j is an operator satisfying $C_j(x_j) = y_j$. Therefore, we obtain

$$\|y_1 + ... + y_r\| = \|C_1(x_1) + ... + C_r(y_r)\| \le \max_i(a(C_i, x_i))\|x_1 + ... + x_r\| \le \|x_1 + ... + x_r\|.$$

By permuting the role of x_i with y_i we also obtain $\|x_1 + ... + x_r\| \le \|y_1 + ... + y_r\|$, so Property $(P_{9,r})$ is satisfied. □

Lemma 3.2.11. *Let X be a Banach space. Let $r \ge 2$ and let $X_1, ..., X_r$ be closed subspaces of X with $X = X_1 \oplus ... \oplus X_r$. For the following conditions,*

(i) Property (P_9) is true for $X_i \oplus (\bigoplus_{j \neq i} X_j)$, $\forall 1 \le i \le r$;

(ii) Property (P_9) is true for $X_i \oplus (X_{i+1} \oplus ... \oplus X_r)$, $\forall 1 \le i < r$;

(iii) Property $(P_{9,r})$ is true for $X = X_1 \oplus X_2 \oplus ... \oplus X_r$.

(1) Property $(P'_{5,r})$ is true for $X = X_1 \oplus X_2 \oplus ... \oplus X_r$;

(2) Property (P'_5) is true for $X_i \oplus (\bigoplus_{j \neq i} X_j)$, $\forall 1 \le i \le r$;

(3) Property (P'_5) is true for $X_i \oplus (X_{i+1} \oplus ... \oplus X_r)$, $\forall 1 \le i < r$.

we have the implications

$$(i) \Rightarrow (ii) \Rightarrow (iii), \ (1) \Leftrightarrow (2) \Rightarrow (3).$$

Proof. $(i) \Rightarrow (ii)$ Let $1 \le i \le r$. Since $X_{i+1} \oplus ... \oplus X_r \subset \bigoplus_{j \neq i} X_j$, item (ii) of Lemma 3.2.4 gives the implication.

- $(ii) \Rightarrow (iii)$ Let $x_j, y_j \in X_j$ be such that $\|x_j\| = \|y_j\|$, $1 \leq j \leq r$. Item (ii) applied to $i = r-1$ gives us $\|x_{r-1} + x_r\| = \|y_{r-1} + y_r\|$. We can then apply item (ii) to $i = r-2$ to obtain

$$\|x_{r-2} + x_{r-1} + x_r\| = \|y_{r-2} + y_{r-1} + y_r\|.$$

Hence, we can reiterate this process with a finite decreasing induction on $1 \leq i < r$ in order to obtain

$$\|x_1 + .. + x_r\| = \|y_1 + .. + y_r\|,$$

which gives item (iii).
- (1) $\Rightarrow$ (2) Let $1 \leq i \leq r$. Let $\lambda, \gamma \in \partial\mathbb{D}$. Let $x_j \in X_j$, $1 \leq j \leq r$. For $j \neq i$, denote $\lambda_j = \lambda$. Denote $\lambda_i = \gamma$. With item (1) we obtain

$$\|\lambda x_i + \gamma(\sum_{j\neq i} x_j)\| = \|\lambda_i x_i + (\sum_{j\neq i} \lambda_j x_j)\| = \|x_i + \sum_{j\neq i} x_j\|,$$

which gives item (2).
- (2) $\Rightarrow$ (3) Let $1 \leq i < r$. Let $\lambda, \gamma \in \partial\mathbb{D}$. For every $i \leq j \leq r$, take $x_j \in X_j$. For $1 \leq j < i$, denote $x_j = 0$. With item (2) we have

$$\|\lambda x_i + \gamma(x_{i+1} + .. + x_r)\| = \|\lambda x_i + \gamma(\sum_{j\neq i} x_j)\| = \|x_i + (\sum_{j\neq i} x_j)\| = \|x_i + (x_{i+1} + .. + x_r)\|,$$

which gives item (3).
- (2) $\Rightarrow$ (1) For $1 \leq j \leq r$, take $x_j \in X_j$ and $\lambda_j \in \partial\mathbb{D}$. For any $y_j \in X_j$ and any $1 \leq i \leq r$, applying item (2) to $y_1, .., y_r$ and to i, with $\lambda = \lambda_i$ and $\gamma = 1$, gives

$$\|y_1 + .. + y_r\| = \|y_1 + .. + y_{i-1} + \lambda_i y_i + y_{i+1} + .. + y_r\|.$$

Therefore, we obtain

$$\|x_1+..+x_r\| = \|x_1+..+x_{r-1}+\lambda_r x_r\| = \|x_1+..+x_{r-2}+\lambda_{r-1}x_{r-1}+\lambda_r x_r\| = .. = \|\lambda_1 x_1+..+\lambda_r x_r\|,$$

proving item (1). □

3.3 Projections and Clarkson equality case for L^p and its subspaces

This Section 3.3 deals with Hermitian projections and L^p-projections on Banach spaces that are either $L^p(\Omega)$ or subspaces of $L^p(\Omega)$. These projections have been completely described on many of these spaces thanks to their specific properties.
When looking at Hermitian projections or L^p-projections on a subspace of dimension 2, their definition can be reformulated into a condition that must be satisfied for every $z \in \mathbb{C}$. Hence we will try to see if a weaker condition (for every $z \in \partial\mathbb{D}$, for $z = \pm 1$) would give the same results. As L^p-projections are a specific case of Hermitian projections we also try to look at subspaces of L^p for which every Hermitian projection is an L^p-projections, or at conditions that ensure the contrary.
In Subsection 3.3.B we study the case $p = 2n$, where the relationship $|f+g|^{2n} = (f+g)^n(\overline{f}+\overline{g})^n$ allows us to obtain a useful additional property.

3.3.A Projections and Clarkson equality case on L^p

Example 3.3.1. (i) For $X = C^0(\Omega)$, with Ω compact and Hausdorff, the projections P satisfying Property (P_4) (Hermitian projection) have the form $P = M_{\chi_A}$, with $A \subset \Omega$ a clopen subset (G closed and open).
Hence, if Ω is connected, there is no non-trivial projection satisfying (P_4) on X [Tor68].

(ii) For $X = L^p(\Omega, \mathcal{F}, \mu, E)$, with E a reflexive Banach space, μ totally σ-finite without atoms, and $1 < p < +\infty$, $p \neq 2$, the projections P satisfying Property (P_4) have the form $P = M_{\chi_A}$ with $A \in \mathcal{F}$ [Lum63]. Hence, every Hermitian projection on X is an L^p-projection.

(iii) For $X = H^p(\mathbb{D})$ or $L^p_\alpha(\mathbb{D})$ with $1 \leq p \leq +\infty$, $p \neq 2$, there is no non-trivial projection P satisfying Property (P_4) [Ber72]. Hence, every Hermitian projection on X is an L^p-projection.

(iv) For $X = C^1([0,1])$, $Lip([0,1])$ or $AC([0,1])$, there is no non-trivial projection P satisfying Property (P_4). [BS74]
Hence, every Hermitian projection on these spaces is a L^∞-projection, and they can be completely described.
These results come from a characterization of Hermitian operators on X ($T \in \mathcal{L}(X)$ with $\|e^{i\alpha T}\| = 1$, $\forall \alpha \in \mathbb{R}$), that we then apply to projections.
This non-existence result can also be proved from a characterization of surjective isometries on X that we then apply in order to show that for P a non-trivial projection, $P + \lambda(I - P)$ is an isometry if and only if $\lambda = \pm 1$.

(v) For $X = C^0(\Omega)$ with Ω locally compact and Hausdorff, every L^∞-projection P on X has the form $P = M_{\chi_A}$ with $A \subset \Omega$ a clopen subset [HWW93, Ex.1.4(a)].

(vi) For $X = \oplus_{n\geq 1}^{l_p} X_n$, with X_n Banach spaces, and $1 \leq p < +\infty$, $p \neq 2$, the Hermitian projections on X have the form $P = \oplus_n P_n$, with P_n a Hermitian projection on X_n [BS74, Thm.2.1] .

(vii) For $X = l_p(\mathbb{N})$ $(1 \leq p \leq +\infty$, $p \neq 2)$ or $c_0(\mathbb{N})$, Hermitian projections on X have the form $P = M_{\chi_A}$, with $A \subset \mathbb{N}$.
For $X = c(\mathbb{N})$, Hermitian projection on X have the form $P = M_{\chi_A}$, with $A \subset \mathbb{N}$ such that $n \mapsto \chi_A(n)$ is constant up to a certain rank [Tor68].
Hence every Hermitian projection on these spaces is an L^p or L^∞-projection.

Remark 3.3.2. If X is a SQ^p space, is every Hermitian projection P on X also an L^p-projection ($\|x\|^p = \|P(x)\|^p + \|(I-P)(x)\|^p$, $\forall x \in X$) ?
The answer is negative regarding subspaces of some $L^p(\Omega)$ when $p = 2n$, $n \geq 2$. Proposition 3.3.10 provides a counter-example in this case.

Lemma 3.3.3. —
Let $X = L^p(\Omega, \mathcal{F}, \mu)$, with $1 \leq p \leq +\infty$, $p \neq 2$. Let $f, g \in X$. Consider the following conditions

(i) f and g have disjoint supports (up to a set of null measure);

(ii) $\|f + zg\| = \|(\|f\|, \|zg\|)\|_{\ell^p}$, $\forall z \in \mathbb{C}$;

(iii) $\|f + zg\| = \|f + |z|g\|$, $\forall z \in \mathbb{C}$.

(1) f and g have disjoint supports (up to a set of null measure);

(2) $\|f + \lambda g\| = \|(\|f\|, \|g\|)\|_{\ell^p}$, $\forall \lambda \in \partial\mathbb{D}$;

(3) $\|f + \lambda g\| = \|f + g\|$, $\forall \lambda \in \partial\mathbb{D}$.

We have the implications

$$(i) \Rightarrow (ii) \Rightarrow (iii),\ (1) \Rightarrow (2) \Rightarrow (3)$$

The implications $(ii) \Rightarrow (i)$ and $(2) \Rightarrow (1)$ are true when $p < +\infty$.
The implications $(ii) \Rightarrow (i)$ and $(2) \Rightarrow (1)$ are false when $p = +\infty$ and $dim(X) \geq 3$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ and $(1) \Rightarrow (2) \Rightarrow (3)$ are immediate.
When $p < +\infty$, the implications $(ii) \Rightarrow (i)$ and $(2) \Rightarrow (1)$ derive from the equality case of the Clarkson inequalities (see Lemma 4.1.12 and Corollary 4.2.3).
When $p = +\infty$ and $dim(X) \geq 3$, the σ-algebra $\mathcal{F}$ contains three sets A_1, A_2, A_3 that are mutually disjoint (up to a set of measure zero) and that satisfy $\mu(A_i) > 0$. By taking $e_i = \frac{1}{\mu(A_i)}\chi_{A_i}$, $\mathrm{Span}(e_1, e_2, e_3)$ is isometrically isomorphic to $l^\infty(\mathbb{C}^3)$. Denote $f = 2e_1 + e_2$ and $g = e_2 + 2e_3$. A short computation gives that $\|f + zg\| = \max(\|f\|, \|zg\|)$, $\forall z \in \mathbb{C}$. However f and g do not have disjoint supports, which concludes the proof. □

Corollary 3.3.4. *Let $X = L^p(\Omega, \mathcal{F}, \mu)$, with $1 \leq p \leq +\infty$. Consider the following assertions*

(i) The implication $(3) \Rightarrow (2)$ in Lemma 3.3.3 is true for X;

(ii) The implication $(iii) \Rightarrow (ii)$ in Lemma 3.3.3 is true for X;

(iii) Every Hermitian projection on a closed subspace of X is an L^p-projection.

Then $(i) \Rightarrow (ii) \Rightarrow (iii)$. Furthermore, conditions (ii) and (iii) are equivalent.

Proof. $(ii) \Rightarrow (iii)$ Let F be a closed subspace of X and let $P \in \mathcal{L}(F)$ be a Hermitian projection. For every $f \in \mathrm{Ker}(P)$, $g \in \mathrm{Ran}(P)$, and $z \in \mathbb{C}$, we have $\|f + zg\| = \|f + |z|g\|$. Hence, if $(iii) \Rightarrow (ii)$ in Lemma 3.3.3 is true, we have $\|f + g\| = \|(\|f\|, \|g\|)\|_{\ell^p}$. Thus P satisfies Proposition (P_{10}) for p, so it is an L^p-projection.
- $(iii) \Rightarrow (ii)$ Let $f, g \in X$ be such that $\|f + zg\| = \|f + |z|g\|$, $\forall z \in \mathbb{C}$. If $f = 0$ or $g = 0$ there is nothing to prove. When $f \neq 0$ and $g \neq 0$, we can see that f and g are not colinear as $|1 + az|\|f\|$ is not equal to $(1 + |az|)\|f\|$ for every $z \in \mathbb{C}$. Hence, $\mathrm{Span}(f, g)$ is of dimension 2. Denote Q the projection on $\mathrm{Span}(f)$ parallel to $\mathrm{Span}(g)$. Let $\lambda, \gamma \in \partial\mathbb{D}$ and $a, b \in \mathbb{C}$. If $a = 0$ then $\|\lambda af + \gamma bg\| = \|bg\| = \|af + bg\|$. If not, we have

$$\begin{aligned}\|\lambda af + \gamma bg\| &= |a|\|f + \frac{\gamma b}{\lambda a}g\| = |a|\|f + \frac{|b|}{|a|}g\| \\ &= |a|\|f + \frac{b}{a}g\| = \|af + bg\|,\end{aligned}$$

so Q is a Hermitian projection. Thus Q is an L^p-projection, so $\|f + zg\| = \|(\|f\|, \|zg\|)\|_{\ell^p}$ for every $z \in \mathbb{C}$, which proves the implication.
- $(i) \Rightarrow (ii)$. Let $f, g \in X$ be such that $\|f + zg\| = \|f + |z|g\|$, $\forall z \in \mathbb{C}$. Let $w \in \mathbb{C}$. For every $\lambda \in \partial\mathbb{D}$ we have

$$\|f + \lambda wg\| = \|f + |\lambda w|g\| = \|f + wg\|,$$

thus the implication $(3) \Rightarrow (2)$ gives $\|f + wg\| = \|(\|f\|, \|wg\|)\|_{\ell^p}$. As this is true for every $w \in \mathbb{C}$, this concludes the proof. □

Remark 3.3.5. We do not know if the conditions (i) and (ii) in Corollary 3.3.4 are equivalent. However, both of them are false for some spaces $L^{2n}(\Omega)$, $n \geq 2$ (see Propositions 3.3.9 and 3.3.10).

Proposition 3.3.6. *Let μ be the Lebesgue measure on $\mathbb{R}$, let I be a non-empty interval of $\mathbb{R}$, and suppose that $1 \leq p < +\infty$, $p \neq 2$. Let $X = L^p(I, \mu)$. Then the conditions (i) and (ii) in Corollary 3.3.4 are false for X.*

Proof. Up to dilating and translating I, we can suppose that I contains $J = [0, 2\pi]$. We then define $f(x) = \chi_J(x)$ and $g(x) = e^{ix}\chi_J(x)$. The maps f and g are in L^p. Let $\lambda = re^{it} \in \mathbb{C}$. Using a change of variables and the 2π-periodicity of $x \mapsto 1 + re^{ix}$ gives us

$$\|f+\lambda g\|_p^p = \int_0^{2\pi} |1+re^{i(t+x)}|^p d\mu(x) = \int_t^{2\pi+t} |1+re^{iu}|^p d\mu(u) = \int_0^{2\pi} |1+re^{is}|^p d\mu(s) = \|f+rg\|_p^p.$$

Hence, f and g satisfy conditions (3) and (iii) of Lemma 3.3.3. However, as f and g do not have disjoint supports, Corollary 4.2.3 implies that we cannot have $\|f \pm g\|^p = \|f\|^p + \|g\|^p$. Hence conditions (2) and (ii) of Lemma 3.3.3 are not satisfied. □

Remark 3.3.7. Let $1 \leq p \leq +\infty$. If we have $L^p(\Omega, \mathcal{F}, \mu)$ with vectors f, g such that $\|f + \lambda g\| = \|f + g\|$, $\forall \lambda \in \partial\mathbb{D}$ but not $\|f + g\| = \|(\|f\|, \|g\|)\|_{\ell^p}$, the same holds true for every space $L^p(\Omega', \mathcal{G}, \nu)$ which contains a closed subspace that is isometrically isomorphic to $L^p(\Omega, \mathcal{F}, \mu)$.
We can also see that the proof of Proposition 3.3.6 that for $F = \mathrm{Span}(f, g)$, F is a subspace of $L^p(I, \mu)$ possessing a Hermitian projection that is not an L^p-projection.

Corollary 3.3.8. *Let $n \geq 1$. Let μ_n be the Lebesgue measure on $\mathbb{R}^n$, let I be a product of intervals in $\mathbb{R}^n$ with $\mu_n(I) > 0$, and suppose that $1 \leq p < +\infty$, $p \neq 2$. Let $X = L^p(I, \mu_n)$. Then the conditions (i) and (ii) in Corollary 3.3.4 are false for X.*

Proof. Write $I = \Pi_{k=1}^n I_k$, with I_k non-empty intervals in $\mathbb{R}$. The map $T : L^p(I_1) \to L^p(I)$ defined by $T(f)(x_1, ..., x_n) = f(x_1)\chi_I(x_1, .., x_n)\Pi_{k=2}^n \mu(I_k)^{-\frac{1}{p}}$ is an isometry. Hence X contains a closed subspace that is isometrically isomorphic to $L^p(I_1, \mu)$, and we can use Remark 3.3.7 to get the desired result. □

3.3.B The case of $L^{2n}(\Omega)$

When p has the form $p = 2n$, with $n \geq 1$, the L^p norm can be expressed differently thanks to the relationship $|f + g|^{2n} = (f + g)^n(\overline{f} + \overline{g})^n$. We will use this relationship to obtain weaker conditions that can give Hermitian projections. We shall also construct Hermitian projections on some subspaces that are not L^p-projections.

Proposition 3.3.9. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $n \geq 1$. Let $X = L^{2n}(\Omega, \mathcal{F}, \mu)$. Let $f, g \in X$. The following are equivalent*

(i) $\|f + \lambda g\|_{2n} = \|f + g\|_{2n}$, $\forall \lambda \in \partial\mathbb{D}$;

(ii) $\|f + zg\|_{2n} = \|f + |z|g\|_{2n}$, $\forall z \in \mathbb{C}$;

(iii) $\sum_{k=0}^{n-l} \binom{n}{k}\binom{n}{l+k} \int_\Omega (g^l|g|^{2k}\overline{f}^l|f|^{2(n-l-k)})(x)d\mu(x) = 0$, $\forall 1 \leq l \leq n$;

(iv) $\|f+zg\|_{2n}^{2n} = \sum_{k=0}^{n} |z|^{2k}\binom{n}{k}^2 \int\limits_\Omega (|f|^{2(n-k)}|g|^{2k})(x)d\mu(x)$, $\forall z \in \mathbb{C}$.

Proof. The implications $(ii) \Rightarrow (i)$ and $(iv) \Rightarrow (ii)$ are immediate. Let us first develop $|f+\lambda g|^{2n}$ in order to show $(iii) \Rightarrow (iv)$. After that we will build a harmonic map to obtain $(i) \Rightarrow (iii)$. Let $z \in \mathbb{C}$. By developing $(f+zg)^n$ we obtain

$$\begin{aligned}|f+zg|^{2n} &= (f+zg)^n\overline{(f+zg)}^n \\ &= \sum_{k=0}^{n}\binom{n}{k}^2|f|^{2(n-k)}|zg|^{2k} + \sum_{l=1}^{n}\sum_{k=0}^{n-l} 2\mathrm{Re}(z^l g^l |zg|^{2k}\bar{f}^l|f|^{2(n-l-k)}\binom{n}{k}\binom{n}{l+k}).\end{aligned}$$

We can notice that all products $(g^l|g|^{2k})(\bar{f}^l|f|^{2(n-l-k)})$ lie in $L^1(\Omega,\mathcal{F},\mu)$ as $(l+2k)+(l+2(n-l-k)) = 2n$. The same is true for $|f|^{2(n-k)}|g|^{2k}$. We can then define

$$h : z \in \mathbb{C} \mapsto 2\sum_{l=1}^{n} Re[z^l \sum_{k=0}^{n-l}\binom{n}{k}\binom{n}{l+k}\int\limits_\Omega (g^l|g|^{2k}\bar{f}^l|f|^{2(n-l-k)})(x)d\mu(x)].$$

The map h is then well-defined and harmonic on $\mathbb{C}$. Using $|\lambda| = 1$, we also notice that for $\lambda \in \partial\mathbb{D}$ we have

$$h(\lambda) + \int_\Omega \sum_{k=0}^{n}\binom{n}{k}^2(|f|^{2(n-k)}|g|^{2k})(x)d\mu(x) = \int_\Omega |f+\lambda g|^{2n}(x)d\mu(x) = \|f+\lambda g\|_{2n}^{2n}.$$

- $(iii) \Rightarrow (iv)$ If item (iii) is true, then for any $z \in \mathbb{C}$ the previous computations give

$$\|f+zg\|_{2n}^{2n} = \int_\Omega \sum_{k=0}^{n}\binom{n}{k}^2(|f|^{2(n-k)}|zg|^{2k})(x)d\mu(x) + 0,$$

so item (iv) is true.
- $(i) \Rightarrow (iii)$ With the previous relationship for $\lambda \in \partial\mathbb{D}$, we can notice that item (i) is equivalent to the fact that h is constant on $\partial\mathbb{D}$. Since h is harmonic, this is equivalent to the fact that h is constant on $\mathbb{D}$, or equivalently to the whole complex plane $\mathbb{C}$. Also, by denoting

$$h_2(z) = 2\sum_{l=1}^{n} z^l \sum_{k=0}^{n-l}\binom{n}{k}\binom{n}{l+k}\int_\Omega (g^l|g|^{2k}\bar{f}^l|f|^{2(n-l-k)})(x)d\mu(x),$$

we see that the map h_2 is well-defined on $\mathbb{C}$, that it is a polynomial in z so it is holomorphic on $\mathbb{C}$, and that is satisfies $h = \mathrm{Re}(h_2)$. Since $h = \mathrm{Re}(h_2)$ is constant on $\mathbb{C}$, the map h_2 is constant on $\mathbb{C}$, equal to $h_2(0) = 0$. As h_2 is a polynomial, each of its coefficients must be zero. Therefore, we obtain

$$\sum_{k=0}^{n-l}\binom{n}{k}\binom{n}{l+k}\int\limits_\Omega (g^l|g|^{2k}\bar{f}^l|f|^{2(n-l-k)})(x)d\mu(x) = 0, \forall 1 \le l \le n,$$

which concludes the proof. □

Proposition 3.3.10. *Let $n \ge 2$. Let $X = l_{2n}(\{0,..,n\})$. Then the implication $(iii) \Rightarrow (ii)$ in Lemma 3.3.3 is false for X. Hence, there are closed subspaces F of X who admit Hermitian projections that are not L^{2n}-projections.*
The same results hold for $Y = l^{2n}(\mathbb{N})$.

Proof. We will use Proposition 3.3.9 in order to imitate the proof of Proposition 3.3.6. For this, define

$$f_m = 1,\; g_m = e^{\frac{2im\pi}{n+1}},\; 0 \leq m \leq n.$$

Define $f := (f_m)_m$ and $g := (g_m)_m$. We then have $f, g \in X$. The quantities in item (iii) of Proposition 3.3.9 for f and g are

$$\sum_{k=0}^{n-l} \binom{n}{k} \binom{n}{l+k} (\sum_{m \geq 0} g_m^l \cdot 1 \cdot 1 \cdot 1),\; \forall 1 \leq l \leq n.$$

Since $e^{\frac{2il\pi}{n+1}} \neq 1$ for every $1 \leq l \leq n$, we obtain

$$\sum_{m \geq 0} g_m^l = \sum_{m=0}^{n} e^{\frac{2iml\pi}{n+1}} = \sum_{m=0}^{n} (e^{\frac{2il\pi}{n+1}})^m = \frac{1-1}{1 - e^{\frac{2il\pi}{n+1}}} = 0.$$

Hence, we can apply Proposition 3.3.9 and obtain with item (i) that

$$\|f + \lambda g\|_{2n} = \|f + g\|_{2n},\; \forall \lambda \in \partial \mathbb{D}.$$

Item (iv) of this Proposition gives us

$$\|f + zg\|_{2n}^{2n} = \sum_{k=0}^{n} |z|^{2k} \binom{n}{k}^2 (n+1).$$

Hence, we can also see that

$$\|f + g\|_{2n}^{2n} > 2(n+1) = \|f\|_{2n}^{2n} + \|g\|_{2n}^{2n},$$

so the implication $(iii) \Rightarrow (ii)$ of Lemma 3.3.3 is not satisfied for X.
This also proves that on the closed subspace $\mathrm{Span}(f, g)$, the projection onto $\mathrm{Span}(f)$ parallel to $\mathrm{Span}(g)$ is a Hermitian projection that is not a L^{2n}-projection.
We obtain the same results for $Y = l^{2n}(\mathbb{N})$ using Remark 3.3.7. □

Chapter 4

L^p-projections on a Banach space

The aim of this chapter is to study L^p-projections, a notion introduced by Cunningham in 1953, on subspaces, quotients, and subspaces of quotients of complex Banach spaces. An L^p-projection on a Banach space X, for $1 \leq p \leq +\infty$, is an idempotent operator P satisfying $\|f\|_X = \|(\|P(f)\|_X, \|(I-P)(f)\|_X)\|_{\ell_p}$ for all $f \in X$. This is an L^p version of the equality $\|f\|^2 = \|Q(f)\|^2 + \|(I-Q)(f)\|^2$, valid for orthogonal projections on Hilbert spaces. For a complex Banach space X and subspaces F, G, we will focus on relationships between the L^p-projections on X and those on F,X/F or G/F. All the results in this chapter are true for $1 < p < +\infty$, $p \neq 2$. The cases $p = 1, 2$ or $+\infty$ can exhibit different behaviour. In this regard, we give a complete description of L^∞-projections on spaces $L^\infty(\Omega)$. For this, we introduce a notion of p-orthogonality for two elements x, y by requiring that $\mathrm{Span}(x, y)$ admits an L^p-projection separating x and y. We also introduce the notion of maximal L^p-projections for X, that is L^p-projections defined on a subspace G of X that cannot be extended to L^p-projections on larger subspaces. We prove results concerning L^p-projections and p-orthogonality of general Banach spaces or on Banach spaces with additional properties. Generalizations of some results to spaces $L^p(\Omega, X)$ as well as some results about L^q-projections on subspaces of $L^p(\Omega)$ are also discussed.

Section 2 mainly focuses on relationships and characterizations of L^p-projections on a subspace F. Section 3 looks at relationships and characterizations of L^p-projections on a quotient X/F and on a subspace of quotient G/F. Section 4 gives generalizations of previous results to some spaces $L^p(\Omega, X)$.

4.1 Introduction and Preliminaries about L^p-projections

Introduction

The notion of L^p-projection has been introduced by Cunningham in 1953 ([Cun53]) and studied, mainly in the cases $p = 1$ and $p = +\infty$, in the papers [Cun53,Cun60,Cun67,CER73]. The general case $1 < p < +\infty$ has been studied by Alfsen-Effros [AE72], Sullivan [Sul70] and Fakhoury [Fak74]. The main characterization results, which were obtained in 1973-1976 by Alfsen-Effros, Behrends, Fakhoury, Sullivan and others, were compiled in the book [BDE+77, Ch.1,2,6].

We will be interested either in general Banach spaces or in Banach spaces with additional properties regarding L^p-projections and p-orthogonality. All these additional properties are true for the spaces $L^p(\Omega)$. The results in this chapter are true for $1 < p < +\infty$, $p \neq 2$ and some of them are even true when $p = 1, 2$ or $+\infty$. However the Hilbert case ($p = 2$) and the non-reflexive cases ($p = 1, +\infty$) can exhibit different behaviours, or do not work well in some contexts. In this regard, section 2.C only deals with the case $p = +\infty$.

In this chapter, a *projection* (or an *idempotent*) is a bounded linear operator P on a Banach space X that satisfies $P^2 = P$. We recall that an L^p-projection is an idempotent satisfying the additional condition of the following definition.

Definition 4.1.1 (L^p-projections)**.** Let X be a Banach space, and let $1 \leq p \leq +\infty$.
A projection P ($P^2 = P$) on X is said to be an L^p-projection if it satisfies the condition

$$\|f\|_X = \|(\|P(f)\|, \|(I-P)(f)\|)\|_p, \text{ for all } f \in X.$$

This means that

$$\begin{cases} \|f\|_X^p = \|P(f)\|_X^p + \|(I-P)(f)\|_X^p, \forall f \in X & \text{when } 1 \leq p < +\infty. \\ \|f\|_X = \max(\|P(f)\|_X, \|(I-P)(f)\|_X), \forall f \in X & \text{when } p = +\infty. \end{cases}$$

We denote by $\mathcal{P}_p(X)$ the set of L^p-projections on X.

Organization.

In the rest of this introductory section we introduce some of the basic results about L^p-projections, giving useful tools for the rest of this chapter.
For a complex Banach space X and subspaces F, G, the following sections will focus on relationships between the L^p-projections on X and those on F,X/F or G/F. Section 4.2 mainly concerns relationships and characterizations of L^p-projections on a subspace F. Section 4.3 looks at relationships and characterizations of L^p-projections on a quotient X/F. Section 4.4 gives generalizations of previous results to some spaces $L^p(\Omega, X)$.
This chapter introduces a notion of *p-orthogonality* for two vectors x, y, that is when $\mathrm{Span}(x, y)$ admits an L^p-projection separating x and y. This p-orthogonality for vectors also implies a notion of *p-orthogonal* for sets. We introduce as well a notion of *maximal L^p-projection* for X, that is L^p-projections defined on a subspace of X that cannot be extended to L^p-projections on a larger subspace.

Preliminaries

Remark 4.1.2. We can first see that 0 and I are always L^p-projections, so $\mathcal{P}_p(X)$ is never empty. The class $\mathcal{P}_p(X)$ of L^p-projections is stronger than the usual class of Hermitian projections. We recall (see for instance [BS74]) that a projection Q is *Hermitian* if $\|e^{i\alpha Q}\| = 1$ for each $\alpha \in \mathbb{R}$ and that a projection Q is Hermitian if and only if $Q + \lambda(I - Q)$ is an isometry for every $\lambda \in \partial\mathbb{D}$ or, equivalently, if $\lambda Q + \gamma(I - Q)$ is an isometry, for any $\lambda, \gamma \in \partial\mathbb{D}$. To see that an L^p-projection P is Hermitian, we note that the L^p-projection condition is also equivalent to

$$\|f + g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p}, \text{ for all } f \in Ran(P),\ g \in Ker(P),$$

where Ran and Ker denote the range and respectively the kernel. Hence, for any $\lambda, \gamma \in \partial\mathbb{D}$, we have

$$\|\lambda f + \gamma g\|_X = \|(\|f\|, \|g\|)\|_{\ell_p} = \|f + g\|_X, \text{ for all } f \in Ran(P),\ g \in Ker(P).$$

Therefore, $\lambda P+\gamma(I-P)$ is an isometry on X, and the L^p-projection P is a Hermitian projection. However, the main characterization results on Hermitian projections in this context mainly concern L^p and H^p spaces(see [Lum63, Ber72, BS74, Tor68]), thus they are of little help in our context.

For the rest of this chapter, we will only consider complex Banach spaces. L^p-projections do not behave differently between real and complex cases except when $p = 1$ or $+\infty$ (see Thm.4.1.9), so most of the results can be easily generalized to the real case when $1 < p < +\infty$, $p \neq 2$.

For a set E, A a subset of E, F a vector space and a map $f : E \to F$, , we define $M_{\chi_A}(f)$ as the multiplication by the characteristic function of A:

$$M_{\chi_A}(f)(x) = f\chi_A(x) = \begin{cases} f(x) & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} .$$

The following facts are recorded here without proofs.

Proposition 4.1.3. *We have*

(i) For $(\Omega, \mathcal{F}, \mu)$ a measure space, $1 \leq p \leq +\infty$, and $A \subset \Omega$ such that $A \cap B \in \mathcal{F}$ for every $B \in \mathcal{F}$ with $\mu(B) < +\infty$, the operator $P = M_{\chi_A}$ is an L^p-projection on $L^p(\Omega, \mathcal{F}, \mu)$.

(ii) For X, Y Banach spaces, $1 \leq p \leq +\infty$, and $T : X \to Y$ an isometric isomorphism, we have $\mathcal{P}_p(Y) = T \circ \mathcal{P}_p(X) \circ T^{-1}$.

(iii) The set of L^2-projections on a Hilbert space H is the set of orthogonal projections.

(iv) For K a compact topological space, the L^∞-projections on $C^0(K)$ have the form $P = M_{\chi_A}$ with A a clopen subset of K.

Proposition 4.1.4. *Let X be a Banach space, and $1 \leq p \leq +\infty$. Let $P, Q \in \mathcal{P}_p(X)$ be such that Ran(P) = Ran(Q). Then, $P = Q$.*

Remark 4.1.5. The set of all L^p-projections P on X is in bijective correspondence to decompositions $X = X_1 \oplus_{\ell_p} X_2$ of X. A ℓ^p-direct sum decomposition like this is called a p-summand of X. Hence, studying the p-summands of a Banach space X amounts to studying the L^p-projections on X. Proposition 4.1.4 indicates that for a p-summand $X = X_1 \oplus_p X_2$, X_2 is the only closed subspace of X that is in ℓ^p-direct sum with X_1.

We now continue to present several results characterizing L^p-projections on a general Banach space X, mainly when $p \neq 2$. Proofs of these results can be found in [BDE+77].

Lemma 4.1.6. *Let X be a Banach space, and consider two real conjugate numbers p and p', with $\frac{1}{p} + \frac{1}{p'} = 1$. Let $P \in \mathcal{L}(X)$ be a projection on X. Then P is an L^p-projection on X if and only if P' is a $L^{p'}$-projection on X'.*

Proposition 4.1.7. *Let X be a Banach space, and $1 \leq p, p' \leq +\infty$, with $\frac{1}{p} + \frac{1}{p'} = 1$ and $p' \neq 1$. Then, all $Q \in \mathcal{P}_{p'}(X')$ are continuous for the weak-* topology $\sigma(X', X)$. Hence, there is $P \in \mathcal{P}_p(X)$ such that $P' = Q$, so $(\mathcal{P}_p(X))' = \mathcal{P}_{p'}(X')$.*

The only pathologic case regarding duality and L^p-projections is $p' = 1$, $p = +\infty$. For example $X = C^0([0, 1])$ has trivial L^∞-projections whereas X' is the space of complex finite Borel measures on $[0, 1]$, which possesses many L^1-projections.

Theorem 4.1.8. *Let X be a Banach space, and $1 \leq p \leq +\infty$, $p \neq 2$. We then have*

(i) All elements of $\mathcal{P}_p(X)$ commute with each other.

(ii) The set $\mathcal{P}_p(X)$ is a commutative Boolean algebra for the operations $(P, Q) \mapsto PQ$, $(P, Q) \mapsto P + Q - PQ$ and $P \mapsto (I - P)$.

(iii) The relationship $P \leq Q \ \Leftrightarrow\ PQ = P$ is an order relationship on $\mathcal{P}_p(X)$.

(iv) When $p \neq +\infty$, every decreasing filtrating net $(P_i)_{i \in I}$ in $\mathcal{P}_p(X)$ is pointwise convergent to an L^p-projection P, with $P = \inf_{i \in I}(P_i)$.

(v) When $p \neq +\infty$, the Boolean algebra $\mathcal{P}_p(X)$ is complete: Every subset $\{P_i,\ i \in I\}$ admits an infimum $\inf_{i \in I}(P_i)$ in $\mathcal{P}_p(X)$. Furthermore, $Ran(\inf_{i \in I}(P_i)) = \bigcap_{i \in I} Ran(P_i)$.

Theorem 4.1.9. *Let X be a Banach space, and $1 \leq p, q \leq +\infty$, with $p \neq q$. Then, at least one of the sets $\mathcal{P}_p(X)$ or $\mathcal{P}_q(X)$ is reduced to $\{0, I\}$.*
The result stays true for a real Banach space Y, unless Y is isometrically isomorphic to $l^1(\mathbb{R}^2) \simeq l^\infty(\mathbb{R}^2)$.

Remark 4.1.10. Theorems 4.1.8 and 4.1.9 show, for $p \neq 2$, the similarity of the set $\mathcal{P}_p(X)$ with the set $\{M_{\chi_A} \colon A \in \mathcal{F}\}$, for $(\Omega, \mathcal{F}, \mu)$ a measure space with μ σ-finite. Theorem 4.1.9 also shows that we do not need to study L^q-projections on a Banach space X whenever there are non-trivial L^p-projections on X, for $1 \leq p, q \leq +\infty$ with $p \neq q$. However, in certain cases, there may exist L^q-projections on subspaces, quotients, or subspaces of quotients of X, like for $X = L^q(\Omega) \oplus_p L^r(\Omega')$ for example. This question will be discussed later on (see Lemma 4.4.3).

Corollary 4.1.11. *Let X be a Banach space, and $1 \leq p < +\infty$, $p \neq 2$. Let E be a subset of X. Then, the set of L^p-projections P on X such that $P(E) = E$ admits a unique minimum with respect to the order relation from Theorem 4.1.8.*
Furthermore, for any $Q \in \mathcal{P}_p(X)$ such that $Q(E) = E$, we have $P \leq Q$.

This minimum in Corollary 4.1.11 is called the *minimal L^p-projection* for E.

We end this section with results focused around L^p-projections on L^p spaces. With Theorem 4.1.8 we can see that sets of L^p-projections on generic Banach spaces have common behaviours. Hence having a complete characterization of the set $\mathcal{P}_p(L^p(\Omega))$ gives a useful example of the structure such a set can have.

Lemma 4.1.12 (Clarkson inequalities). *Let $1 \leq p < +\infty$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For every f, g in $L^p(\Omega, \mathcal{F}, \mu)$, we have*

(i) $\|f + g\|_p^p + \|f - g\|_p^p \leq 2(\|f\|_p^p + \|g\|_p^p)$ if $1 \leq p \leq 2$

(ii) $\|f + g\|_p^p + \|f - g\|_p^p \geq 2(\|f\|_p^p + \|g\|_p^p)$ if $2 \leq p$

Moreover, if $p \neq 2$ there is equality in the above inequalities if and only if $\mu(supp(f) \cap supp(g)) = 0$, that is if f and g have a disjoint support.

A proof of these inequalities can be found in [Roy88, Ch15-7,Lem 22,p.416].
The equality case in Clarkson inequalities is the main ingredient in the proof of the two following characterizations of L^p-projections on L^p-spaces.
This equality case has no equivalent for $p = +\infty$, as one can see in Proposition 4.2.24.

Theorem 4.1.13. *Let $1 \leq p < +\infty$, $p \neq 2$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space, with μ being σ-finite. Let $P \in \mathcal{P}_p(L^p(\Omega))$. Then there exists $A \in \mathcal{F}$ such that $P = M_{\chi_A}$.*

Theorem 4.1.14. *Let $1 \leq p < +\infty$, $p \neq 2$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $P \in \mathcal{P}_p(L^p(\Omega))$. Then we have $P = M_{\chi_A}$, with $A \subset \Omega$ such that $A \cap B \in \mathcal{F}$ for every $B \in \mathcal{F}$ with $\mu(B) < +\infty$.*

Theorem 4.1.15. *Let $1 \leq p < +\infty$, $p \neq 2$ and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let Y be a finite dimensional Banach space. Let $P \in \mathcal{P}_p(L^p(\Omega, Y))$. Denote $\mathcal{P}_p(Y) = \{T_i,\, i \in I\}$. Then we have $P = \sum_{i \in I} P_i \otimes T_i$, where P_i are L^p-projections on $L^p(\Omega)$ such that $\sum_{i \in I} P_i = I$, that is Ran(P_i) are in direct sum in $L^p(\Omega)$.*

Remark 4.1.16. We refer to Daniel Li's thesis [Li79] or to [BDE+77] for a proof of Theorems 4.1.13, 4.1.14 and 4.1.15. The proof of Theorem 4.1.15 uses results about the Stonean space associated to $\mathcal{P}_p(X)$ in order to make an L^p-projection P on $L^p(\Omega, Y)$ correspond to an L^p-projection P' on a space of continuous functions, and then describes the form of the latter L^p-projection by noticing that P' composed with any evaluation operator is again an L^p-projection. The construction made in this proof formalizes the impression that, for $P \in \mathcal{P}_p(L^p(\Omega, Y))$ and for any $w \in \Omega$, $f \in L^p(\Omega, Y)$, we should have $\|f(w)\| = \|(\|P(f)(w)\|, \|(I-P)(f)(w)\|)\|_p$ (even though these quantities are not valid in that case). Most of the tools used in these proofs behave differently in the case $p = +\infty$, even though this underlying idea stays similar (see Section 4.2.C). We also prove in this chapter that $L^\infty(\Omega)$ cannot admit other L^∞-projections than multiplication operators (Theorem 4.2.25).

4.2 The p-Orthogonality Relationship

For a Banach space X and F a closed subspace of X, we will focus in this section on relationships between L^p-projections on F and those on X, either for a general complex Banach space X or for a space X satisfying additional properties. Section 4.2.A introduces and studies the notions of *p-orthogonality* for vectors and of *p-orthogonal sets*. Two properties for Banach spaces regarding p-orthogonality for vectors are also defined and studied. All the results in this section are valid for $1 \leq p < +\infty$, $p \neq 2$, while some of them are true for $p = +\infty$. Section 4.2.B gives some examples and Section 4.2.C focuses on the case $p = +\infty$, one of its main results being the description of $\mathcal{P}_\infty(L^\infty(\Omega))$. Lastly, Section 4.2.$D$ introduces and studies the notion of maximal L^p-projections: L^p-projections on a subspace of X that cannot be extended to L^p-projections on a larger subspace, for $1 \leq p \leq +\infty$, $p \neq 2$. This section also focuses on the number of maximal L^p-projections for a subspace $F \subset X$, especially in the finite dimensional case.

4.2.A p-orthogonality, L^p-projections on subspaces, maximal L^p-projections

We introduce the following definition.

Definition 4.2.1 (p-orthogonality)**.** Let X be a Banach space, and $1 \leq p \leq +\infty$. Let $f, g \in X$. The elements f and g are said to be *p-orthogonal*, denoted by $f \perp_p g$, if

$$\begin{cases} \|f + zg\|^p = \|f\|^p + |z|^p \|g\|^p, \ \forall z \in \mathbb{C}, \text{ when } p < +\infty; \\ \|f + zg\| = \max(\|f\|, |z|\|g\|), \ \forall z \in \mathbb{C}, \text{ when } p = +\infty. \end{cases}$$

If $f \neq 0$ and $g \neq 0$, this condition is equivalent to the fact that Span(f, g) has dimension 2 and that the projection on Span(f) parallel to Span(g) is an L^p-projection on Span(f, g).

Remark 4.2.2. This p-orthogonality relationship is symmetric. We also have

$$f \perp_p f \Leftrightarrow f = 0 \text{ and } f \perp_p g \Leftrightarrow f \perp_p wg, \text{ for any } w \in \mathbb{C}^*.$$

However, $f \perp_p g$ and $f \perp_p h$ does not imply $f \perp_p (g+h)$ in general. See item 4.2.20 for a counter-example.

A similar notion of orthogonality can be found in a paper of Berkson [Ber72, Section 3], although it concerns Boolean algebras of Hermitian operators. We also recall the *Birkhoff-James orthogonality*, defined as

$$x \perp_{BJ} y \Leftrightarrow \|x + ty\| \geq \|x\| \text{ for every } t \in \mathbb{R}.$$

If $x \perp_p y$, then one can see that $x \perp_{BJ} y$ and $y \perp_{BJ} x$. Hence the p-orthogonality, defined via L^p-projections, is stronger than the Birkhoff-James orthogonality (the latter is defined using norm one projections). We refer for instance to [Ran01, 5.f] and [BP88], and warn the reader that there are many other notions of orthogonality in the literature.

Corollary 4.2.3. *Let $1 \leq p < +\infty$, $p \neq 2$, and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Take $X = L^p(\Omega, \mathcal{F}, \mu)$ and let $f, g \in X$. We then have the equivalences*

(i) $f \perp_p g$;

(ii) $\|f + e^{it} g\|^p = \|f\|^p + \|g\|^p$, $\forall t \in \mathbb{R}$;

(iii) $\|f \pm g\|^p = \|f\|^p + \|g\|^p$;

(iv) $\|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p)$;

(v) f and g have disjoint supports (up to a set of measure 0).

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$ are immediate.
- $(v) \Rightarrow (i)$ If $\mu(\operatorname{supp}(f) \cap \operatorname{supp}(g)) = 0$, then for any $z \in \mathbb{C}$ we have

$$\|f + zg\|^p = \int_\Omega |(f + zg)(x)|^p d\mu(x) = \|f\|^p + \|zg\|^p,$$

so $f \perp_p g$.
- $(iv) \Rightarrow (v)$ If $\|f + g\|^p + \|f - g\|^p = 2(\|f\|^p + \|g\|^p)$, then f and g satisfy the equality case in Clarkson inequalities 4.1.12, which implies that $\mu(\operatorname{supp}(f) \cap \operatorname{supp}(g)) = 0$. □

Definition 4.2.4 (p-orthogonal)**.** Let X be a Banach space, and $1 \leq p \leq +\infty$. Let E be a subset of X. We define $E^{\perp_p}$ the *p-orthogonal* of E as

$$E^{\perp_p} := \{f \in X \colon f \perp_p g, \forall g \in E\}.$$

Remark 4.2.5. Since the map $f \mapsto \|f + zg\| - \|(\|f\|, |z|\|g\|)\|_p$ is continuous for every $z \in \mathbb{C}$, the set $E^{\perp_p}$ is closed. We also have $E \subset (E^{\perp_p})^{\perp_p}$, and $E \subset F \Rightarrow F^{\perp_p} \subset E^{\perp_p}$, similarly to orthogonal sets in Hilbert spaces or in dual spaces. Hence, for P an L^p-projection leaving E invariant, we have $\operatorname{Ker}(P) \subset E^{\perp_p}$. However, $E^{\perp_p}$ is not a linear subspace and is not always equal to $\operatorname{Ker}(P)$, as we will see in Counter-examples 4.2.20 and 4.2.21.

Lemma 4.2.6. [BDE+77] *Let X be a Banach space, and $1 \leq p \leq +\infty$. Let $Q \in \mathcal{P}_p(X)$ be an L^p-projection. We have $Ran(Q)^{\perp_p} = Ker(Q)$ and $Ker(Q)^{\perp_p} = Ran(Q)$.*

Proof. We can first notice that for any $h_1 \in \mathrm{Ran}(Q)$, $h_2 \in \mathrm{Ker}(Q)$, we have $h_1 \perp_p h_2$. This implies that $\mathrm{Ker}(Q) \subset \mathrm{Ran}(Q)^{\perp_p}$. Let $f \in \mathrm{Ran}(Q)^{\perp_p}$.
Suppose first that $p < +\infty$. We have $f = Q(f) + (I - Q)(f)$, so

$$\|f\|^p = \|Q(f)\|^p + \|(I - Q)(f)\|^p.$$

But we also have $(I - Q)(f) = f - Q(f)$, so

$$\|(I - Q)(f)\|^p = \|f\|^p + \|Q(f)\|^p.$$

Hence, $Q(f) = 0$, so $f \in \mathrm{Ker}(Q)$ and $\mathrm{Ran}(Q)^{\perp_p} = \mathrm{Ker}(Q)$.
Suppose now that $p = +\infty$. Let $z \in \mathbb{C}$. We have $f + zQ(f) = (1 + z)Q(f) + (I - Q)(f)$, so

$$\max(\|f\|, \|zQ(f)\|) = \|f + zQ(f)\| = \max(\|(1 + z)Q(f)\|, \|(I - Q)(f)\|).$$

As this must be true for every $z \in \mathbb{C}$, we have $Q(f) = 0$, so $f \in \mathrm{Ker}(Q)$ and $\mathrm{Ran}(Q)^{\perp_p} = \mathrm{Ker}(Q)$.
The other equality is obtained by taking $Q' = (I - Q)$. □

We introduce properties on a Banach space X for $1 \leq p \leq +\infty$ regarding the behaviour of L^p-projections and of the p-orthogonality of vectors. These properties hold true in the case of L^p-spaces, as we will see in Proposition 4.2.12 below.

Property 4.2.7 (Extension of p-orthogonality to X)**.** For any $f, g \in X$ such that $f \perp_p g$, there exists $P \in \mathcal{P}_p(X)$ such that $P(f) = f$ and $P(g) = 0$.

Property 4.2.8 (Linearity of p-orthogonality on X)**.** For any $f, g, h \in X$ such that $f \perp_p g$ and $f \perp_p h$, we have $f \perp_p (g + h)$.

Proposition 4.2.9. *Let X be a Banach space, and $1 \leq p \leq +\infty$, $p \neq 2$. Then, Property 4.2.7 implies Property 4.2.8*

Proof. Let $f, g, h \in X$ such that $f \perp_p g$ and $f \perp_p h$. Then there exists P_1 and P_2 two L^p-projections such that $P_1(f) = f, P_1(g) = 0$, $P_2(f) = f, P_2(h) = 0$. Let $Q = P_1 P_2$. Since $p \neq 2$, P_1 and P_2 commute, so we have $Q(f) = P_1 P_2(f) = f$, $Q(g) = P_1 P_2(g) = P_2 P_1(g) = 0$, and $Q(h) = P_1 P_2(h) = 0$. Hence, $g, h \in \mathrm{Ker}(Q)$, so $g + h \in \mathrm{Ker}(Q)$. Since $f \in \mathrm{Ran}(Q)$, this implies in turn that $f \perp_p g + h$. □

Proposition 4.2.10. *Let X be a Banach space, and $1 \leq p < +\infty$, $p \neq 2$. The following are equivalent*

(i) X satisfies Property 4.2.7 for p;

(ii) For any subsets E_1, E_2 of X, such that $f \perp_p g$ for every $f \in E_1$, $g \in E_2$, there exists $P \in \mathcal{P}_p(X)$ such that $P(E_1) = E_1$ and $P(E_2) = \{0\}$.

Furthermore, if one of them is true, then for any subspace F of X and for any $P \in \mathcal{P}_p(F)$, there exists $Q \in \mathcal{P}_p(X)$ such that $P = Q|_F$.

Proof. By definition, item (ii) implies Property 4.2.7 for p. For the converse implication, let E_1,E_2 be subsets of X that are p-orthogonal. Since $p < +\infty$, let $f \in E_1$ and denote P_f the minimal L^p-projection for f. For any $g \in E_2$, we have an L^p-projection Q such that $Q(f) = f$ and $Q(g) = 0$. Hence, $P_f \leq Q$ by Corollary 4.1.11, so $P_f Q = P_f$. This implies that $P_f(g) = P_f Q(g) = 0$, so $g \in \mathrm{Ker}(P_f)$. Therefore, $E_2 \subset \mathrm{Ker}(P_f)$.
Let us now denote by P the minimal L^p-projection for E_2. By minimality of P, we have $P \leq (I - P_f)$ that is $P(I - P_f) = P$. This implies that $P(f) = P(I - P_f)(f) = 0$. Therefore we have $E_1 \subset \mathrm{Ker}(P)$, so $E_1 \subset \mathrm{Ran}(I - P)$ and $E_2 \subset \mathrm{Ker}(I - P)$. Thus the conditions of item (ii) are satisfied.
If we now take a subspace F of X and an L^p-projection R on F, we can apply the condition of item (ii) to $E_1 = \mathrm{Ran}(R)$ and $E_2 = \mathrm{Ker}(P)$ to get $S \in \mathcal{P}_p(X)$ such that $S(E_1) = E_1$ and $S(E_2) = \{0\}$. As $F = \mathrm{Ran}(R) \oplus \mathrm{Ker}(R)$ we have $S(F) \subset F$, and S coincides with R on the subspace F, so $S|_F = R$. □

Remark 4.2.11. Proposition 4.2.10 turns out to be also true for $p = +\infty$, but its proof requires additional information for this case. (see Prop.4.2.29)
If X satisfies Property 4.2.8, then so do all the subspaces F of X. This is however not true in general for Property 4.2.7. (see 4.2.21)

Proposition 4.2.12. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $1 \leq p < +\infty$, $p \neq 2$. Then $L^p(\Omega, \mathcal{F}, \mu)$ satisfies Property 4.2.7 for p.*

Proof. Let $f, g \in L^p(\Omega)$ such that $f \perp_p g$. According to Corollary 4.2.3, f and g have a disjoint support. Hence, if we take $A = \mathrm{supp}(f)$ and $P = M_{\chi_A}$, then P is an L^p-projection on $L^p(\Omega)$ such that $P(f) = f$ and $P(g) = 0$. □

Proposition 4.2.12 is not true in general for L^∞-spaces (see Cor.4.2.27). The result in the case $p = 1$ can also be found in the book of Harmand, Werner and Werner [HWW93, Prop 1.21]. It allows us to fully describe L^p-projections on subspaces of L^p-spaces using $\mathcal{P}_p(L^p(\Omega))$.

Corollary 4.2.13 (L^p-projections on subspaces of $L^p(\Omega)$)**.** *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $1 \leq p < +\infty$, $p \neq 2$. Let F be a subspace of $L^p(\Omega, \mathcal{F}, \mu)$ and let $P \in \mathcal{P}_p(F)$. Then there exists $A \subset \Omega$ satisfying $A \cap B \in \mathcal{F}$ for every $B \in \mathcal{F}$ with $\mu(B) < +\infty$ such that $P = M_{\chi_A}$, and we have $F = M_{\chi_A}(F) \oplus_p M_{\chi_{A^C}}(F)$. Furthermore, F admits non-trivial L^p-projections if and only if it admits a non-trivial decomposition of this form.*

Proof. Proposition 4.2.12 tells us that $L^p(\Omega)$ satisfies Property 4.2.7 for p. We can then apply Proposition 4.2.10 and Theorem 4.1.14 in order to obtain that $P = Q|_F$ with $Q = M_{\chi_A}$. Thus we have $P = M_{\chi_A}$, which proves the Corollary. □

For a general Banach space X we have no notion of support for an element x unlike in $L^p(\Omega)$. However, when X satisfies Property 4.2.7 the minimal L^p-projection for x plays a similar role regarding p-orthogonality. The following Lemma gives a similar equivalence as $(iv) \Leftrightarrow (v)$ from Corollary 4.2.3.

Lemma 4.2.14. *Let X be a Banach space, $1 \leq p < +\infty$, $p \neq 2$. Suppose that X satisfies Property 4.2.7 for p. Let $x, y \in X$ and let P, Q be the minimal L^p-projections for x, y respectively. Then, the following are equivalent*

(i) $x \perp_p y$;

(ii) $PQ = 0$;

(iii) $Ran(P) \cap Ran(Q) = \{0\}$.

Proof. $(i) \Rightarrow (ii)$ Since X satisfies Property 4.2.7 for p and $x \perp_p y$, we can use Proposition 4.2.10 to have $R \in \mathcal{P}_p(X)$ such that $R(x) = x$ and $R(y) = 0$. Since P and Q are minimal, Corollary 4.1.11 gives $PR = P$ and $Q(I - R) = Q$. Thus

$$PQ = PRQ(I - R) = PR(I - R)Q = 0.$$

- $(ii) \Rightarrow (i)$ We have $P(x) = x$ and $Q(y) = y$. Therefore $P(y) = P(Q(y)) = 0$, and thus $x \perp_p y$.
- $(ii) \Leftrightarrow (iii)$ This equivalence comes from the fact that P and Q are commuting projections. □

Proposition 4.2.15 (L^p-projections on specific subspaces and quotients of X)**.** [Li79] *Let X be a Banach space, $1 \leq p \leq +\infty$, $p \neq 2$. Let $P \in \mathcal{P}_p(X)$. Then, we have*

(i) The Boolean algebra $\mathcal{P}_p(Ran(P))$ is isomorphic to $\mathcal{P}_p(X) \circ P$;

(ii) The canonical projection $\pi : X \to X/Ran(P)$ induces an isometric isomorphism between $Ker(P)$ and $X/Ran(P)$. Hence, the Boolean algebra $\mathcal{P}_p(X/Ran(P))$ is isomorphic to $\mathcal{P}_p(X) \circ (I - P)$.

Proof. (i) Let $T \in \mathcal{P}_p(X)$. Since $p \neq 2$, T commutes with P so $T(\mathrm{Ran}(P)) \subset \mathrm{Ran}(P)$. Hence, the map $Q_T : y \in \mathrm{Ran}(P) \mapsto T(y)$ is well-defined and is an L^p-projection on $\mathrm{Ran}(P)$.
Let $Q \in \mathcal{P}_p(\mathrm{Ran}(P))$. We then have

$$\begin{aligned} X &= \mathrm{Ker}(P) \oplus_p \mathrm{Ran}(P) = \mathrm{Ker}(P) \oplus_p (\mathrm{Ker}(Q) \oplus_p \mathrm{Ran}(Q)) \\ &= (\mathrm{Ker}(P) \oplus_p \mathrm{Ker}(Q)) \oplus_p \mathrm{Ran}(Q). \end{aligned}$$

Thus, the projection T on $\mathrm{Ran}(Q)$ parallel to $(\mathrm{Ker}(P) \oplus \mathrm{Ker}(Q))$ is an L^p-projection on X. Since $p \neq 2$, T commutes with P and $TP = T$. By construction, we get $Q_T = Q$. Therefore, there is a bijection between $\mathcal{P}_p(\mathrm{Ran}(P))$ and $\{T \circ P,\, T \in \mathcal{P}_p(X)\}$, and the commutativity of these sets makes this bijection an isomorphism of commutative Boolean algebras.
(ii) As $X = \mathrm{Ker}(P) \oplus_p \mathrm{Ran}(P)$, the quotient map $\pi : X \to X/\mathrm{Ran}(P)$ induces a linear bijection T between $\mathrm{Ker}(P)$ and $X/\mathrm{Ran}(P)$. Let $x \in \mathrm{Ker}(P)$. We have

$$\|T(x)\| = \|\pi(x)\| = \inf_{h \in \mathrm{Ran}(P)} \{\|x - h\|\} = \inf_{h \in \mathrm{Ran}(P)} \{\|(\|(I - P)(x)\|, \| - h\|)\|_p\} = \|x\|.$$

Therefore the linear bijection T is isometric. Hence, using item (ii) of Proposition 4.1.3, we have that $\mathcal{P}_p(X/\mathrm{Ran}(P)) = T\mathcal{P}_p(\mathrm{Ker}(P))T^{-1}$. Thus there is an isomorphism of Boolean algebras between $\mathcal{P}_p(X/\mathrm{Ran}(P))$ and $\mathcal{P}_p(\mathrm{Ker}(P))$. Note now that $\mathcal{P}_p(Ker(P))$ is in turn isomorphic to $\mathcal{P}_p(X) \circ (I - P)$. □

Remark 4.2.16. When $p = 2$, the same arguments show that $\mathcal{P}_2(\mathrm{Ran}(P))$ is in bijection with the set $\{TP,\, T \in \mathcal{P}_2(X)\colon TP = PT\}$, which is in general neither commutative nor a Boolean algebra.

Proposition 4.2.17. *Let $(X_i)_{i \in I}$ be a of family Banach spaces. Suppose $1 \leq p \leq +\infty$, $p \neq 2$ and let $Y = \Pi^{\ell_p}_{i \in I} X_i$. Let $P_i \in \mathcal{L}(Y)$ be the projection on X_i parallel to the product of X_j, $j \neq i$. Then the map $\phi : P \in \mathcal{P}_p(Y) \mapsto (PP_i|_{X_i})_{i \in I} \in \Pi_{i \in I} \mathcal{P}_p(X_i)$ is well defined and is an isomorphism of Boolean algebras.*

Proof. Each operator P_i is an L^p-projection on Y. Let $P \in \mathcal{P}_p(Y)$. As $p \neq 2$, P commutes with each P_i, so $P(X_i) \subset X_i$, and $PP_i|_{X_i}$ defines an L^p-projection on X_i. The commutativity also implies that for $P' \in \mathcal{P}_p(Y)$, we have $(PP')P_i|_{X_i} = (P'P_i|_{X_i})(PP_i|_{X_i})$, so $\phi(PP') = \phi(P)\phi(P')$. Thus ϕ turns to be a morphism of Boolean algebras. Let $P \in \mathrm{Ker}(\phi)$ and $y = (x_i)_{i\in I} \in \mathrm{Ran}(P)$. For every $j \in I$, we have $P_j(y) \in Y_j$. This gives

$$0 = PP_j(P_j(y)) = PP_j(y) = P_jP(y) = P_j(y).$$

Hence we have $y_j = 0$ for every $j \in J$, so $y = 0$. This means that $\mathrm{Ran}(P) = \{0\}$, so $P = 0$ and ϕ is injective. Now, let $Q = (Q_i)_{i\in I} \in \Pi_{i\in I}\mathcal{P}_p(X_i)$. The map $P_Q : (x_i)_i \in Y \mapsto (Q_i(x_i)) \in Y$ is well defined and is a norm one projection on Y. We also have

$$\begin{aligned}\|(x_i)_i\|_Y &= \|(\|x_i\|_{X_i})_i\|_{\ell_p} = \|(\|(\|Q_i(x_i)\|, \|(I_{X_i} - Q_i)(x_i)\|)\|_{\ell_p})_i\|_{\ell_p} \\ &= \|(\|(\|Q_i(x_i)\|)_i\|_{\ell_p}, \|(\|(I_{X_i} - Q_i)(x_i)\|)_i\|_{\ell_p})\|_{\ell_p} \\ &= \|(\|(Q_i(x_i))_i\|_Y, \|((I_{X_i} - Q_i)(x_i))_i\|_Y)\|_{\ell_p}.\end{aligned}$$

Hence P_Q is an L^p-projection on Y. By construction of ϕ and P_Q we have $\phi(P_Q) = (Q_i)_i$, so ϕ is bijective. This concludes the proof. □

Proposition 4.2.18 (Finite dimensional p-orthogonal decomposition)**.** *Let X be a finite dimensional Banach space, with $X \neq 0$ and $1 \leq p \leq +\infty$, $p \neq 2$. Then, we have*

(i) $Card(\mathcal{P}_p(X)) = 2^m$, for some $1 \leq m \leq dim(X)$, where $Card(E)$ is the cardinality of E;

(ii) For $m = \log_2(Card(\mathcal{P}_p(X)))$, there exist subspaces $X_1, ..., X_m$ of X such that

$$X = X_1 \oplus_p .. \oplus_p X_m \text{ with } X_i \neq \{0\} \text{ and } \mathcal{P}_p(X_i) = \{0, I\};$$

(iii) With the direct sum of item (ii), denote P_i the projection on X_i parallel to $\oplus_{j\neq i}X_j$. Then $\mathcal{P}_p(X)$ is generated as a Boolean algebra by the family $\{P_1, .., P_m\}$;

(iv) If X satisfies Property 4.2.7, then every X_i in the direct sum of item (ii) satisfies

$$x\perp_p y,\ x, y \in X_i \Rightarrow x = 0 \text{ or } y = 0.$$

Proof. (i) Let $n = dim(X)$. Operators acting on X can then be identified to $n\times n$ matrices with complex entries (for a choice of a basis for X). Thus, $\mathcal{P}_p(X)$ identifies to a set of commutative and diagonalizable matrices with eigenvalues in $\{0, 1\}$. These matrices are then jointly diagonalizable: there exists a change of basis of X that turns all these matrices into diagonal ones. This means that there can be at most $Card(\{0, 1\})^n = 2^n$ elements in this set. As the cardinal of a finite commutative Boolean algebra is 2^m for some $m \geq 0$, we obtain the conclusion of item (i).
- (ii) We will prove the result with an induction on $m \geq 1$. If $m = 1$ then $\mathcal{P}_p(X) = \{0, I\}$ and the result is true. Suppose that the result is true for any Banach space Y with $Card(\mathcal{P}_p(Y)) \leq 2^m$, and let X be a Banach space with $Card(\mathcal{P}_p(Y)) = 2^{m+1}$. As $m+1 > 1$, X possesses non-trivial L^p-projections. Let $P \in \mathcal{P}_p(X)$ that is non-trivial. We then have $X = \mathrm{Ran}(P) \oplus_p \mathrm{Ker}(P)$, and Proposition 4.2.17 gives

$$2^{m+1} = Card(\mathcal{P}_p(X)) = Card(\mathcal{P}_p(\mathrm{Ran}(P)))Card(\mathcal{P}_p(\mathrm{Ker}(P))).$$

Since both Ran(P) and Ker(P) are different from $\{0\}$, these subspaces possess at least 2 L^p-projections. The previous equation then implies that they cannot possess more than 2^m L^p-projections. As

$$Card(\mathcal{P}_p(\text{Ran}(P))) = 2^k \text{ and } Card(\mathcal{P}_p(\text{Ker}(P))) = 2^{m+1-k},$$

we can apply the induction hypothesis to Ran(P) and Ker(P) to get subspaces $F_1, .., F_k$ and $G_{k+1}, ..., G_{m+1}$ such that $F_i, G_j \neq \{0\}$, $\mathcal{P}_p(F_i) = \{0, I\}$, $\mathcal{P}_p(G_j) = \{0, I\}$, and

$$X = \text{Ran}(P) \oplus_p \text{Ker}(P) = (F_1 \oplus_p .. \oplus_p F_k) \oplus_p (G_{k+1} \oplus_p .. \oplus_p G_{m+1}),$$

which concludes the proof by induction.
- (iii) Since $X = X_1 \oplus_p .. \oplus_p X_m$, all P_i are L^p-projections on X. Since every X_i is not reduced to $\{0\}$, every P_i is non-zero. For any set $E \subset \{1, .., m\}$, we can see that the projection on $\oplus_{i \in E} X_i$ parallel to $\oplus_{j \notin E} X_j$ is well-defined and lies in the Boolean sub-algebra generated by the family $\{P_1, ..., P_m\}$. Since all these projections are distinct, the Boolean sub-algebra generated by the family $\{P_1, .., P_m\}$ has at least $2^m = Card(\mathcal{P}_p(X))$ elements, so it is equal to $\mathcal{P}_p(X)$.
- (iv) Let $1 \leq i \leq m$. Let $x, y \in X_i$ be such that $x \perp_p y$, and let $Q \in \mathcal{P}_p(X)$ be such that $Q(x) = x$ and $Q(y) = 0$. As for every $1 \leq j \leq m$, we have $P_j(X_i) = X_i$ or $\{0\}$, we can use item (iii) to see that we either have $Q(X_i) = X_i$ or $\{0\}$. If we have $Q(X_i) = X_i$ this implies that $0 = Q(y) = y$. If we have $Q(X_i) \neq X_i$ then $Q(X_i) = \{0\}$ so $x = Q(x) = 0$. Therefore, we have $x = 0$ or $y = 0$. □

Remark 4.2.19. Concerning item (iv) of Proposition 4.2.18, we have not been able to answer the following question when $1 < p \leq +\infty$. For X a Banach space of the form $X = X_1 \oplus_p .. \oplus_p X_m$ satisfying

$$x_i \perp_p y_i,\ x_i, y_i \in X_i \Rightarrow x_i = 0 \text{ or } y_i = 0,$$

does X satisfy Property 4.2.7 for p ?

When $p = 1$, if we take $x, y \in X$ with $x \perp_p y$, for any $z \in \mathbb{C}$ we have

$$\sum_i \|x_i + zy_i\| = \|\sum_i x_i + zy_i\| = \|x + zy\| = \|x\| + \|zy\| = \sum_i \|x_i\| + \sum_i \|zy_i\|,$$

and the triangular inequality applied to every $x_i + zy_i$ forces us to have $\|x_i + zy_i\| = \|x_i\| + \|zy_i\|$. Hence, we end up with $x_i \perp_p y_i$ for every i, so $x_i = 0$ or $y_i = 0$ for every i. By taking $J = \{i\colon\ x_i \neq 0\}$ and P the projection on $\bigoplus_{j \in J} X_j$ parallel to $\bigoplus_{j \notin J} X_j$, we can see that P is a L^1-projection such that $P(x) = x$ an $P(y) = 0$.

4.2.B Counter-examples for p-orthogonality

We collect in this section several examples.

Counter-Example 4.2.20 (A Banach space not satisfying Property 4.2.8)**.** Let $X = \mathbb{C}^3$, and $\{e_1, e_2, e_3\}$ be its canonical basis. Let $1 \leq p < +\infty$. Take

$$E = \{e^{ia} t.e_1 + e^{ib}(1 - t^p)^{\frac{1}{p}}.e_2;\ e^{ia} t.e_3 + e^{ib}(1 - t^p)^{\frac{1}{p}}.e_2;\ e^{ic}(e_1 + e_2 + e_3),\ a, b, c \in \mathbb{R},\ t \in [0, 1]\}$$
$$K = \text{Conv}(E).$$

Then, K is a compact convex set that contains 0 in its interior and that is invariant under multiplication by λ, for any $\lambda \in \partial\mathbb{D}$. Hence there exists a norm N on X whose closed unit ball is K. By construction, $\mathrm{Span}(e_1, e_2)$ and $\mathrm{Span}(e_2, e_3)$ are isometric to $\ell^p(\mathbb{C}^2)$, therefore we have $e_2 \perp_p e_1$ and $e_2 \perp_p e_3$. Since $e_1 + e_2 + e_3$, e_2 and $\frac{e_1+e_3}{2}$ are in the boundary of K, we have

$$N(e_1 + e_2 + e_3) = N(e_2) = N(\frac{e_1 + e_3}{2}) = 1.$$

Thus, we have

$$N(e_2 + (e_1 + e_3)) = 1 < 2 = N(e_1 + e_3) \leq \|(N(e_2), N(e_1 + e_3))\|_p.$$

Hence, e_2 is not p-orthogonal to $e_1 + e_3$, so X does not satisfy Property 4.2.8.

Counter-Example 4.2.21 (A Subspace not satisfying Property 4.2.7 nor Proposition 4.2.10)**.** Let $1 \leq p < +\infty$ and $X = \ell^p(\mathbb{C}^4)$. Take $f = (1, -1, 0, 0)$, $g = (0, 0, 1, -1)$, $h = (1, 1, 1, 1)$, and $F = \mathrm{Span}(f, g, h)$. As X is an L^p-space, it satisfies Property 4.2.7. However, its closed subspace F does not. Indeed, on F we have $f \perp_p h$, but since $h^{\perp_p} = \mathrm{Span}(f)$ and $f^{\perp_p} = \mathrm{Span}(h)$, F cannot possess any L^p-projection P such that $P(f) = f$ and $P(g) = 0$ as we would have

$$dim(\mathrm{Ran}(P)) + dim(\mathrm{Ker}(P)) \leq 1 + 1 = 2 < 3 = dim(F).$$

Furthermore, every element x in F that is not in $\mathrm{Span}(f)$ nor $\mathrm{Span}(g)$ satisfies $x^{\perp_p} = \{0\}$ as there is no element in F outside 0 that has a support disjoint with $\mathrm{supp}(x)$. Hence, $\mathcal{P}_p(F) = \{0, I\}$, whereas the subspace $\mathrm{Span}(f, g)$ clearly possesses non-trivial L^p-projections. Thus, the second part of Proposition 4.2.10 is also false for F and F does not satisfy Property 4.2.7.

Counter-Example 4.2.22 (A Banach Space satisfying Property 4.2.7 but not Clarkson's equality case)**.** Let $1 \leq p < +\infty$, $X = \mathbb{C}^2$ and let $\{e_1, e_2\}$ be its canonical basis. Take

$$K = \mathrm{Conv}(\{e^{ia}e_1;\, e^{ia}e_2;\, \frac{e^{ia}e_1 + e^{ib}e_2}{2^{\frac{1}{p}}},\, a, b \in \mathbb{R}\}).$$

Then, K is a compact convex set that contains 0 in its interior and that is invariant under multiplication by λ, for any $\lambda \in \partial\mathbb{D}$. Hence there exists a norm N on X whose closed unit ball is K. By construction, e_1, e_2 and $\frac{1}{2^{\frac{1}{p}}}(e_1 + e^{ib}e_2)$ are in the boundary of K, so these elements are of norm one. Thus, we have

$$N(e_1 + e^{it}e_2)^p = 2 = N(e_1)^p + N(e_2)^p,\ \forall t \in \mathbb{R}.$$

Since the unit ball for N is not uniformly convex, it cannot be equal to an ℓ^p unit ball for some $1 < p < +\infty$. Hence, $\mathrm{Span}(e_1, e_2)$ cannot be isometrically isomorphic to $\ell^p(\mathbb{C}^2)$, so e_1 and e_2 are not p-orthogonal. Thus, $\mathcal{P}_p(X)$ is trivial, as well as $\mathcal{P}_p(F)$ for every subspace F of X. Therefore, X satisfies Property 4.2.7 but it does not possess anything similar to the equality case in Clarkson inequalities as the implication $(iii) \Rightarrow (iv)$ of Corollary 4.2.3 is false on X.

Counter-Example 4.2.23 (A Banach space not satisfying Property 4.2.8, of finite dimension, but with $\alpha(X)$ not finite)**.** Let $X = \mathbb{C}^3$, and $\{e_1, e_2, e_3\}$ be its canonical basis. Let $1 \leq p \leq +\infty$.

We refer to 4.2.30 for the definition of the cardinality $\alpha(X)$. Take

$$\begin{aligned} E = &\{e^{ia}t.e_2 + e^{ib}(1-t^p)^{\frac{1}{p}}[\cos(s)e_1 + \sin(s)e_3];\ e^{ia}(\frac{1}{2}e_2 + (1-\frac{1}{2^p})^{\frac{1}{p}}(e_1-e_3);\\ &e^{ia}te_1 + e^{ib}\sqrt{1-t^2}e_3,\ a,b\in\mathbb{R},\ 0\le s\le\frac{\pi}{2}\}\\ K = &\ \mathrm{Conv}(E). \end{aligned}$$

Then, K is a compact convex set that contains 0 in its interior and that is invariant under multiplication by λ, for any $\lambda\in\partial\mathbb{D}$. Hence there exists a norm N on X whose closed unit ball is K. By construction, $\mathrm{Span}(e_1,e_3)$ is isometric to $l_2(\mathbb{C}^2)$ and $\mathrm{Span}(e_2,\cos(s)e_1+\sin(s)e_3)$ is isometric to $\ell^p(\mathbb{C}^2)$ for every $0\le s\le\frac{\pi}{2}$. Since $\frac{\sqrt{2}}{2}(e_1-e_3)$, e_2 and $\frac{1}{2}e_2+(1-\frac{1}{2^p})^{\frac{1}{p}}(e_1-e_3)$ are in the boundary of K, we have

$$N(e_1-e_3)=\sqrt(2),\ N(e_2)=N(\frac{1}{2}e_2+(1-\frac{1}{2^p})^{\frac{1}{p}}(e_1-e_3))=1.$$

This gives

$$N(\frac{1}{2}e_2)^p + N((1-\frac{1}{2^p})^{\frac{1}{p}}(e_1-e_3))^p = \frac{1}{2^p}+\sqrt{2}(1-\frac{1}{2^p}) > 1 = 1^p,$$

so the vectors e_2 and e_1-e_3 are not p-orthogonal. Since e_2 is p-orthogonal to e_1 and e_3 we can see that X does not satisfy Property 4.2.8 for p.
Since e_2 is not p-orthogonal to $\mathrm{Span}(e_1,e_3)$, then for almost every $0\le s\le\frac{\pi}{2}$ any non-trivial L^p-projection on $\mathrm{Span}(e_2,\cos(s)e_1+\sin(s)e_3)$ does not extend to $\mathrm{Span}(e_2,e_1,e_3)=X$. Therefore, according to Definition 4.2.30, almost all non-trivial L^p-projections on these spaces are maximal, so $\alpha(X)$ is not finite.

4.2.C The case of L^∞-projections

This section focuses on results regarding L^∞-projections. We describe ∞-orthogonality in $L^\infty(\Omega)$ and we use it to determine $\mathcal{P}_\infty(L^\infty(\Omega))$, along with an equivalent form of Proposition 4.2.10 for $p=+\infty$.

Proposition 4.2.24. *Let $(\Omega,\mathcal{F},\mu)$ be a measure space, and let $f\in L^\infty(\Omega)$, $f\neq 0$. Then,*

(i) $f^{\perp_\infty} = \{g : |g(x)| \le \frac{\|f\|-|f(x)|}{\|f\|}\|g\|,\ \forall_{a.e.}x\in\Omega\} = \{g\neq 0 : \frac{|g|}{\|g\|}+\frac{|f|}{\|f\|}\le_{a.e.} 1\}\cup\{0\}$;

(ii) For $g\in f^{\perp_\infty}$ and $B_n\in\mathcal{F}$ such that $\mu(B_n)>0$ and $\|(\|f\|-|f|)\chi_{B_n}\|\to_n 0$, we have $\|g\chi_{B_n}\|\to_n 0$;

*(iii) We have $f^{\perp_\infty}=\{0\}$ if and only if $\|\|f\|-|f|\|<\|f\|$.
When $\Omega=\{1,..,n\}$, this is equivalent to $|f(i)|\neq 0$ for every $i\in\Omega$.
When $\Omega=\mathbb{N}$, this is equivalent to 0 not being in the closure of $\{f(n),\ n\ge 0\}$;*

(iv) $f^{\perp_\infty}$ is a non-zero subspace if and only if $|f| =_{a.e.} \|f\|\chi_A$ for $A\in\mathcal{F}$ with $\mu(A^C)>0$.

Proof. All the elements of $L^\infty(\Omega)$ that we will consider here will be associated to a representative that takes finite values everywhere.
- (i) Let $g\in f^{\perp_\infty}$. If $g=0$ there is nothing to prove. Suppose that $g\neq 0$. Since the ∞-orthogonality is homogeneous, we can divide f and g by their respective norms and suppose

that $\|f\| = \|g\| = 1$. As $\mathbb{C}$ is separable, there is $A \in \mathcal{F}$ with $\mu(A^C) = 0$ such that for every $x \in A$ and for every $z \in \mathbb{C}$, we have $|f(x) + zg(x)| \leq \|f + zg\|$. We also recall that for $x \in A$ and for any $r \geq 0$, we have $z \in \mathbb{C}$ with $|z| = r$ such that $|f(x) + zg(x)| = |f(x)| + r|g(x)|$. Therefore, for $r = 1$ we have a $z \in \mathbb{C}$ such that

$$|f(x) + zg(x)| = |f(x)| + |g(x)| \leq \|f + zg\| = \max(\|f\|, \|zg\|) = \max(1, 1) = 1.$$

Hence, we obtain

$$|g(x)| \leq 1 - |f(x)|, \, \forall x \in A,$$

which is the condition in the statement when $\|f\| = \|g\| = 1$. We now need to prove that for every g non-zero that satisfies this condition, g lies in $f^{\perp_\infty}$. By homogeneity of both sets, we can suppose that $\|g\| = 1$. Hence, we need to show that

$$\|f + zg\| = \max(\|f\|, \|zg\|) = \max(1, |z|), \, \forall z \in \mathbb{C}.$$

Let $z \in \mathbb{C}$. For almost every $x \in \Omega$, we have

$$\begin{aligned} |f(x) + zg(x)| &\leq |f(x)| + |z||g(x)| = (|f(x)| + |g(x)|) + (|z| - 1)|g(x)| \leq 1 + (|z| - 1)|g(x)| \\ &\leq \begin{matrix} 1 + (|z| - 1) = |z|, \text{ if } |z| \geq 1; \\ 1 + 0 = 1, \text{ if } |z| \leq 1; \end{matrix} \end{aligned}$$

so $\|f + zg\| \leq_{a.e.} \max(1, |z|)$. Suppose that $|z| \geq 1$. The properties of the norm $\|\cdot\|_\infty$ imply the existence of sets $A_n \in \mathcal{F}$ such that $\mu(A_n) > 0$ and $\alpha(n) := \|(\|g\| - |g|)\chi_{A_n}\| \to_n 0$. Therefore, for almost every $x \in A_n$, the condition on g gives us

$$\begin{aligned} |f(x) + zg(x)| &\geq |z||g(x)| - |f(x)| = |z|\|g\| - |z|(\|g\| - |g(x)|) - |f(x)| \\ &\geq |z| - |z|\alpha(n) - (1 - |g(x)|) \geq |z| - |z|\alpha(n) - \alpha(n). \end{aligned}$$

Hence we obtain

$$|z| \geq \|f + zg\| \geq \|(f + zg)\chi_{A_n}\| \geq |z| - |z|\alpha(n) - \alpha(n) \to_n |z|,$$

so $\|f + zg\| = |z|$.
Suppose now that $|z| \leq 1$. The properties of the norm $\|\cdot\|_\infty$ imply the existence of sets $B_n \in \mathcal{F}$ such that $\mu(B_n) > 0$ and $\beta(n) := \|(\|f\| - |f|)\chi_{B_n}\| \to_n 0$. Therefore, for almost every $x \in B_n$, the condition on g gives us

$$\begin{aligned} |f(x) + zg(x)| &\geq |f(x)| - |z||g(x)| = \|f\| - (\|f\| - |f(x)|) - |z||g(x)| \\ &\geq 1 - \beta(n) - |z|(1 - |f(x)|) \geq 1 - \beta(n) - |z|\beta(n). \end{aligned}$$

Hence we obtain

$$1 \geq \|f + zg\| \geq \|(f + zg)\chi_{B_n}\| \geq 1 - \beta(n) - |z|\beta(n) \to_n 1,$$

so $\|f + zg\| = 1$. This proves that $f \perp_\infty g$ and concludes the proof of item (i).
- (ii) Let $B_n \in \mathcal{F}$ such that $\mu(B_n) > 0$ and $\|(\|f\| - |f|)|_{B_n}\| \to_n 0$ and $g \in f^{\perp_\infty}$. Then, for almost every $x \in B_n$ we have

$$|g(x)| \leq \frac{\|g\|}{\|f\|}(\|f\| - |f(x)|) \leq \frac{\|g\|}{\|f\|}\|(\|f\| - |f|)\chi_{B_n}\|,$$

so $\|g\chi_{B_n}\| \to_n 0$.
- (iii) Suppose that we have a vector g in $f^{\perp_\infty}$ that is non-zero. Then we have $f \in g^{\perp_\infty}$. We also have $A_n \in \mathcal{F}$ such that $\mu(A_n) > 0$ and $\|(\|g\| - |g|)\chi_{A_n}\| \to_n 0$. Since g is non-zero, item (ii) implies that $\|f\chi_{A_n}\| \to_n 0$. Therefore, we have

$$\|f\| \geq \|\|f\| - |f|\| \geq \|(\|f\| - |f|)\chi_{A_n}\| = \|f\| - \|f\chi_{A_n}\| \to_n \|f\|,$$

so we obtain $\|\|f\| - |f|\| = \|f\|$. Conversely, suppose now that $\|\|f\| - |f|\| = \|f\|$. Take $g = \frac{\|f\|-|f|}{\|f\|}$. Then $g \in L^\infty(\Omega)$ and $\|g\| = \frac{\|f\|}{\|f\|} = 1$. Thus,

$$|g| =_{a.e} \frac{\|f\| - |f|}{\|f\|} = \frac{\|f\| - |f|}{\|f\|}\|g\|,$$

so $g \in f^{\perp_\infty}$ according to item (i), and the set $f^{\perp_\infty}$ is not reduced to 0.
The remaining statements of item (iii) for $\Omega = \{1, .., n\}$ and $\Omega = \mathbb{N}$ are simplifications of the condition $\|\|f\| - |f|\| < \|f\|$ in these cases.
- (iv) When $|f| =_{a.e.} \|f\|\chi_A$, item (i) tells us that $f^{\perp_\infty} = M_{\chi_{A^C}}(L^\infty)$, which is a subspace. Such a subspace is non-zero if and only if $\mu(A^C) > 0$. We now take f such that $|f|$ does not have the form $\|f\|\chi_A$. If $\|\|f\| - |f|\| < \|f\|$ then item (iii) tells us that $f^{\perp_\infty} = \{0\}$, so we can also suppose that $\|\|f\| - |f|\| = \|f\|$. Then, as seen previously, the function $g = \frac{\|f\|-|f|}{\|f\|}$ has a norm of 1 and lies in $f^{\perp_\infty}$. Since $|f|$ is not equal to any $\|f\|\chi_A$, we can find $\epsilon > 0$ and $B \in \mathcal{F}$ such that $\mu(B) > 0$ and $\epsilon \leq |f| \leq \|f\| - \epsilon$ almost everywhere on B. We now take $h = \frac{\|f\|-|f|}{\|f\|} - \frac{\epsilon}{\|f\|}\chi_B$. As we have $|f| + \epsilon \leq \|f\|$ a.e. on B, we can see that $h \geq 0$ a.e. on Ω. As h coincides with g a.e. on B^C, the properties of g and B also imply that $\|h\| = \|g\| = 1$. Hence, for almost every x we have

$$|h(x)| = h(x) \leq \frac{\|f\| - |f(x)|}{\|f\|} = \frac{\|f\| - |f|}{\|f\|}\|h\|,$$

so h is also in $f^{\perp_\infty}$. However $g - h$ does not lie in $f^{\perp_\infty}$. Indeed, if we had $f \perp_\infty g - h$, then f would belong to $(g - h)^{\perp_\infty} = L^\infty(B^C)$, which is not true since $f > 0$ a.e. on B. Therefore $f^{\perp_\infty}$ is not a subspace in this case. □

L^∞-projections have been totally characterized on certain subspaces of L^∞ such as $C^0(\Omega)$ (continuous maps on a locally compact and Hausdorff set) [HWW93, Ex.1.4(a)]; $L^\infty(\Omega)$ when Ω is σ-finite [CER73][HWW93, Thm.1.9]; $Lip([0, 1])$, $AC([0, 1])$ or $C^1([0, 1])$ (Lipschitz, absolutely continuous, or C^1 maps over $[0, 1]$) [BS74]; $l^\infty(\mathbb{N})$, $c_0(\mathbb{N})$, or $c(\mathbb{N})$ [Tor68]. These results either come from a characterization of Hermitian operators ($P \in \mathcal{L}(X)$: $\|e^{i\alpha P}\| = 1$, $\forall \alpha \in \mathbb{R}$) on the said spaces, or from a characterization of surjective isometries (a projection P is Hermitian if and only if $P + \lambda(I - P)$ is an isometry for every $\lambda \in \partial\mathbb{D}$).
The following theorem characterizes L^∞-projections on every L^∞ space using tools and results mainly coming from the $+\infty$-orthgonality relationship.

Theorem 4.2.25 (L^∞-projections on $L^\infty(\Omega)$)**.** *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. Then*

$$\mathcal{P}_\infty(L^\infty(\Omega)) = \{M_{\chi_A},\ A \in \mathcal{F}\}.$$

Proof. Notice first that any projection of the form M_{χ_A}, $A \in \mathcal{F}$, is a L^∞-projection.
Let $P \in \mathcal{P}_\infty(L^\infty(\Omega))$ that is non-trivial. Let $f \in \mathrm{Ran}(P)$ and $g \in \mathrm{Ker}(P)$ that are non-zero.

We will first show that f and g have disjoint supports (up to a set of measure zero). Take $A = \text{supp}(f) \cap \text{supp}(g)$ and suppose that $\mu(A) > 0$. Denote $f_1 = \chi_A f$ and $g_1 = \chi_A g$. The L^∞-projection $Q = PM_{\chi_A}$ gives

$$Q(f_1) = PM_{\chi_A}M_{\chi_A}(f) = PM_{\chi_A}(f) = M_{\chi_A}P(f) = M_{\chi_A}(f) = f_1.$$

Similarly, we obtain $Q(g_1) = M_{\chi_A}P(g) = 0$. Hence f_1 and g_1 are ∞-orthogonal and have a support of A (up to a set of measure zero). Since the ∞-orthogonality is homogeneous and f_1, g_1 are non-zero, we can divide them by their respective norm to suppose that $\|f_1\| = \|g_1\| = 1$. Hence, there exists $\epsilon > 0$ small enough such that the set

$$B = \{x \in A\colon\ \epsilon \leq |f_1(x)|\}$$

has a strictly positive measure.
Let us now consider $f_2 = f_1\chi_B$ and $g_2 = g_1\chi_B$. Since A is a support for f_1 and g_1 and that $B \subset A$, f_2 and g_2 are non-zero and the set B is a support of these maps. With the L^∞-projection $R = QM_{\chi_B}$, a small computation gives $R(f_2) = f_2$ and $R(g_2) = 0$, so f_2 and g_2 are ∞-orthogonal. However, since $|f_2| \geq \epsilon$ almost everywhere on B, we can see that

$$\| \|f_2\| - |f_2| \|_{L^\infty(B)} \leq \|f_2\|_{L^\infty(B)} - \epsilon < \|f_2\|_{L^\infty(B)}.$$

Considering now $L^\infty(B)$, we can then apply item (iii) of Proposition 4.2.24 to obtain $f_2^{\perp_\infty} = \{0\}$ in $L^\infty(B)$, which contradicts the fact that in $L^\infty(B)$ we have $g_2 \neq 0$ and $f_2 \perp_\infty g_2$. Thus, we have $\mu(A) = 0$, so the initial f and g have disjoint supports.
Let us now take $f = P(\chi_\Omega)$, and $g = (I-P)(\chi_\Omega)$. We now know that the supports of f and g are disjoint. Since $f + g = \chi_\Omega$ and $\text{supp}(\chi_\Omega) = \Omega$, then $\text{supp}(f)^C$ is a support for g (up to a set of measure zero). For every $g' \in \text{Ker}(P)$ the previous argument tells us that $\text{supp}(g')$ is contained in $\text{supp}(f)^C$. For every $f' \in \text{Ran}(P)$, $\text{supp}(f')$ is then contained in $\text{supp}(g)^C = (\text{supp}(f)^C)^C = \text{supp}(f)$. Thus the L^∞-projection $M_{\chi_{\text{supp}(f)}}$ coincides with P on $\text{Ran}(P)$ and on $\text{Ker}(P)$, hence $P = M_{\chi_{\text{supp}(f)}}$, which concludes the proof. □

Remark 4.2.26. The ideas in the proof of this theorem can be used with some extra computations in order to show that for $X = c_0(\mathbb{N}), c(\mathbb{N})$, or $c_{00}(\mathbb{N})$, we have $\mathcal{P}_\infty(X) = \{P \in \mathcal{P}_\infty(l^\infty(\mathbb{N}))\colon\ P(X) \subset X\}$.

Corollary 4.2.27. *Let X be a Banach space such that $Card(\mathcal{P}_\infty(X)) > 4$. Then X does not satisfy Property 4.2.8 for $p = +\infty$.*
This is in particular true for $X = l^\infty(\{0, .., n-1\})$, with $n \geq 3$.

Proof. We will show that X possesses a subspace isometrically isomorphic to $l^\infty(\{0,1,2\})$. As $Card(\mathcal{P}_\infty(X)) > 4$, we can find L^∞-projections P, Q on X such that $P \notin \{0, I\}$ and $Q \notin \{0, I, P, (I-P)\}$. Since P and Q commute, we have $Q(\text{Ran}(P)) \subset \text{Ran}(P)$ and $Q(\text{Ker}(P)) \subset \text{Ker}(P)$, so $Q|_{\text{Ker}(P)}$ and $Q|_{\text{Ran}(P)}$ are also L^∞-projections, with at least one of them non-trivial. Hence, we have

$$\begin{aligned} X &= \text{Ran}(P) \oplus_\infty \text{Ker}(P) \\ &= (Q(\text{Ran}(P)) \oplus_\infty (I-Q)(\text{Ran}(P))) \oplus_\infty (Q(\text{Ker}(P)) \oplus_\infty (I-Q)(\text{Ker}(P))) \\ &= E_1 \oplus_\infty E_2 \oplus_\infty E_3 \oplus_\infty E_4. \end{aligned}$$

The choice of P and Q implies that at most one E_j is reduced to $\{0\}$. Up to reordering, let us suppose that the subspaces E_1, E_2, E_3 are not reduced to $\{0\}$, and take $f_i \in E_i$ with $\|f_i\| = 1$. Thus, we have

$$\|af_1 + bf_2 + cf_3\| = \max(\|af_1 + bf_2\|, \|cf_3\|) = \max(\max(\|af_1\|, \|bf_2\|), |c|)$$
$$= \max(|a|, |b|, |c|) = \|(a, b, c)\|_\infty.$$

Hence, $\mathrm{Span}(f_1, f_2, f_3)$ is isometrically isomorphic to $l^\infty(\{0, 1, 2\})$. By using Proposition 4.2.24 for $f = (2, 1, 0)$ on $l^\infty(\{0, 1, 2\})$ we can see that $f^{\perp_{+\infty}}$ is not a subspace. Hence $\mathrm{Span}(f_1, f_2, f_3)$ does not satisfy Property 4.2.8 for $p = +\infty$ and Remark 4.2.11 implies that X does not satisfy this Property too. □

Remark 4.2.28. With Corollary 4.2.27 we can see that for $p = +\infty$, the question in Remark 4.2.19 turns out to be false in general. Indeed, we can see that the ∞-orthogonality is trivial on $\mathbb{C}$ but $l^\infty(\{0, 1, 2\}) = \mathbb{C} \oplus_\infty \mathbb{C} \oplus_\infty \mathbb{C}$ does not satisfy Property 4.2.8, so it does not satisfy Property 4.2.7.

Corollary 4.2.27 also allows us to extend the results of Proposition 4.2.10 to the case $p = +\infty$, as we state below.

Proposition 4.2.29. *Let X be a Banach space, and $p = +\infty$. The following are equivalent*

(i) X satisfies Property 4.2.7 for p;

(ii) For any subsets E_1, E_2 of X, such that $f \perp_p g$ for every $f \in E_1$, $g \in E_2$, there exists $P \in \mathcal{P}_p(X)$ such that $P(E_1) = E_1$ and $P(E_2) = \{0\}$.

Furthermore, if one of them is true, then there exist closed subspaces X_1, X_2 satisfying $f \perp_\infty g \Rightarrow f = 0$ or $g = 0$, such that $X = X_1 \oplus_\infty X_2$.

Proof. We can see that item (ii) implies Property 4.2.7. Hence, suppose that Property 4.2.7 is true for X and $+\infty$. Thus, Property 4.2.8 is true for X and $+\infty$, according to Proposition 4.2.9. Therefore, $Card(\mathcal{P}_\infty(X)) \leq 4$, according to Corollary 4.2.27. We then have $\mathcal{P}_\infty(X) = \{0, I\}$ or $\{0, I, P, (I - P)\}$. If $\mathcal{P}_\infty(X) = \{0, I\}$, then $f \perp_\infty g$ implies that $f = 0$ or $g = 0$, which means that for any set E we have $E^{\perp_\infty} = \{0\}$. Hence X satisfies item (ii) as every ∞-orthogonality between sets is trivial, and for $X_1 = X$, $X_2 = \{0\}$ we also have the desired decomposition.
Suppose now that $\mathcal{P}_\infty(X) = \{0, I, P, (I - P)\}$. Denote $X_1 = \mathrm{Ker}(P)$, $X_2 = \mathrm{Ran}(P)$. Let $E_1, E_2 \subset X$ be such that $E_1 \perp_\infty E_2$. If $E_1 = \{0\}$ then E_1 and E_2 are separated by $Q = 0$. If $E_2 = \{0\}$ then E_1 and E_2 are separated by $Q = I$. Suppose now that $E_1 \neq \{0\}$ and $E_2 \neq \{0\}$. Let $f \in E_1$,$g \in E_2$ that are non-zero. We have $f \perp_\infty g$. Hence, there exists $Q \in \mathcal{P}_\infty(X)$ such that $Q(f) = f$ and $Q(g) = 0$. Since f, g are non-zero, we must have $Q = P$ or $Q = (I - P)$, that is $f \in X_1$,$g \in X_2$ or $f \in X_2$,$g \in X_1$.
Up to reordering, suppose that $f \in X_1$ and $g \in X_2$. Thus, for any $f' \in E_1$ we have $f' \perp_\infty g$ so the L^∞-projection Q such that $Q(f') = f'$ and $Q(g) = 0$ is either 0 or P. Similarly, for any $g' \in E_2$ we have $f \perp_\infty g'$ so the L^∞-projection Q such that $Q(f) = f$ and $Q(g') = 0$ is either 0 or P. This implies that $E_1 \subset X_1$ and $E_2 \subset X_2$ so $P(E_1) = E_1$ and $P(E_2) = \{0\}$. Therefore, X satisfies item (ii).
We can also notice that the subspaces $X_1 = \mathrm{Ker}(P)$ and $X_2 = \mathrm{Ran}(P)$ are non-trivial and that they have no L^∞-projection other than 0 and I, as it is stated by item (i) of Proposition 4.2.15. Thus, similarly to the beginning of the proof, for $f, g \in X_1$ (resp X_2) that are ∞-orthogonal, we have $f = 0$ or $g = 0$, $X = X_1 \oplus_\infty X_2$ gives the desired decomposition. □

4.2.D Maximal L^p-projections for a Banach space

In this section we introduce the notion of maximal L^p-projections for a Banach space X, that is L^p-projections on a subspace of X that cannot be extended to L^p-projections on a larger subspace. This section will study these projections and focus on upper bounds for their number in certain situations, especially in the finite dimensional case.
Most of the results are stated for $1 \leq p \leq +\infty$, $p \neq 2$. The initial results are true for $p = 2$ (up to Lemma 4.2.37), but not the later ones who rely on properties of $\mathcal{P}_p(X)$. As the previous section highlighted differences in the case $p = +\infty$, (see Proposition 4.2.24 and Proposition 4.2.29), some of these later results are also not stated for $p = +\infty$. However equivalent information for $p = +\infty$ is given with item (ii) of Proposition 4.2.50.

Definition 4.2.30 (Maximal L^p-projections)**.** Let X be a Banach space, and $1 \leq p \leq +\infty$. Let F be a closed subspace of X, and let $P \in \mathcal{P}_p(F)$. The L^p-projection P is said to be *maximal for X* if there exists no subspaces G and L^p-projection Q on G such that $F \subsetneq G$ and $Q|_F = P$. We also define

$$\alpha(F) := Card(\{P : P \text{ is a maximal } L^p\text{-projection for } F\}),$$

where $Card(\cdot)$ refers to the cardinality map.

Let us consider P a maximal L^p-projection on X, defined on a subspace G, that is non-trivial. Then for every x in $\mathrm{Ker}(P)$ (resp. $\mathrm{Ran}(P)$) we have $\mathrm{Ran}(P) \subset x^{\perp_p}$ (resp. $\mathrm{Ker}(P) \subset x^{\perp_p}$). As G is spanned by $\mathrm{Ran}(P)$ and $\mathrm{Ker}(P)$, we obtain that G is spanned by vectors x of X whose p-orthogonal is not reduced to $\{0\}$. This fact led us to the following definition.

Definition 4.2.31. Let X be a Banach space, and let p be such that $1 \leq p \leq +\infty$. We define the subspace

$$X_{(p)} := \mathrm{Span}(\{x \in X : x^{\perp_p} \neq \{0\}\}).$$

Hence $X_{(p)}$ is the subspace of X spanned by all the vectors whose p-orthogonal is not reduced to $\{0\}$.

Remark 4.2.32. When F satisfies Property 4.2.7 for p, all L^p-projections that are maximal for F are an element of $\mathcal{P}_p(F)$ according to Proposition 4.2.10. Thus $\alpha(F) = Card(\mathcal{P}_p(F))$. In this case, we also have $F_{(p)} = F$ if $\mathcal{P}_p(F) \neq \{0, I\}$, or $F_{(p)} = \{0\}$ if not. As we saw in Counter-Example 4.2.21, we can construct a subspace G of F possessing L^p-projections and find some elements $(f_i)_{i \in I}$ that do not "behave" well regarding the L^p-projections of G. Using these elements, we can construct a subspace H of F that contains G but that only possesses trivial L^p-projections. Hence, we focus our study on the L^p-projections that may exist on subspaces of H. We give below some general results concerning L^p-projections that are maximal for H, and give an upper bound of $\alpha(H)$ when $dim(H)$ is finite.

The next lemmas and corollaries show how the subspace $X_{(p)}$ appears in the study of maximal L^p-projections on X.

Lemma 4.2.33. *Let X be a Banach space, and $1 \leq p \leq +\infty$.*

(i) Let F be a closed subspace of X and let $P \in \mathcal{P}_p(F)$. Then, there exists a subspace G and $Q \in \mathcal{P}_p(G)$ such that $F \subset G$, Q is a maximal L^p-projection for X, and $Q|_F = P$;

(ii) *Let $x \in X$ such that $x^{\perp_p} \neq \{0\}$. Then, there exists Q a maximal L^p-projection for X such that $Q(x) = x$;*

(iii) *Let H, F be closed subspaces of X with $H \subset F$ and let $P \in \mathcal{P}_p(H)$ be a maximal L^p-projection for F. Then, for any subspace G of X containing H and any $Q \in \mathcal{P}_p(G)$ such that $Q|_H = P$, we have*

$$Ran(P) = (F \cap Ran(Q)) \text{ and } Ker(P) = (F \cap Ker(Q));$$

(iv) *Let F be a closed subspace of X. Then $\alpha(F) \leq \alpha(X)$.*

Proof. (i) Let G_1 be a vector subspace included in $\mathrm{Ran}(P)^{\perp_p}$, that contains $\mathrm{Ker}(P)$, and that is maximal for the inclusion of sets. As $\mathrm{Ran}(P)^{\perp_p}$ is closed, it contains $\overline{G_1}$, so $\overline{G_1} = G_1$ by maximality and G_1 is closed.
Similarly, let G_2 be a vector subspace included in $G_1^{\perp_p}$, that contains $\mathrm{Ran}(P)$, and that is maximal for the inclusion of sets. As $G_1^{\perp_p}$ is closed, it contains $\overline{G_2}$, so $\overline{G_2} = G_2$ by maximality and G_2 is closed.
Hence, the subspace $G = G_1 \oplus G_2$ admits an L^p-projection Q such that $Q(G_1) = \{0\}$ and $Q(G_2) = G_2$. We also have $F \subset G$ and $Q|_F = P$. The maximality of G_1 and G_2 implies that Q cannot be extended on a larger subspace, so it is a maximal L^p-projection for X.
- (ii) If $x = 0$, then choose $Q = 0$. If not, since $x^{\perp_p} \neq \{0\}$ there is $y \in X$ that is non-zero such that $x \perp_p y$, and the subspace $\mathrm{Span}(x, y)$ admits a L^p-projection P such that $P(x) = x$ and $P(y) = 0$. We then apply item (i) to P in order to get the desired result.
- (iii) As $F \cap \mathrm{Ran}(Q)$ is a closed subspace of F, containing $\mathrm{Ran}(P)$, and contained in $\mathrm{Ker}(P)^{\perp_p} \cap F$, the maximality of P implies that $F \cap \mathrm{Ran}(Q) = \mathrm{Ran}(P)$. A similar reasoning for $\mathrm{Ker}(Q)$ and $\mathrm{Ker}(P)$ gives $F \cap \mathrm{Ker}(Q) = \mathrm{Ker}(P)$.
- (iv) Let P be a maximal L^p-projection for F, defined on the subspace H of F. With item (i) we have a subspace G and Q a maximal L^p-projection for X such that $Q|_H = P$. Item (iii) gives us $\mathrm{Ran}(P) = (F \cap \mathrm{Ran}(Q))$ and $\mathrm{Ker}(P) = (F \cap \mathrm{Ker}(Q))$. Therefore, for P_1, P_2 two different maximal L^p-projections for F, there is no maximal L^p-projection Q for X that extends both P_1 and P_2. Hence, $\alpha(F) \leq \alpha(X)$. □

Remark 4.2.34. The construction in the proof of Lemma 4.2.33 does not give every maximal L^p-projection on a Banach space X. For example let $(\Omega, \mathcal{F}, \mu)$ a measure space that is not σ-finite such that $\Omega = \Omega_1 \sqcup \Omega_2$ with $\Omega_i \in \mathcal{F}$ that are not σ-finite. Then, for any $f \in L^p(\Omega)$ the L^p-projection P that is built with the construction in the proof will be equal to $M_{\chi_{\mathrm{supp}(f)}}$, and such an L^p-projection cannot be equal to the L^p-projection $M_{\chi_{\Omega_1}}$ as $\mathrm{supp}(f)$ is σ-finite whereas Ω_1 is not. The projection $I - P$ is also not equal to $M_{\chi_{\Omega_1}}$ as $\Omega_2 = \Omega_1^C$ is not σ-finite too.

Corollary 4.2.35. *Let X be a Banach space, and $1 \leq p \leq +\infty$. We have the equivalences*

(i) *The p-orthogonality on X is trivial, that is $x \perp_p y \Rightarrow x = 0$ or $y = 0$;*

(ii) *$X_{(p)} = \{0\}$;*

(iii) *$\alpha(X) = 2$ if $X \neq \{0\}$ or $\alpha(X) = 1$ if $X = \{0\}$.*

In such a case, X satisfies Property 4.2.7 for p.

Proof. The equivalence $(i) \Leftrightarrow (ii)$ comes immediately from the definition of the subspace $X_{(p)}$. We can also see that $\alpha(X) = 1$ can only happen when $X = \{0\}$, in which case the equivalences are immediate. Therefore we suppose that $X \neq \{0\}$ for the rest of the proof.
- $(i) \Rightarrow (iii)$ Let P be a maximal L^p-projection for X. Suppose that P is non-trivial. Thus $\mathrm{Ran}(P)$ and $\mathrm{Ker}(P)$ are not reduced to $\{0\}$ and we can find x, y that are non-zero such that $P(x) = x$ and $P(y) = 0$. This implies that $x \perp_p y$, which is not possible. Therefore the maximal L^p-projections for X are only the trivial ones, and $\alpha(X) = 2$.
- $(iii) \Rightarrow (i)$ Let $x, y \in X$ be such that $x \perp_p y$, we can apply item (ii) of Lemma 4.2.33 to obtain an L^p-projection Q that is maximal for X and such that $Q(x) = x$, $Q(y) = 0$. Since $\alpha(X) = 2$ we either have $Q = 0$ or $Q = I$, which is equivalent to $x = 0$ or $y = 0$.
In such a case, every p-orthogonality relationship has the form $0 \perp_p y$ and $x \perp_p 0$, so it is extended by either $P = 0$ or $P = I$. Thus X satisfies Property 4.2.7. Lastly, the subspace $Y = \{0\}$ possesses 0 as its single L^p-projection, so $\alpha(Y) = 1$, which concludes the proof. □

Lemma 4.2.36. *Let X be a Banach space, and $1 \leq p \leq +\infty$. Let F be a subspace of X.*

(i) Let P be a non-trivial L^p-projection that is maximal for F. Then P is maximal for $F_{(p)}$;

(ii) Let Q be a non-trivial L^p-projection that is maximal for $F_{(p)}$. Then Q is maximal for F;

(iii) If $F_{(p)} \neq \{0\}$ or $F = F_{(p)}$, then $\alpha(F) = \alpha(F_{(p)})$.

Proof. (i) Since P is non-trivial, for any element x in $\mathrm{Ran}(P)$ or $\mathrm{Ker}(P)$ we can see that $x^{\perp_p} \cap F \neq \{0\}$. Hence, such a x lies in $F_{(p)}$. Since $F_{(p)}$ is a subspace, $\mathrm{Ran}(P) \oplus_p \mathrm{Ker}(P)$ lies in $F_{(p)}$. As we have $F_{(p)} \subset F$, the maximality of P for F implies that P is maximal for $F_{(p)}$.
- (ii) By applying item (i) of Lemma 4.2.33 to Q and F, we obtain a subspace H and $R \in \mathcal{P}_p(H)$ such that $\mathrm{Ran}(Q) \oplus_p \mathrm{Ker}(Q) \subset H \subset F$, $R|_{\mathrm{Ran}(Q) \oplus \mathrm{Ker}(Q)} = Q$, and R is maximal for F. Item (i) of this Lemma implies that $H \subset F_{(p)}$. Since Q is maximal for $F_{(p)}$, we obtain $H = \mathrm{Ran}(Q) \oplus \mathrm{Ker}(Q)$ and $R = H$. Thus, Q is maximal for F.
- (iii) The previous items show that the set of L^p-projections that are non-trivial and maximal for F is equal to the set of L^p-projections that are non-trivial and maximal for $F_{(p)}$. If $F_{(p)} \neq \{0\}$ or $F = F_{(p)}$ then F and $F_{(p)}$ have the same amount of trivial L^p-projections (either 2 or 1). Thus their sets of maximal L^p-projections are in bijection, so $\alpha(F) = \alpha(F_{(p)})$. □

Lemma 4.2.37 (Maximal L^p-projections on a subspace)**.** *Let X be a Banach space, and $1 \leq p \leq +\infty$. Let F be a closed subspace of X. Let G be a closed subspace of X and $R \in \mathcal{P}_p(G)$. The following are equivalent*

(i) R defines a non-trivial L^p-projection on a subspace H of F;

(ii) $(F \cap R(F)) \neq \{0\}$ and $(F \cap (I - R)(F)) \neq \{0\}$.

If X satisfies Property 4.2.7 for p, the non-trivial L^p-projections on a subspace of F can all be obtained from those on $G = X$.

Proof. $(i) \Rightarrow (ii)$ Since $R|_H$ is a projection, we have $R(H) \subset H$. As $R(H), (I - R)(H)$ are included in H and H is included in F, we have $R(H) \subset (R(F) \cap F)$ and $(I - R)(H) \subset ((I - R)(F) \cap F)$. Since $R|_H$ is non-trivial, both of these subspaces are not reduced to $\{0\}$.
- $(ii) \Rightarrow (i)$ Conversely, take $H = (F \cap R(F)) \oplus_p (F \cap (I - R)(F))$. We have $H \subset F$ and $R(H) \subset H$, so $R|_H$ is an L^p-projection. It is also non-trivial since $R(H) \neq \{0\}$ and $(I - R)(H) \neq \{0\}$.

When X satisfies Property 4.2.7 for p, every L^p-projection on a subspace G extends to an L^p-projection on X, so looking at every $R \in \mathcal{P}_p(X)$ is enough. $\square$

Remark 4.2.38. Every maximal L^p-projection for F can be obtained this way. However, even if $(F \cap R(F)) \neq \{0\}$, $(F \cap (I-R)(F)) \neq \{0\}$ and $H = (F \cap R(F)) \oplus_p (F \cap (I-R)(F))$, the projection $P|_H$ may not be a maximal L^p-projection for F. For example, take $X = \ell^p(\mathbb{C}^4)$, $f_1 = (0,0,0,1)$, $f_2 = (0,-1,1,0)$, $f_3 = (1,1,1,0)$ and $F = \mathrm{Span}(f_1, f_2, f_3)$. For $P = M_{\chi_{\{2,3\}}}$ we can see that $H = \mathrm{Span}(f_2) \oplus_p \mathrm{Span}(f_1)$, but $F = \mathrm{Span}(f_2, f_3) \oplus_p \mathrm{Span}(f_1)$ so $P|_H$ is not a maximal L^p-projection for F.

Lemma 4.2.39. *Let X be a Banach space, and $1 \leq p \leq +\infty$, $p \neq 2$. Suppose that X satisfies Property 4.2.7 for p. Let F be a closed subspace of X. Let $x, y \in F$ with $x \perp_p y$. Let $P \in \mathcal{P}_p(F)$. Then, we have $P(x) \perp_p P(y)$.*

Proof. Since X satisfies Property 4.2.7 for p, there exist $Q, R \in \mathcal{P}_p(X)$ such that $Q|_F = P$ and $R(x) = x$, $R(y) = 0$. As $p \neq 2$, the projections Q and R commute. Hence, we have

$$P(x) = Q(x) = Q(R(x)) = R(Q(x)) = R(P(x)).$$

Thus, $P(x) \in \mathrm{Ran}(R)$, so $P(x) \perp_p y$. Since Property 4.2.7 implies the linearity of the p-orthogonality on X according to Proposition 4.2.9 and since $P(x) \perp_p (I-P)(y)$, we end up with $P(x) \perp_p y - (I-P)(y) = P(y)$. $\square$

Remark 4.2.40. Is Lemma 4.2.39 always true if we only suppose that F satisfies Property 4.2.8 for p, or for any F ? This question is analogous to the one in Remark 4.2.19, albeit in a more general context.

Corollary 4.2.41. *Let X be a Banach space, and $1 \leq p \leq +\infty$, $p \neq 2$. Suppose that X satisfies Property 4.2.7 for p. Let F be a closed subspace of X. Suppose that $F = F_1 \oplus_p \ldots \oplus_p F_k$. Let $x = x_1 + \ldots + x_k \in F$. Then, we have*

$$x^{\perp_p} = \{y_1 + .. + y_k \,:\, y_i \in x_i^{\perp_p} \cap F_i\} = \sum_{j=1}^{k} x_j^{\perp_p} \cap F_j.$$

Proof. Let $1 \leq i \leq k$, and $y_i \in x_i^{\perp_p} \cap F_i$. Then, for any $1 \leq j \leq k$ we have $y_i \perp_p x_j$. Hence, we have $y_i \perp_p x_1 + .. + x_k = x$. Thus, we have $x \perp_p y_1 + .. + y_k$. For the converse, if we take $y \in F$ such that $y \perp_p x$, then by denoting P_i the projection on F_i parallel to $\oplus_{j \neq i} F_j$, P_i is an L^p-projection on F and Lemma 4.2.39 tells us that $y_i = P_i(y) \perp_p P_i(x) = x_i$. Hence, y_i lies in $x_i^{\perp_p} \cap F_i$. $\square$

Corollary 4.2.42 (Maximal L^p-projections for p-orthogonal sums)**.** *Let X be a Banach space, and $1 \leq p \leq +\infty$, $p \neq 2$. Suppose that X satisfies Property 4.2.7 for p. Let F be a closed subspace of X. Suppose that $F = F_1 \oplus_p \ldots \oplus_p F_k$. Denote P_i the L^p-projection on F_i parallel to $\oplus_{j \neq i} F_j$. Let $G \subset F$ and $Q \in \mathcal{P}_p(G)$. Denote $G_i = P_i(G)$ and $Q_i = P_i Q : G \to G_i$. Then*

- *(i) If Q is a maximal L^p-projection for F then $G = G_1 \oplus_p .. \oplus_p G_k$, $Q = Q_1 + .. + Q_k$, and all Q_i are maximal L^p-projections for F_i;*
- *(ii) If $G = G_1 \oplus_p .. \oplus_p G_k$ and all $Q_i|_{G_i}$ are maximal L^p-projections for F_i, then Q is a maximal L^p-projection for F;*

(iii) $\alpha(F) = \Pi_{i=1}^k \alpha(F_i)$.

Proof. (i) By construction we have $G_i \subset F_i$ and $I_F = P_1 + .. + P_k$. Suppose first that Q is maximal for F. Let $1 \leq i \leq k$. Since X satisfies Property 4.2.7, there are $\widetilde{Q}, \widetilde{P_i} \in \mathcal{P}_p(X)$ such that $\widetilde{Q}|_G = Q$ and $\widetilde{P_i}|_F = P_i$. Let $y \in P_i(G)$. There is $z \in G$ such that $y = P_i(z)$. We then have

$$\widetilde{Q}(y) = \widetilde{Q}(P_i(z)) = \widetilde{Q}(\widetilde{P_i}(z)) = \widetilde{P_i}(\widetilde{Q}(z)) = P_i(Q(z)) \in P_i(G).$$

Denote $G' = P_1(G) \oplus_p .. \oplus_p P_k(G)$. Thus $G \subset G' \subset F$ and $\widetilde{Q}$ leaves G' invariant, so $\widetilde{Q}|_{G'} \in \mathcal{P}_p(G')$. As we also have $\widetilde{Q}|_G = Q$, the maximality of Q for F implies that $G = G'$. Hence, $G = G_1 \oplus_p .. \oplus_p G_k$ and $Q = Q_1 + ... + Q_k$.
Suppose now that there is a i such that Q_i is not maximal for F_i. Up to reordering, we can choose $i = 1$. With item (i) of Lemma 4.2.33 we have a subspace H_1 such that $G_1 \subsetneq H_1 \subset F_1$, and $R_1 \in \mathcal{P}_p(H_1)$ such that $R_1|_{G_1} = Q_1$. Thus, if we now consider $H = H_1 \oplus_p G_2 \oplus_p .. \oplus_p G_k$ and define $R \in \mathcal{L}(H)$ with $R(x_1 + .. + x_k) = R_1(x_1) + \sum_{j=2}^k Q_j(x_j)$, we can see that H strictly contains G, that R is an L^p-projection on H, and that $R|_G = Q$, which contradicts the maximality of Q for F.
- (ii) Suppose that Q is not maximal for F. Then, according to item (i) of Lemma 4.2.33 there exists a subset H such that $G \subsetneq H \subset F$, and $R \in \mathcal{P}_p(H)$ such that $R|_G = Q$ and that R is maximal for F. Item (i) of this Corollary implies that $H = P_1(H) \oplus_p .. \oplus_p P_k(H)$. As H strictly contains G, this decomposition implies that there must be at least one i such that $P_i(H)$ strictly contains $P_i(G)$. We can then see that $P_i(G) \subsetneq P_i(H) \subset F_i$, that $R|_{P_i(H)}$ is an L^p-projection on $P_i(H)$ and that $(R|_{P_i(H)})|_{P_i(G)} = Q|_{P_i(G)} = Q_i|_{G_i}$. This contradicts the maximality of Q_i, so Q is maximal for F.
- (iii) Items (i) and (ii) give a bijective correspondence between L^p-projections Q that are maximal for F and k-uplets $(Q_1, ..., Q_k)$ of L^p-projections such that Q_i is maximal for F_i. Therefore, we get $\alpha(F) = \Pi_{i=1}^k \alpha(F_i)$. □

Remark 4.2.43. Producing examples using subspaces of an L^p-space as above means that the statements of Lemma 4.2.39 and Corollary 4.2.41, will always be valid. Hence, these subspaces seem to only be usable as good models for the behaviour of L^p-projections on subspaces F for a space X satisfying Property 4.2.7, whereas L^p-projections on subspaces F for a general Banach space X (like ones that would only satisfy Property 4.2.8) could exhibit different behaviours due to the additional freedom regarding the structure of the norm on X.

For the following Propositions, we recall that for a subspace F, $F_{(p)}$ denotes the subspace of F spanned by every vector x whose p-orthogonal $x^{\perp_p}$ is not reduced to $\{0\}$.

Proposition 4.2.44. *Let $1 \leq p < +\infty$, $p \neq 2$, $n \geq 4$ and $X = \ell^p(\mathbb{C}^n)$. Denote $(e_i)_i$ the canonical basis of X. For $1 \leq i < n$, denote $f_i = e_i + e_{i+1}$. Take $F = Span(f_i, 1 \leq i < n)$. Then, we have $F = F_{(p)}$, $dim(F) = n - 1$, and*

$$\alpha(F) = Card(\{P\colon P \text{ is a maximal } L^p\text{-projection for } F\}) = 2^n - 2n.$$

Thus, for $n \geq 5$, there is strictly more than $2^{dim(F)}$ L^p-projections that are maximal for F.

Proof. By construction, we have $dim(F) \leq n - 1$. We can see that $F + \mathbb{C}.e_1$ contains every e_i, so $F + \mathbb{C}e_1 = X$. Hence $dim(F) = n - 1$ and $e_1 \notin F$. Similarly, we can see that $e_i \notin F$ for every $1 \leq i \leq n$. For every $1 \leq i < j \leq n$, denote $g_{i,j} = e_i + (-1)^{j-i} e_j$. A computation gives

$g_{i,j} = \sum_{k=0}^{j-i-1}(-1)^k f_{i+k}$, thus $g_{i,j} \in F$.
As $n \geq 4$, for any $1 \leq i < n$ we have k, l such that $k < l$ and $k, l, i, i+1$ are all distinct. Thus, $f_i \perp_p g_{k,l}$. Since both elements are non-zero and in F, we deduce that $f_i \in F_{(p)}$ by definition of $F_{(p)}$. As $F_{(p)}$ is a subspace of F, we get $F_{(p)} = F$.
Let $P \in \mathcal{P}_p(X)$, with $P \neq 0, I$. We then have $A \subset \{1, .., n\}$ such that $P = M_{\chi_A}$. Write A as $A = \{m_1, ..., m_r\}$ with $1 \leq r \leq n-1$ and $m_1 < m_2 < .. < m_r$. We will prove that $dim(F \cap P(F)) = r - 1$ and

$$F \cap P(F) = F \cap \text{Ran}(P) = \{0\} \text{ if } r = 1,$$
$$F \cap P(F) = F \cap \text{Ran}(P) = \text{Span}(g_{m_1,m_2}, ..., g_{m_{r-1},m_r}) \text{ if } 2 \leq r \leq n.$$

We recall that $F \cap P(F) = F \cap \text{Ran}(P)$ since P is a projection. If $r = 1$ we have $\text{Ran}(P) = \text{Span}(e_{m_1})$, so $F \cap \text{Ran}(P) = \{0\}$ as $e_{m_1} \notin F$.
Suppose now that $r \geq 2$. We have $\text{Ran}(P) = Span(e_{m_1}, ..., e_{m_r})$, so we obtain

$$\text{Span}(g_{m_1,m_2}, ..., g_{m_{r-1},m_r}) \subset F \cap \text{Ran}(P).$$

We also see that $F \neq \text{Ran}(P)$ as every e_{m_i} does not belong to F, thus

$$dim(F \cap \text{Ran}(P)) \leq dim(\text{Ran}(P)) - 1 = r - 1.$$

We can see that $\text{Span}(g_{m_1,m_2}, ..., g_{m_{r-1},m_r}) + \mathbb{C}e_{m_1}$ contains every e_{m_i}, so this subspace is equal to $\text{Ran}(P)$ and $dim(\text{Span}(g_{m_1,m_2}, ..., g_{m_{r-1},m_r})) \geq r - 1$. Therefore, we have

$$F \cap \text{Ran}(P) = \text{Span}(g_{m_1,m_2}, ..., g_{m_{r-1},m_r}) \text{ and } dim(F \cap \text{Ran}(P)) = r - 1.$$

Using this result, we obtain

$$dim((F \cap P(F)) \oplus (F \cap (I-P)(F))) = (r-1) + ((n-r)-1) = n - 2.$$

Since X satisfies Property 4.2.7 for p, for any non-trivial L^p-projection Q on F there exists a non-trivial projection $R \in \mathcal{P}_p(X)$ that extends Q. This implies that $F = (F \cap P(F)) \oplus (F \cap (I-P)(F))$ with $dim(F) = n-1$, which contradicts the previous result. Therefore, $\mathcal{P}_p(F) = \{0, I\}$.
We now consider $P \in \mathcal{P}_p(X)$ that is non-trivial, with $P = M_{\chi_A}$, $A = \{m_1, ..., m_r\}$. If $r = 1$ or $r = n-1$ then we either have $P \cap P(F) = \{0\}$, or $P \cap (I-P)(F) = \{0\}$.
If not, with $2 \leq r \leq n-2$ we have $P \cap P(F) \neq \{0\}$ and $P \cap (I-P)(F) \neq \{0\}$, so P defines a non-trivial L^p-projection on the subspace

$$F_P := (F \cap P(F)) \oplus_p (F \cap (I-P)(F)).$$

As $dim(F_P) = n - 2 = dim(F) - 1$, and as F does not possess any non-trivial L^p-projection, we can see that $P|_{F_P}$ is a maximal L^p-projection for F.
Furthermore, all these L^p-projections $P|_{F_P}$ are different. Indeed, since $g_{i,j}$ lies in $\text{Ran}(M_{\chi_B})$ if and only if $i, j \in B$, we can see that the only L^p-projection R on X such that $F \cap R(F) = \text{Span}(g_{m_1,m_2}, ..., g_{m_{r-1},m_r})$ is $R = M_{\chi_B} = P$ with $B = \{m_1, ..., m_r\}$.
By using Lemma 4.2.37 and Remark 4.2.38 we can conclude that there are as many maximal L^p-projections that are non-trivial for F as there are subsets A of $\{1, .., n\}$ with $A = \{m_1, ..., m_r\}$ and $2 \leq r \leq n-2$. This quantity is then equal to

$$2^n - \binom{0}{n} - \binom{n}{n} - \binom{1}{n} - \binom{n-1}{n} = 2^n - 2 - 2n.$$

If we add 0 and I, we conclude that there are $2^n - 2n$ maximal L^p-projections for F. When $n = 4$ we have $2^n - 2n = 8 = 2^{n-1}$ but when $n \geq 5$ we have $2n < 2^{n-1}$ so $2^n - 2n > 2^{n-1}$. □

Remark 4.2.45. We can generalize the result of Proposition 4.2.44 by replacing $\ell^p(\mathbb{C}^n) = \text{Span}(e_1, ..., e_n)$ with $\text{Span}(h_1, ..., h_n)$, with $h_1, ..., h_n$ vectors of a Banach space Y, such that $h_i \perp_p h_j$ for every $i \neq j$. As we can build an isometric isomorphism between this space and $\ell^p(\mathbb{C}^n)$ it satisfies Property 4.2.7 for p and. And as the proof of Proposition 4.2.44 only used the p-orthogonality of the family $\{e_1, ..., e_n\}$, it can be mimicked for $\{h_1, ..., h_n\}$ to get the desired result.

Proposition 4.2.46. *Let X be a Banach space, and $1 \leq p < +\infty$, $p \neq 2$. Suppose that X satisfies Property 4.2.7. Let F be a subspace of X with $dim(F_{(p)}) \leq 3$. Then, we have $\alpha(F) \leq 2^{\max(dim(F_{(p)}),1)}$.*

Proof. We recall that item (iii) of Lemma 4.2.36 gives $\alpha(F) = \alpha(F_{(p)})$ whenever $F_{(p)} \neq \{0\}$ and that Corollary 4.2.35 gives $\alpha(F) = 2$ or 1 when $F_{(p)} = \{0\}$. Therefore we will focus on $F_{(p)}$ and discuss cases depending on $dim(F_{(p)})$.
If $F_{(p)} = \{0\}$ we have $\alpha(F) \leq 2 = 2^{\max(0,1)}$.
If $F_{(p)} \neq \{0\}$ then we have $x \in F$ be a vector which is non-zero and such that $x^{\perp_p} \cap F \neq \{0\}$. Let $y \in F$ that is non-zero and such that $y \perp_p x$. Then, $y^{\perp_p} \cap F \neq \{0\}$. Therefore both x and y lie in $F_{(p)}$. As y cannot be colinear to x, we cannot have $dim(F_{(p)}) = 1$.
If $dim(F_{(p)}) = 2$, the vectors x and y chosen previously span the subspace $F_{(p)}$. Hence $F_{(p)} = \text{Span}(x, y) = \text{Span}(x) \oplus_p \text{Span}(y)$ and $F_{(p)}$ is isometrically isomorphic to $\ell^p(\mathbb{C}^2)$. Item (iii) of Corollary 4.2.42 then implies

$$\alpha(F_{(p)}) = \alpha(\text{Span}(x))\alpha(\text{Span}(y)) = 2.2 = 2^{\max(dim(F_{(p)}),1)}.$$

If $dim(F_{(p)}) = 3$, the vectors x and y chosen previously only form a linearly independant family in $F_{(p)}$. By definition of $F_{(p)}$, there exists a vector $z \in F$ such that $z^{\perp_p} \cap F \neq \{0\}$ and such that $z \notin \text{Span}(x, y)$. Thus, $\{x, y, z\}$ is a basis for $F_{(p)}$. We will discuss cases depending on the vectors in $z^{\perp_p} \cap F$. For this, take $P \in \mathcal{P}_p(X)$ an L^p-projection that extends the p-orthogonality between x and y, with $P(x) = x$ and $P(y) = 0$.
If $z \perp_p ax$, $a \neq 0$, we then have $F_{(p)} = \text{Span}(x) \oplus_p \text{Span}(y, z)$. Item (iii) of Corollary 4.2.42 then implies

$$\alpha(F_{(p)}) = \alpha(\text{Span}(x))\alpha(\text{Span}(y, z)) \leq 2.2^{max(2,1)} \leq 2^3 = 2^{max(dim(F_{(p)}),1)}.$$

If $z \perp_p by$, $b \neq 0$ we get the same upper bound by symmetry between x and y.
If $z \perp_p ax + by$, $a, b \neq 0$, then Lemma 4.2.39 implies that $P(z) \perp_p aP(x)$. Since $a \neq 0$ and since the p-orthogonality is linear on X, we obtain

$$x = P(x) \perp_p P(z) + (I - P)(z) = z,$$

which brings us to the case $z \perp_p x$ and gives us the desired result.
We are then left with the case where $z^{\perp_p} \cap \text{Span}(x, y) = \{0\}$. Thus, an element in F that is non-zero and p-orthogonal to z has the form $w = ax + by + cz$, with $c \neq 0$. Up to dividing by c we can suppose that $c = 1$. Lemma 4.2.39 then gives

$$P(z) \perp_p P(w) = ax + P(z) \text{ and } (I - P)(z) \perp_p (I - P)(w) = by + (I - P)(z).$$

If we have $a = 0$ we get $P(z) = 0$, so $z = (I - P)(z) \perp_p x$, which is impossible. Similarly, if $b = 0$ we get $(I - P)(z) = 0$, so $z = P(z) \perp_p y$ which is impossible. As $a, b \neq 0$, up to replacing x by

ax and y by by we can suppose that $a = b = 1$, so $w = x + y + z$. Take

$$G = \text{Span}(P(z), P(z) + x, (I - P)(z), (I - P)(z) + y).$$

As in the previous use of Lemma 4.2.39, these vectors form a p-orthogonal family. Denote

$$h_1 = P(z) + x,\ h_2 = -P(z),\ h_3 = -(I - P)(z),\ h_4 = (I - P)(z) + y.$$

We then have $G = \text{Span}(h_1, h_2, h_3, h_4)$ and $x = h_1 + h_2$,$y = h_4 + h_3$ and $z = -(h_2 + h_3)$, so G contains $F_{(p)} = \text{Span}(x, y, z)$ and $F_{(p)} = \text{Span}(h_1 + h_2, h_3 + h_4, h_2 + h_3)$. We can then use Remark 4.2.45 to apply Proposition 4.2.44 to G and $F_{(p)}$ and get

$$\alpha(F_{(p)}) = 2^4 - 2.4 = 8 = 2^{\max(dim(F_{(p)}),1)},$$

which concludes the proof. □

Proposition 4.2.47. *Let X be a Banach space, and $1 \le p \le +\infty$, $p \ne 2$. Suppose that X satisfies Property 4.2.7 for p. Let F be a subspace of X of finite dimension. Then, there exists a subspace H of X such that $F \subset H$, $dim(H) \le 2^{\max(dim(F)-1,0)}$, and H satisfies Property 4.2.7 for p.*

Proof. We will prove the result with an induction on $n = dim(F)$.
If $n = 1$ then for any $x, y \in F$ such that $x \perp_p y$, we must have $x = 0$ or $y = 0$ as x and y are colinear. Thus F satisfies Property 4.2.7 for p and we can take $H = F$.
Let $n \ge 2$. Suppose that the result is true for any subspace of X of dimension less or equal to $n - 1$. If all the maximal L^p-projections for F are defined on F, then Corollary 4.2.35 implies that F satisfies Property 4.2.7 for p and we can take $H = F$.
If not, let P be a maximal L^p-projection for F that is not defined on F. Such a projection P is defined on $\text{Ran}(P) \oplus \text{Ker}(P) \subsetneq F$. Since X satisfies Property 4.2.7, there exists $Q \in \mathcal{P}_p(X)$ that extends P. Denote $G = Q(F) \oplus (I - Q)(F)$. Item (i) of Lemma 4.2.37 gives us

$$\text{Ker}(P) = F \cap \text{Ker}(Q) \text{ and } \text{Ran}(P) = F \cap \text{Ran}(Q).$$

By using the rank-nullity Theorem for linear maps $Q : F \to Q(F)$ and $(I - Q) : F \to (I - Q)(F)$ we get

$$\begin{aligned} dim(Q(F)) &= dim(F) - dim(F \cap \text{Ker}(Q)) = dim(F) - dim(\text{Ker}(P)) \\ dim((I - Q)(F)) &= dim(F) - dim(F \cap \text{Ran}(Q)) = dim(F) - dim(\text{Ran}(P)). \end{aligned}$$

As P is non-trivial we have $\text{Ker}(P), \text{Ran}(P) \ne \{0\}$, so $dim(Q(F)), dim((I - Q)(F)) \le n - 1$. We can then apply the induction hypothesis to $Q(F)$ and $(I - Q)(F)$ to obtain subspaces H_1, H_2 such that $Q(F) \subset H_1$, $(I - Q)(F) \subset H_2$ and H_1, H_2 satisfy Property 4.2.7. Denote $H = H_1 \oplus_p H_2$. We then have $F \subset Q(F) \oplus (I - Q)(F) \subset H$, so we need to show that H satisfies Property 4.2.7. Let $x, y \in H$ be such that $x \perp_p y$. Lemma 4.2.39 tells us that

$$Q(x) \perp_p Q(y) \text{ and } (I - Q)(x) \perp_p (I - Q)(y).$$

Since $Q(x), Q(y)$ are in H_1, there is $P_1 \in \mathcal{P}_p(H_1)$ such that

$$P_1(Q(x)) = Q(x) \text{ and } P_1(Q(y)) = 0.$$

Since $(I-Q)(x), (I-Q)(y)$ are in H_2, there is $P_2 \in \mathcal{P}_p(H_2)$ such that

$$P_2((I-Q)(x)) = (I-Q)(x) \text{ and } P_1((I-Q)(y)) = 0.$$

Proposition 4.2.17 gives us $R \in \mathcal{P}_p(H)$ such that $R(h_1+h_2) = P_1(h_1) + P_2(h_2)$ for any $h_1 \in H_1$, $h_2 \in H_2$. Therefore, we have

$$\begin{aligned} R(x) &= R(Q(x) + (I-Q)(x)) = Q(x) + (I-Q)(x) = x, \\ R(y) &= R(Q(y) + (I-Q)(y)) = 0 + 0 = 0, \end{aligned}$$

and H satisfies Property 4.2.7 for p. We also have

$$\begin{aligned} dim(H) = dim(H_1) + dim(H_2) &\leq 2^{\max(dim(Q(F)-1,0)} + 2^{\max(dim((I-Q)(F))-1,0)} \\ &\leq 2^{dim(Q(F))-1} + 2^{dim((I-Q)(F))-1} \leq 2^{dim(F)-1-1} + 2^{dim(F)-1-1} = 2^{dim(F)-1}, \end{aligned}$$

which proves the upper bound on $dim(H)$ and concludes the proof. □

Remark 4.2.48. Proposition 4.2.47 gives us a first upper bound for $\alpha(F)$, that is $\alpha(F) \leq 2^{2^{\max(dim(F)-1,0)}}$. But its main interest is to allow us to look at F as a subspace of G, with G satisfying Property 4.2.7 and of finite dimension. Therefore we can apply Proposition 4.2.18 and Corollary 4.2.42 to write G as $G = G_1 \oplus_p .. \oplus_p G_m$, with $\alpha(G_i) = 2$ and $\alpha(G) = Card(\mathcal{P}_p(G)) = 2^m$. Then, F as a subspace of G is generated by elements of the form $f = f_{i_1} + .. + f_{i_r}$ with $1 \leq r \leq n$, $1 \leq i_1 < .. < i_r \leq n$, and $f_{i_k} \in G_{i_k}$. This is similar to Propositions 4.2.44 and 4.2.46, where we can count the maximal L^p-projections for F by looking at the behaviour of all L^p-projections of G with respect to F.

The previous results and examples motivate the following conjecture.

Conjecture 4.2.49. Let X be a Banach space, and $1 \leq p < +\infty$, $p \neq 2$. Suppose that X satisfies Property 4.2.7. Let F be a subspace of X of finite dimension such that $\mathcal{P}_p(F_{(p)}) = \{0, I\}$. Then, we have

$$\alpha(F) = Card(\{P\colon P \text{ is a maximal } L^p\text{-projection for } F\}) \leq 2^{dim(F_{(p)})+1}.$$

If this conjecture is true, we can then obtain a better upper bound for $\alpha(F)$ when F has finite dimension and F is a subspace of a Banach space X that satisfies Property 4.2.7. We can also treat the case $p = +\infty$, which gives a simpler result.

Proposition 4.2.50. *Let X be a Banach space. Let F be a subspace of X of finite dimension.*

(i) Let $1 \leq p < +\infty$, $p \neq 2$. Suppose that X satisfies Property 4.2.7 for p and that Conjecture 4.2.49 is true. For $Card(\mathcal{P}_p(F_{(p)})) = 2^m$, we then have

$$\alpha(F) \leq 2^{\min(\frac{5}{4}dim(F_{(p)}), dim(F_{(p)})+m)}.$$

(ii) Let $p = +\infty$. Suppose that X satisfies Property 4.2.7 for p. For $X = X_1 \oplus_\infty X_2$, we then have one of the following

$$\begin{aligned} &\alpha(F) = 1 \text{ if } F = \{0\}; \\ &\alpha(F) = 2 \text{ if } F = \{0\} \text{ and } (F \cap X_1 = \{0\} \text{ or } F \cap X_2 = \{0\}); \\ &\alpha(F) = 4 \text{ if } F \cap X_1 \neq \{0\} \text{ and } F \cap X_2 \neq \{0\}. \end{aligned}$$

Proof. (i) We have $\alpha(F) = \alpha(F_{(p)})$, so we only need to prove the result when $F = F_{(p)}$. Since F has a finite dimension, $Card(\mathcal{P}_p(F))$ is finite and is a power of 2 according to Proposition 4.2.18. Furthermore, for m such that $Card(\mathcal{P}_p(F_{(p)})) = 2^m$, we then have subspaces $F_1, ..., F_m$ of F such that

$$F = F_1 \oplus_p .. \oplus_p F_m,\ F_i \neq \{0\},\ \text{and}\ \mathcal{P}_p(F_i) = \{0, I\}.$$

Furthermore, Corollary 4.2.42 tells us that

$$\alpha(F) = \Pi_{i=1}^m \alpha(F_i).$$

Now, if $1 \leq dim(F_i) \leq 3$, we can apply the result of Proposition 4.2.46 to get

$$\alpha(F_i) \leq 2^{\max(dim((F_i)_p),1)} \leq 2^{\max(dim(F_i),1)} = 2^{dim(F_i)}.$$

If not, with $dim(F_i) \geq 4$, Conjecture 4.2.49 can be applied to F_i to obtain

$$\alpha(F_i) \leq 2^{dim((F_i)_p)+1} \leq 2^{dim(F_i)+1}.$$

Denote

$$E = \{i \in \{1, .., m\} \colon\ dim(F_i) \geq 4\}.$$

Since $dim(F_{(p)}) = dim(F) = \sum_{i=1}^m dim(F_i)$, we have $Card(E) \leq \frac{dim(F_{(p)})}{4}$, so

$$Card(E) \leq \min(\frac{dim(F_{(p)})}{4}, m).$$

Combining the previous results gives

$$\begin{aligned}\alpha(F) &= (\Pi_{i\in E}\alpha(F_i))(\Pi_{j\notin E}\alpha(F_j)) \leq (\Pi_{i\in E}2^{dim(F_i)+1})(\Pi_{j\notin E}2^{dim(F_i)}) \\ &\leq 2^{Card(E)+\sum_{k=1}^m dim(F_k)} = 2^{Card(E)+dim(F_{(p)})} \\ &\leq 2^{dim(F_{(p)})+\min(\frac{dim(F_{(p)})}{4},m)} = 2^{\min(\frac{5}{4}dim(F_{(p)}),dim(F_{(p)})+m)},\end{aligned}$$

which concludes the proof.
- (ii) We apply Corollary 4.2.27 to obtain that $\alpha(X) = \mathcal{P}_p(X) = 1$,2 or 4. Since $\alpha(F) \leq \alpha(X) \leq 4$, we have $\alpha(F) = 1$,2 or 4. Proposition 4.2.29 allows us to write X as $X = X_1 \oplus_\infty X_2$, with $\alpha(X_i) = 1$ if $X_i = \{0\}$ or $\alpha(X_i) = 2$.
If $F = \{0\}$, then $\alpha(F) = 1$. If $F \cap X_1 \neq \{0\}$ and $F \cap X_2 \neq \{0\}$, then $F \neq \{0\}$ and F has non-zero elements that are ∞-orthogonal, so it possesses non-trivial maximal L^p-projections. Therefore $\alpha(F) = 4$. If $F \neq \{0\}$ and $F \cap X_1 = \{0\}$ or $F \cap X_2 = \{0\}$, then Proposition 4.2.29 tells us that the ∞-orthogonality is trivial on F, thus $\alpha(F) = 2$. □

4.3 L^p-Projections on Quotient Spaces and Subspaces of Quotients

The aim of this section is to link p-orthogonality and L^p-projections between a Banach space X and quotient spaces X/F. We will introduce a third property for X regarding p-orthogonality between X/F and X. All the results will be valid for $1 < p < +\infty$, $p \neq 2$. Unlike Section 4.2, the case $p = 1$ does not behave well with respect to quotients (see Counter-example 4.3.7). The initial results are also true for $p = 2$ (up to Lemma 4.3.12), but the Hilbertian case is excluded as soon as properties of the Boolean algebra $\mathcal{P}_p(X)$ are required. We quickly drop the case $p = +\infty$ as the properties that we can add to X, F or $P(F)$ give very specific behaviours that require special care (in a similar way to Proposition 4.2.29 or item (ii) of Proposition 4.2.50).

4.3.A L^p-projections on quotient spaces

Definition 4.3.1 (Representative of minimal norm of a quotient, metric projection on a subspace)**.** Let $1 < p < +\infty$, $p \neq 2$. Let F be a closed subspace of X such that every element of X/F admits a unique representative of minimal norm. For $\overline{x} \in X/F$ we define $\mathrm{Rep}_F(\overline{x}) \in X$ the representative of $\overline{x}$ of minimal norm. For $x \in X$ we denote $\mathrm{Proj}(x, F) \in F$ the metric projection of x onto the closed convex set F.
We then have $\mathrm{Rep}_F(\overline{x}) = x - \mathrm{Proj}(x, F)$ and

$$\|\mathrm{Rep}_F(\overline{x})\| = \|\overline{x}\| = \inf_{a \in F}(\|x - a\|) = \|x - \mathrm{Proj}(x, F)\|.$$

Also, $\mathrm{Rep}_F(X/F) = \{x \in X\colon\ \mathrm{Proj}(x, F) = 0\}$.
If there is no ambiguity regarding the quotient space, the map $\mathrm{Rep}_F : X/F \to X$ will be abbreviated as Rep in the rest of the chapter.

Remark 4.3.2. While some results can be applied to subspaces F of a Banach space X with no condition on the quotient X/F, most of them will require the unicity of any representative of minimal norm of $\overline{x} \in X/F$, or equivalently the existence and unicity of a metric projection of every $x \in X$ onto F. Even though L^p-projections give decompositions into p-orthogonal subspaces, it is not true that for any subspace F and any L^p-projection P, the existence and uniqueness of the metric projection on F implies the existence and uniqueness of the metric projection on $P(F)$. It is however true with additional conditions between P and F (see Lemma 4.3.12).
We also remark that in general the set $\mathrm{Rep}_F(X/F)$ is not a subspace of X. Further results will show that in certain conditions the set $\mathrm{Rep}_F(X/F)$ possesses two subsets A, B such that $A + B = \mathrm{Rep}_F(X/F)$ and $B \subset A^{\perp_p}$. The p-orthogonality between A and B ensures that every element of $\mathrm{Rep}_F(X/F)$ has a unique decomposition as a sum $A+B$. Hence, such a decomposition will be denoted as $\mathrm{Rep}_F(X/F) = A \oplus_p B$, even though the set A and B are not subspaces, in order to match the p-orthogonal decompositions that happen on Banach spaces.

Lemma 4.3.3. *Let $1 < p \leq +\infty$. Let X be a Banach space and let F be a closed subspace of X. Let $x \in X$, let G be a subspace of X containing F and x, and let $P \in \mathcal{P}_p(G)$ be such that $P(x) = x$. The following are equivalent*

(i) $\inf_{a \in F} \|x - a\| = \|x\|$;

(ii) $\inf_{a \in F} \|x - P(a)\| = \|x\|$.

If the metric projections on F and $P(F)$ are well-defined, then we also have the equivalence :

(1) *Proj*$(x, F) = 0$;

(2) *Proj*$(x, P(F)) = 0$.

Proof.
- $(ii) \Rightarrow (i)$. Suppose first that $p < +\infty$. For any $a \in F$, we have

$$\|x - a\|^p = \|x - P(a)\|^p + \|(I - P)(a)\|^p \geq \|x\|^p + \|(I - P)(a)\|^p \geq \|x\|^p.$$

Thus, $\|x - 0\|^p = \inf_{a \in F}(\|x - a\|^p)$. When $p = +\infty$, we similarly have

$$\|x - a\| = \max(\|x - P(a)\|, \|(I - P)(a)\|) \geq \|x - P(a)\| \geq \|x\|,$$

so we get the same conclusion.
- $(i) \Rightarrow (ii)$. Suppose first that $p < +\infty$, and that $\inf_{a\in F} \|x - P(a)\| < \|x\|$. Hence, we have $a \in F$ non zero such that

$$\|x - P(a)\|^p < \|x\|^p.$$

We will show by a convexity argument that there exists $0 < \lambda < 1$ such that $\|x - \lambda.a\|^p < \|x\|^p$. In order to do this, we define the following maps

$$g(\lambda) := \|x - \lambda P(a)\|^p \text{ and } h(\lambda) := \|x - \lambda a\|^p = g(\lambda) + |\lambda|^p.\|(I - P)(a)\|^p$$

Now, for every $b, c \in X$, the map $r \in \mathbb{R} \mapsto \|b + cr\| \in [0, +\infty[$ is convex. The map $y \in [0, +\infty[\mapsto y^p \in \mathbb{R}$ is convex with a positive derivative on $[0, +\infty[$. Thus, the composed map $r \mapsto \|b + cr\|^p$ is convex on $\mathbb{R}$, so g and h are convex on $\mathbb{R}$. It follows that these maps have right derivatives at every point, denoted by g'_r, h'_r.
As we have

$$g(1) = \|x - P(a)\|^p < \|x\|^p = g(0),$$

we must have $g'_r(0) < 0$. Also, since $p > 1$, the map $r \mapsto |r|^p$ has a derivative of 0 at 0. Thus $h_r(0)' = g_r(0)' + 0 < 0$, which means that h is decreasing on a neighbourhood of 0. This gives a $0 < \lambda < 1$ such that

$$\|x - \lambda.a\|^p = h(\lambda) < h(0) = \|x\|^p,$$

which proves the equivalence in this case.
When $p = +\infty$, if $\inf_{a\in F} \|x - P(a)\| < \|x\|$ we then have $a \in F$ non zero such that

$$\|x - P(a)\| < \|x\|.$$

Since the map $s : r \in \mathbb{R} \mapsto \|x - rP(a)\|$ is convex with $s(0) > s(1)$, s is decreasing on an interval $[0, t]$, for some $0 < t \leq 1$. Let $r > 0$ be such that $r\|(I - P)(a)\| < \|x\|$ and $r < t$. Then, we have

$$\|x - ra\| = \max(\|x - rP(a)\|, r\|(I - P)(a)\|) = \max(s(r), r\|(I - P)(a)\|) < \|x\|,$$

which proves the equivalence in this case.
- (1) $\Leftrightarrow$ (2) When the metric projection on a subspace G is well-defined, $\mathrm{Proj}(x, G)$ is the unique $g \in G$ such that $\inf_{a\in G} \|x - a\| = \|x - g\|$. Since it is the case for F and $P(F)$, the equivalence $(i) \Leftrightarrow (ii)$ concludes the proof. □

We can then use Lemma 4.3.3 to transfer some p-orthogonality properties X to X/F.

Corollary 4.3.4. *Let $1 < p < +\infty$. Let X be a Banach space and let F be a closed subspace of X. Let $x, y \in X$ be such that $\|\overline{x}\| = \|x\|$, $\|\overline{y}\| = \|y\|$ and $x \perp_p y$. Suppose that there exists $P \in \mathcal{P}_p(X)$ such that $P(x) = x$ and $P(y) = 0$. Then*

(i) We have $\overline{x} \perp_p \overline{y}$ and $\|\overline{ax + by}\| = \|ax + by\|$ for all $a, b \in \mathbb{C}$;

(ii) If the metric projection onto F is well-defined, then we also have Proj$(ax + by, F) = 0$ for all $a, b \in \mathbb{C}$.

Proof. (i) Up to changing x by ax and y by by, we will show that $\|\overline{x + y}\| = \|x + y\|$. Let $u \in F$. By applying Lemma 4.3.3 we obtain

$$\|x + y - u\|^p = \|x - P(u)\|^p + \|y - (I - P)(u)\|^p \geq \|x\|^p + \|y\|^p = \|x + y\|^p.$$

Thus, $\|x+y-0\|^p = \inf_{u\in F}(\|x+y-u\|^p)$, so $\|\overline{x+y}\| = \|x+y\|$. Therefore, for any $z \in \mathbb{C}$, we get

$$\|\overline{x} + z\overline{y}\|^p = \|x+zy\|^p = \|x\|^p + \|zy\|^p = \|\overline{x}\|^p + \|z\overline{y}\|^p,$$

so $\overline{x}$ and $\overline{y}$ are p-orthogonal.
- (ii) If the metric projection onto F is well-defined, then point (i) implies that $\mathrm{Proj}(ax+by, F) = 0$. □

The next corollary generalizes the second equivalence of Lemma 4.3.3 for a broader use.

Corollary 4.3.5. *Let $1 < p < +\infty$. Let X be a Banach space. Let F be a closed subspace of X. Let $x \in X$ and $P \in \mathcal{P}_p(X)$ be such that $P(x) = x$. Suppose that every $x \in X$ possesses at least one metric projection onto F. Then, the following are equivalent :*

(i) $\inf_{a\in F} \|x-a\| = \|x-\alpha\|$, *for* $\alpha \in F \cap P(F)$*;*

(ii) $\inf_{a\in F} \|x-a\| = \inf_{b\in P(F)} \|x-b\|$*;*

(iii) $\inf_{b\in P(F)} \|x-b\| = \|x-\beta\|$, *for* $\beta \in F$*.*

If the metric projections onto F and $P(F)$ are well-defined, then the following are equivalent :

(1) *Proj*$(x,F) \in P(F)$*;*

(2) *Proj*$(x,F) =$ *Proj*$(x,P(F)) \in (F\cap P(F))$*;*

(3) *Proj*$(x,P(F)) \in F$*.*

Proof. $(i) \Rightarrow (ii)$ Take $y = x-\alpha$. Since $\alpha \in F\cap P(F)$ we have $P(y) = y$ and $\inf_{a\in F}\|y-a\| = \|y\|$. Hence, Lemma 4.3.3 gives

$$\inf_{b\in P(F)} \|x-b\| = \inf_{b\in P(F)} \|y-b\| = \|y\| = \inf_{a\in F} \|y-a\| = \inf_{a\in F} \|x-a\|.$$

- $(ii) \Rightarrow (iii)$ Since $\inf_{a\in F}\|x-a\|$ is attained for some $\beta \in F$, we obtain item (iii).
- $(iii) \Rightarrow (i)$ Since for any $a \in F$ we have

$$\|x-a\|^p = \|x-P(a)\|^p + \|(I-P)(a)\|^p \geq \|x-P(a)\|^p,$$

we obtain $\inf_{a\in F}\|x-a\| \geq \inf_{b\in P(F)}\|x-b\|$. Thus, item (iii) implies

$$\inf_{b\in P(F)} \|x-b\| = \|x-\beta\| = \inf_{a\in F} \|x-a\|.$$

Since $\|x-\beta\|^p = \|x-P(\beta)\|^p + \|(I-P)(\beta)\|^p \geq \inf_{b\in P(F)}\|x-b\|^p + 0$, we must have $\|(I-P)(\beta)\|^p = 0$, that is $P(\beta) = \beta$, which implies that $\beta \in F\cap P(F)$ and gives item (i).
We suppose now that the metric projections onto F and $P(F)$ are well-defined. The implications $(2) \Rightarrow (1)$ and $(2) \Rightarrow (3)$ are immediate.
- $(1) \Rightarrow (2)$ As $\mathrm{Proj}(x,F) \in F$, we have $\mathrm{Proj}(x,F) \in F\cap P(F)$. Thus item (2) comes from the implication $(i) \Rightarrow (ii)$.
- $(3) \Rightarrow (2)$ The implication $(iii) \Rightarrow (ii)$ tells us that $\mathrm{Proj}(x,P(F)) = \mathrm{Proj}(x,F)$, so $\mathrm{Proj}(x,F)$ belongs to $P(F)\cap F$, which concludes the proof. □

Remark 4.3.6. The following counter-example shows that Lemma 4.3.3, which is central in this section, is not true for $p = 1$.
We also remark that similarly to Lemma 4.3.3, Corollaries 4.3.4 and 4.3.5 are true for $p = +\infty$. The only difference in the proofs between $1 < p < +\infty$ and $p = +\infty$ is the fact that $\|a + b\|^p = \|a\|^p + \|b\|^p$ must be replaced by $\|a + b\| = \max(\|a\|, \|b\|)$.

Counter-Example 4.3.7. Lemma 4.3.3 is not true when $p = 1$. Indeed, if we take $X = l^1(\mathbb{C}^3)$, $M \geq 1$, $a = (0, 1, M)$, $x = (1, 1, 0)$, $F = \text{Span}(a)$, and the L^1-projection $P = M_{\chi_{\{1,2\}}}$, then we have

$$\inf_{v \in F}(\|x - v\|) = 2 = \|x\| \text{ but } \inf_{u \in P(F)}(\|x - u\|) = 1 < \|x\|.$$

This is because the right derivative of $r \mapsto |r|$ at 0 is not zero.

We also give a result that is exclusive to the case $p = +\infty$ before ruling it out for the rest of the section.

Lemma 4.3.8. *Let X be a Banach space and $p = +\infty$. Let F be a closed subspace of X with $F \neq X$. Let $P \in \mathcal{P}_\infty(X)$ that is non-trivial. Suppose that the metric projection on F is well-defined and that a metric projection on $P(F)$ and $(I - P)(F)$ exists. Then,*

- *(i) If $P(F) \neq \{0\}$ and $(I - P)(F) \neq \{0\}$, then metric projections on $P(F)$ and $(I - P)(F)$ are not unique.*
- *(ii) If $F \cap Ker(P) \neq \{0\}$, then $P(F) = P(X)$ and for every $x \in Ran(P) \setminus (F \cap Ran(P))$, we have $x - Proj(x, F) \notin P(X)$.*
- *(iii) If $F \cap Ran(P) \neq \{0\}$, then $(I - P)(F) = (I - P)(X)$ and for every $x \in Ker(P) \setminus (F \cap Ker(P))$, we have $x - Proj(x, F) \notin (I - P)(X)$.*

Proof. (i) Since P is non-trivial, $\text{Ker}(P)$ and $\text{Ran}(P)$ are not reduced to $\{0\}$. If $P(F) \neq \{0\}$ then metric projections on $P(F)$ are not unique. Indeed, for $x \in \text{Ker}(P)$ we have

$$\inf_{f \in P(F)} \|x - f\| = \inf_{f \in \text{Ran}(P)} \max(\|x\|, \|f\|) = \|x\|,$$

but this infimum is also attained for every $f \in \text{Ran}(P)$ such that $\|f\| \leq \|x\|$. The same argument shows that if $(I - P)(F) \neq \{0\}$ then metric projections on $(I - P)(F)$ are not unique.
- $(ii), (iii)$ Suppose that $F \cap \text{Ker}(P) \neq \{0\}$. Let $x \in \text{Ran}(P)$. We then have $\alpha \in P(F)$ such that $\inf_{a \in P(F)} \|x - a\| = \|x - \alpha\|$. Take $y = x - \alpha$. Then Lemma 4.3.3 tells us that $\inf_{f \in F} \|y - f\| = \|y\|$. As we saw previously, for any $f \in F \cap \text{Ker}(P)$ such that $\|f\| \leq \|y\|$, we have $\|y - f\| = \|y\|$. Since $F \cap \text{Ker}(P) \neq \{0\}$, by unicity of the metric projection on F we must have $y = 0$, that is $x = \alpha \in P(F)$, so $P(X) = P(F)$. Now, let $x \in \text{Ran}(P) \setminus (F \cap \text{Ran}(P))$. If $x - \text{Proj}(x, F) \in P(X)$ then this vector is non-zero. Also, for any $f \in F \cap \text{Ker}(P)$ we have

$$\|x - \text{Proj}(x, F) - f\| = \max(\|x - \text{Proj}(x, F)\|, \|f\|) > 0.$$

Thus there exists $f \neq 0$ small enough such that $\|x - (\text{Proj}(x, F) + f)\| = \|x - \text{Proj}(x, F)\|$, which contradicts the unicity of $\text{Proj}(x, F)$. Hence $x - \text{Proj}(x, F) \notin P(X)$ and item (i) is proved. Item (ii) is obtained by mimicking the proof with $(I - P)$ instead of P. □

Let X be a Banach space, let F be a closed subspace of X, and let $1 \leq p \leq +\infty$. Similarly to Property 4.2.7, we introduce a property linking the p-orthogonality between the quotient X/F and X.

Property 4.3.9 (Extension of p-orthogonality from a quotient)**.** For any $\overline{x}, \overline{y} \in X/F$ such that $\overline{x} \perp_p \overline{y}$, there exists $x, y \in X$ representatives of $\overline{x}, \overline{y}$ of minimal norm such that $x \perp_p y$.

The next proposition shows that Corollary 4.3.4 admits a converse when $X = L^p(\Omega)$, which is obtained through a generalization of the equality case in Clarkson inequalities for quotients of $L^p(\Omega)$.

Proposition 4.3.10 (Clarkson equality case for quotients of L^p)**.** *Let $1 < p < +\infty$, $p \neq 2$, and let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let F be a closed subspace of $L^p(\Omega)$. Then $L^p(\Omega)$ satisfies Property 4.3.9 for p and F. Also, for any $x, y \in L^p$ such that Proj$(x, F) = 0$ and Proj$(y, F) = 0$, the following are equivalent*

(i) $\|\overline{x+y}\|^p + \|\overline{x-y}\|^p = 2(\|\overline{x}\|^p + \|\overline{y}\|^p)$;

(ii) $\|\overline{x} \pm \overline{y}\|^p = \|\overline{x}\|^p + \|\overline{y}\|^p$;

(iii) $\overline{x} \perp_p \overline{y}$;

(iv) $x \perp_p y$;

(v) Proj$(ax + by, F) = 0$, $\forall a, b \in \mathbb{C}$ *and* $x \perp_p y$.

Proof. Since $1 < p < +\infty$, every class in $L^p(\Omega)/F$ admits a unique representative of minimal norm and Proj$(., F)$ is well defined.
- $(iii) \Rightarrow (ii) \Rightarrow (i)$ is immediate.
- $(iv) \Leftrightarrow (v)$ has already been obtained in Corollary 4.3.5.
- $(v) \Rightarrow (iii)$ Since Proj$(ax + by, F) = 0$, we have $\|\overline{ax + by}\| = \|ax + by\|$ for any $a, b \in \mathbb{C}$. As $x \perp_p y$, we obtain

$$\|\overline{x} + z\overline{y}\|^p = \|x + zy\|^p = \|x\|^p + |z|^p \|y\|^p = \|\overline{x}\|^p + |z|^p \|\overline{y}\|^p.$$

- $(i) \Rightarrow (iv)$ We will separate the cases $1 < p < 2$ and $2 < p$.
Suppose first that $1 < p < 2$. Since $\|\overline{h}\| \leq \|h\|$, the Clarkson inequality gives

$$\|\overline{x+y}\|^p + \|\overline{x-y}\|^p \leq \|x+y\|^p + \|x-y\|^p \leq 2(\|x\|^p + \|y\|^p).$$

Since the leftmost and rightmost terms are equal, we are in the equality case of the Clarkson inequalities. Thus, x and y are p-orthogonal.
Suppose now that $2 < p$. Let h, k be the representatives of $\overline{x+y}, \overline{x-y}$ of minimal norm. Then, $\frac{h+k}{2}, \frac{h-k}{2}$ are representatives of $\bar{x}, \bar{y}$. We will show that they are the ones of minimal norm, i.e. x and y. We have

$$\begin{aligned} 2(\inf_{u \in F} \|\frac{h+k}{2} - u\|^p + \inf_{v \in F} \|\frac{h-k}{2} - v\|^p) &= 2(\|\bar{x}\|^p + \|\bar{y}\|^p) \\ &= \|\overline{x+y}\|^p + \|\overline{x-y}\|^p = \|h\|^p + \|k\|^p. \end{aligned}$$

and the Clarkson inequality gives

$$\|h\|^p + \|k\|^p \geq 2(\|\frac{h-k}{2}\|^p + \|\frac{h+k}{2}\|^p).$$

Thus,

$$\|\frac{h-k}{2}\|^p + \|\frac{h+k}{2}\|^p \leq \inf_{u\in F} \|\frac{h+k}{2} - u\|^p + \inf_{v\in F} \|\frac{h-k}{2} - v\|^p.$$

Since we have an equality, this means that $\frac{h+k}{2}, \frac{h-k}{2}$ are the representatives of $\bar{x}, \bar{y}$ of minimal norm, i.e. x and y. Hence, $x = \frac{h+k}{2}$ and $y = \frac{h-k}{2}$, so $x+y = h$ and $x - y = k$. Therefore we obtain

$$\|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p).$$

And the equality case in the Clarkson inequality implies that $x \perp_p y$. □

Remark 4.3.11. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ of Proposition 4.3.10 are also true when $p = 2$. Indeed, this case amounts to orthogonality in a Hilbert space setting, where such equivalences can be easily proven using orthogonality and inner products. However, as the quotient of a Hilbert space is a Hilbert space, these spaces have no significant interest in this section.
The equivalence $(iii) \Leftrightarrow (iv)$ proves to be very useful in studying the L^p-projections on $L^p(\Omega)/F$. It turns out that we can study $\mathcal{P}_p(X/F)$ for broader Banach spaces X as long as they satisfy a similar property.

Lemma 4.3.12. *Let X be a Banach space and $1 < p < +\infty$. Let F be a closed subspace such that every element of X/F admits an unique representative of minimal norm. Let $P \in \mathcal{P}_p(X)$ be such that $P(F) \subset F$. Then,*

(i) The metric projection on $P(F)$ is well-defined, so the maps $Rep_{P(F)} : X/P(F) \to X$ and $Proj(., P(F)) : X \to P(F)$ are well-defined. The same is true for $(I-P)$;

(ii) For any $x \in X$ we have $Proj(x, P(F)) = Proj(P(x), P(F)) = Proj(P(x), F)$;

(iii) For any $x \in X$ we have $Proj(x, F) = Proj(x, P(F)) + Proj(x, (I-P)(F))$.

Proof. Since $P(F) \subset F$ we have $F = P(F) \oplus_p (I-P)(F)$. We recall that every element of X/F admits unique representatives of minimal norm if and only if for every element x of X, the metric projection of x on F is unique.
Let $x \in X$. We then have

$$\begin{aligned}\|P(x) - \mathrm{Proj}(P(x), F)\|^p &= \inf_{f\in F} \|P(x) - f\|^p = \inf_{a+b \in P(F)\oplus(I-P)(F)} (\|P(x) - a\|^p + \|b\|^p) \\ &= \inf_{a\in P(F)} \|P(x) - a\|^p.\end{aligned}$$

Since we also have

$$\begin{aligned}\|P(x) - \mathrm{Proj}(P(x), F)\|^p &= \|P(x) - P(\mathrm{Proj}(P(x), F))\|^p + \|(I-P)(\mathrm{Proj}(P(x), F))\|^p \\ &\geq \inf_{a\in P(F)} \|P(x) - a\|^p + 0,\end{aligned}$$

we then get $(I-P)(\mathrm{Proj}(P(x),F))=0$, that is $\mathrm{Proj}(P(x),F)\in P(F)$. Thus $P(x)$ possesses a metric projection on $P(F)$. Therefore,

$$\begin{aligned}\|x-\mathrm{Proj}(P(x),F)\|^p &= \|P(x)-\mathrm{Proj}(P(x),F)\|^p+\|(I-P)(x)\|^p\\ &= \inf_{a\in P(F)}\|P(x)-a\|^p+\|(I-P)(x)\|^p=\inf_{a\in P(F)}\|x-a\|^p,\end{aligned}$$

so x possesses a metric projection on $P(F)$.
Now, let $f_1\in P(F)$ be such that $\|x-f_1\|=\inf_{a\in P(F)}\|x-a\|$. We can see that

$$\begin{aligned}\|P(x)-f_1\|^p &= \|x-f_1\|^p-\|(I-P)(x)\|^p=\|x-\mathrm{Proj}(P(x),F)\|^p-\|(I-P)(x)\|^p\\ &= \|P(x)-\mathrm{Proj}(P(x),F)\|^p,\end{aligned}$$

So $f_1=\mathrm{Proj}(P(x),F)$ by unicity of the metric projection on F. Thus, the metric projection of x on $P(F)$ is unique. As we also obtained $\mathrm{Proj}(x,P(F))=\mathrm{Proj}(P(x),F)=\mathrm{Proj}(P(x),P(F))$, item (ii) is proved. Since we have $(I-P)(F)\subset F$, we can mimic the proof for $(I-P)$ to prove that the metric projection on $(I-P)(F)$ is well-defined, and that $\mathrm{Proj}(x,(I-P)(F))=\mathrm{Proj}((I-P)(x),F)=\mathrm{Proj}((I-P)(x),(I-P)(F))$. Thus item (i) is proved.
- (iii) As the metric projections on $P(F)$ and $(I-P)(F)$ are well-defined, we obtain

$$\begin{aligned}\|x-\mathrm{Proj}(x,F)\|^p &= \inf_{a\in F}\|x-a\|^p=\inf_{a\in P(F)\oplus(I-P)(F)}\|(P(x)+(I-P)(x))\\ &\qquad -(P(a)+(I-P)(a))\|^p\\ &= \inf_{b\in P(F)}\|P(x)-b\|^p+\inf_{c\in(I-P)(F)}\|(I-P)(x)-c\|^p\\ &= \|P(x)-\mathrm{Proj}(P(x),P(F))\|^p+\|(I-P)(x)\\ &\qquad -\mathrm{Proj}((I-P)(x),(I-P)(F))\|^p\\ &= \|x-(\mathrm{Proj}(P(x),P(F))+\mathrm{Proj}((I-P)(x),(I-P)(F)))\|^p.\end{aligned}$$

Hence, by unicity of the metric projection on F, we get

$$\begin{aligned}\mathrm{Proj}(x,F) &= \mathrm{Proj}(P(x),P(F))+\mathrm{Proj}((I-P)(x),(I-P)(F))\\ &= \mathrm{Proj}(x,P(F))+\mathrm{Proj}(x,(I-P)(F)).\end{aligned}$$

The proof is now complete. □

The following proposition gives, for a subspace F, the form of some L^p-projections on X/F.

Proposition 4.3.13. *Let X be a Banach space and let $1<p<+\infty$, $p\neq 2$. Let F be a closed subspace of X, and let $P\in\mathcal{P}_p(X)$ be such that $P(F)\subset F$. Then,*

(i) $X/F\simeq P(X)/P(F)\oplus_p(I-P)(X)/(I-P)(F)$;
If the metric projection on F is well-defined, then

$$Rep(X/F)=Rep(P(X)/P(F))\oplus_p Rep((I-P)(X)/(I-P)(F)).$$

(ii) There exists an L^p-projection P' on X/F such that $P'(\overline{x})=\overline{P(x)}$;

(iii) P' is non-trivial if and only if $P(F)\neq P(X)$ and $(I-P)(F)\neq(I-P)(X)$;

(iv) *Let $P_F \in \mathcal{P}_p(X)$ be the maximal L^p-projection such that $Ran(P_F) \subset F$. Then X/F is isometrically isomorphic to $(I - P_F)(X)/(I - P_F)(F)$.*

(v) *Denote $\phi : P \in \{Q \in \mathcal{P}_p(X)\colon Q(F) \subset F\} \mapsto P' \in \mathcal{P}_p(X/F)$. Then ϕ is a morphism of commutative Boolean algebras, and $Ker(\phi) = \mathcal{P}_p(X) \circ P_F$. Hence, $\phi(P_1) = \phi(P_2)$ if and only if $(I - P_F)P_1 = (I - P_F)P_2$, and ϕ is injective if and only if $P_F = 0$.*
In general, $Ran(\phi) = \phi(\{Q \in \mathcal{P}_p(X)\colon QP_F = 0,\ P(F) \subset F\})$ and ϕ is injective on this set.

Proof. (i) As $P(F) \subset F$, we also have $(I - P)(F) \subset F$. Thus F can be decomposed as

$$F = P(F) \oplus_p (I - P)(F).$$

Since the same is true for X, we have $X = P(X) \oplus_p (I - P)(X)$. As $P(F) \subset P(X)$ and $(I - P)(F) \subset (I - P)(X)$, the p-orthogonal decompositions of X and F imply that

$$\begin{aligned} X/F &= (P(X) \oplus_p (I - P)(X))/(P(F) \oplus_p (I - P)(F)) \\ &\simeq P(X)/P(F) \oplus_p (I - P)(X)/(I - P)(F). \end{aligned}$$

Suppose now that the metric projection on F is well-defined. Thus, every element of X/F admits a representative of minimal norm. With item (i) of Lemma 4.3.12, the same is true for $P(F)$ and $(I - P)(F)$. Let $x \in \mathrm{Rep}(X/F)$. Item (iii) of Lemma 4.3.12 gives

$$\begin{aligned} 0 = \mathrm{Proj}(x, F) &= \mathrm{Proj}(x, P(F)) + \mathrm{Proj}(x, (I - P)(F)) \\ &= \mathrm{Proj}(P(x), P(F)) + \mathrm{Proj}((I - P)(x), (I - P)(F)). \end{aligned}$$

Hence, $\mathrm{Proj}(P(x), P(F)) = 0 = \mathrm{Proj}((I-P)(x), (I-P)(F))$. Thus $P(x)$ and $(I-P)(x)$ belong to $\mathrm{Rep}(X/F)$ according to Lemma 4.3.3. This implies that

$$P(\mathrm{Rep}(X/F)) \subset \mathrm{Rep}(X/F) \text{ and } (I - P)(\mathrm{Rep}(X/F)) \subset \mathrm{Rep}(X/F).$$

Therefore, we get $\mathrm{Rep}(X/F) = P(\mathrm{Rep}(X/F)) + (I - P)(\mathrm{Rep}(X/F))$. For $x \in \mathrm{Rep}(X/F)$, the previous computation gave $\mathrm{Proj}(P(x), P(F)) = 0$ so $P(x) \in \mathrm{Rep}(P(X)/P(F))$. Hence $P(\mathrm{Rep}(X/F)) \subset \mathrm{Rep}(P(X)/P(F))$. Conversely, let $y \in \mathrm{Rep}(P(X)/P(F))$. We then have $P(y) = y$ and $\mathrm{Proj}(y, P(F)) = 0$, so $\mathrm{Proj}(y, F) = 0$. Thus, $y \in \mathrm{Ran}(P)$ and $y \in \mathrm{Rep}(X/F)$, so $y \in P(\mathrm{Rep}(X/F))$. The same reasoning can be applied to $I - P$, in order to prove that $(I - P)(\mathrm{Rep}(X/F)) = \mathrm{Rep}((I - P)(X)/(I - P)(F))$. Combining the relationships gives

$$\begin{aligned} \mathrm{Rep}(X/F) &= P(\mathrm{Rep}(X/F)) \oplus_p (I - P)(\mathrm{Rep}(X/F)) \\ &= \mathrm{Rep}(P(X)/P(F)) \oplus_p \mathrm{Rep}((I - P)(X)/(I - P)(F)). \end{aligned}$$

- (ii) Let us define P' the projection on $P(X)/P(F) \oplus \{0\}$ parallel to $\{0\} \oplus (I-P)(X)/(I-P)(F)$. Then P' is an L^p-projection on X/F. Now, let $x \in X$. As we have $x = P(x) + (I - P)(x)$ and $F = P(F) \oplus (I - P)(F)$, the class $\overline{x} = (x\ mod(F))$ is equal to the class

$$P(x)\ mod(P(F)) + (I - P)(x)\ mod((I - P)(F))).$$

Hence, the class $\overline{P(x)} = (P(x)\ mod(F))$ is equal to $P(x)\ mod(P(F)) + 0\ mod((I-P)(F))$ which is equal to

$$P'[P(x)\ mod(P(F)) + (I - P)(x)\ mod((I - P)(F))] = P'(\overline{x}),$$

by definition of P'. Therefore, we have $P'(\overline{x}) = \overline{P(x)}$.
- (iii) The L^p-projection P' is non-trivial on X/F if and only if $P(X)/P(F) \oplus \{0\} = \mathrm{Ran}(P') \neq \{0\}$ and $\{0\} \oplus (I-P)(X)/(I-P)(F) = \mathrm{Ker}(P') \neq \{0\}$. This is equivalent to $P(F) \neq P(X)$ and $(I-P)(F) \neq (I-P)(X)$.
- (iv) The set $\mathcal{P}_{p,F}(X) := \{Q \in \mathcal{P}_p(X)\colon \mathrm{Ran}(Q) \subset F\}$ contains 0 so it is non-empty. Let $(Q_i)_{i\in I}$ be net converging to Q in the strong operator topology. Thus, for any $x \in X$, the net $(Q_i(x))_{i\in I}$ converges to $Q(x)$. Since $Q_i(x)$ lies in F and F is closed, we get $Q(x) \in F$, therefore $\mathrm{Ran}(Q) \subset F$.
As $1 \leq p < +\infty$, $p \neq 2$, the set $\mathcal{P}_{p,F}(X)$ admits a supremum P_F that is unique, according to Theorem 4.1.8. Since this set is closed for the net convergence with the strong operator topology, we have $P_F \in \mathcal{P}_{p,F}(X)$. Since $P_F(X) \subset F$ we have $P_F(F) \subset P_F(X) \subset P_F(F)$, so $P_F(F) = P_F(X)$ and $P_F(F) \subset F$. Hence, item (i) gives

$$X/F \simeq P_F(X)/P_F(X) \oplus_p (I-P_F)(X)/(I-P_F)(F) \simeq (I-P_F)(X)/(I-P_F)(F).$$

- (v) The set $\{Q \in \mathcal{P}_p(X)\colon Q(F) \subset F\}$ is a Boolean sub-algebra of $\mathcal{P}_p(X)$. The linearity of the quotient map $x \mapsto \overline{x}$ and the relationship of item (ii) between P and $\phi(P)$ ensure that $\phi(I) = I$, and that ϕ is invariant for the operations $Q \mapsto I - Q$, $(Q,R) \mapsto QR$, $(Q,R) \mapsto Q + R - QR$. Thus, ϕ is a morphism of Boolean algebras.
Let $Q \in \mathcal{P}_p(X)$ be such that $Q(F) \subset F$. From the construction of item (ii), $\phi(Q)$ identifies as the projection on $Q(X)/Q(F) \oplus \{0\}$ parallel to $\{0\} \oplus (I-Q)(X)/(I-Q)(F)$. As Q and P_F commute and leave F invariant, and as $F = P_F(X) \oplus_p (I-P_F)(F)$, we can decompose the spaces $Q(X)$ and $Q(F)$ as follows :

$$Q(X) = Q(P_F(X) \oplus_p (I-P_F)(X)) = QP_F(X) \oplus_p Q(I-P_F)(X),$$
$$Q(F) = Q(P_F(X) \oplus_p (I-P_F)(F)) = QP_F(X) \oplus_p Q(I-P_F)(F).$$

Hence, we get

$$\mathrm{Ran}(\phi(Q)) \simeq Q(X)/Q(F) \simeq Q((I-P_F)(X))/Q((I-P_F)(F)).$$

Therefore, $\phi(Q) = 0$ if and only if $Q((I-P_F)(X)) = Q((I-P_F)(F))$. If this condition is true, then the projection $R = Q + P_F - QP_F$ satisfies

$$\mathrm{Ran}(R) = P_F(X) + Q(X) = P_F(X) + Q((I-P_F)(X)) = P_F(F) + Q((I-P_F)(F)) \subset F.$$

Thus R lies in $\mathcal{P}_{p,F}(X)$. As we also have $P_F R = P_F$, that is $P_F \leq R$, this implies that $R = P_F$ by maximality of P_F in $\mathcal{P}_{p,F}(X)$. The condition $P_F + Q - QP_F = P_F$ is equivalent $Q(I-P_F) = 0$, which is in turn equivalent to $Q = SP_F$ for some $S \in \mathcal{P}_p(X)$.
Therefore, $\mathrm{Ker}(\phi) = \mathcal{P}_p(X) \circ P_F$. Since ϕ is a morphism of commutative Boolean algebras, we then have $\phi(P_1) = \phi(P_2)$ if and only if $(I-P_F)P_1 = (I-P_F)P_2$. For any Q such that $Q(F) \subset F$, take $S = Q(I-P_F)$. Then, $SP_F = 0$, $S(F) \subset F$, and $S(I-P_F) = Q(I-P_F)$, so $\phi(S) = \phi(Q)$. Thus,

$$\mathrm{Ran}(\phi) = \phi(\{Q \in \mathcal{P}_p(X)\colon QP_F = 0,\, Q(F) \subset F\}).$$

For P_1, P_2 in this set such that $\phi(P_1) = \phi(P_2)$, we have $P_1(I-P_F) = P_2(I-P_F)$. Since $P_1P_F = 0 = P_2P_F$, we get $P_1 = P_1(I-P_F) + P_1P_F = P_2$, so ϕ is injective on this set. ☐

Remark 4.3.14. The maximality of the L^p-projection P_F implies that $P_{(I-P_F)(F)} = 0$. By considering $(I-P_F)(X)/(I-P_F)(F)$, we can reduce the study to a quotient for which the map ϕ is injective.
The set $\{P \in \mathcal{P}_p(X)\colon PP_F = 0,\ P(F) \subset F\}$ is the Boolean algebra of L^p-projections that have a "disjoint support" with P_F. Hence, one can wonder if the map ϕ is bijective. The answer is false in general, but it turns out to be true when X satisfies Properties 4.2.7 and 4.3.9 and when quotients of X have unique representatives of minimal norm, as we will see in Counter-example 4.3.15 and Proposition 4.3.16.

Counter-Example 4.3.15 (Non-surjectivity of the morphism ϕ)**.** Let $1 < p < +\infty$, $p \neq 2$. In $\ell^p(\mathbb{C}^4)$, take $x = (-1,1,0,0)$, $y = (0,0,-1,1)$, $f = (1,1,1,1)$. Denote $G = \mathrm{Span}(x,y,f)$ and $F = \mathrm{Span}(f)$ as in Counter-example 4.2.21. A computation shows that $\mathrm{Proj}(x,F) = 0$, $\mathrm{Proj}(y,F) = 0$. As we have $x \perp_p y$, we have $\mathrm{Proj}(ax+by,F) = 0$, according to Corollary 4.3.4. Hence, G/F is isometrically isomorphic to $\mathrm{Span}(x,y)$, which possesses non-trivial L^p-projections. However, G has trivial L^p-projections as its only elements who have a non-zero p-orthogonal are scalar multiples of x and y. Therefore the morphism ϕ of Boolean algebras of Proposition 4.3.13 is not surjective in this case, even though Properties 4.2.8 and 4.3.9 are satisfied for G,F and p. Furthermore, since f has a maximal support in $\ell^p(\mathbb{C}^4)$, we can use item (i) of Proposition 4.3.16 to obtain $\mathcal{P}_p(\ell^p(\mathbb{C}^4)/F) = \{0, I\}$. This implies that $\ell^p(\mathbb{C}^4)/F$ does not satisfy Property 4.2.7 as $\mathcal{P}_p(G/F)$ is non-trivial, even though $\ell^p(\mathbb{C}^4)$ satisfies Property 4.2.7 and 4.3.9.

Proposition 4.3.16. *Let $1 < p < +\infty$, $p \neq 2$. Let X be a Banach space satisfying Property 4.2.7. Let F be a closed subspace of X such that every element of X/F admits a unique representative of minimal norm. Suppose that Property 4.3.9 is satisfied for X,F and p. Denote*

$$\phi : P \in \{Q \in \mathcal{P}_p(X)\colon Q(F) \subset F\} \mapsto \phi(P) \in \mathcal{P}_p(X/F)$$

the morphism of commutative Boolean algebras from Proposition 4.3.13, with $\phi(P)$ satisfying $\phi(P)(\overline{x}) = \overline{P(x)}$ for every $x \in X$. Then,

- *(i) The morphism ϕ is surjective, every L^p-projection of X/F can be associated to an L^p-projection P on X such that $P(F) \subset F$;*
- *(ii) The Boolean algebra $\mathcal{P}_p(X/F)$ is isomorphic to $\{P \in \mathcal{P}_p(X)\colon PP_F = 0,\ P(F) \subset F\}$;*
- *(iii) Denote P_F the maximal L^p-projection of X such that $Ran(P_F) \subset F$. The space X/F admits non-trivial L^p-projections if and only if there exist L^p-projections P such that $P_F < P < I$ and $P(F) \subset F$.*

Proof. (i) Let $Q \in \mathcal{P}_p(X/F)$. Then $X/F = \mathrm{Ran}(Q) \oplus_p \mathrm{Ker}(Q)$. Denote $Rep : X/F \to X$ the map that sends $\overline{f}$ to its representative of minimal norm. As Property 4.3.9 is satisfied for X,F and p, we have $\mathrm{Rep}_F(\mathrm{Ran}(Q)) \perp_p \mathrm{Rep}(\mathrm{Ker}(Q))$ and Corollary 4.3.4 implies that $\mathrm{Rep}(X/F) = \mathrm{Rep}(\mathrm{Ran}(Q)) \oplus_p \mathrm{Rep}(\mathrm{Ker}(Q))$. Property 4.2.7 and Proposition 4.2.10 imply the existence of $P \in \mathcal{P}_p(X)$ such that

$$P(\mathrm{Rep}(\mathrm{Ran}(Q))) = \mathrm{Rep}(\mathrm{Ran}(Q)) \text{ and } P(\mathrm{Rep}(\mathrm{Ker}(Q))) = \{0\}.$$

We will show that $P(F) \subset F$ and that $\phi(P) = Q$.
Let $x \in P(F)$. We can write $x = g_1 + g_2 + f$, with $g_1 \in \mathrm{Rep}(\mathrm{Ran}(Q))$, $g_2 \in \mathrm{Rep}(\mathrm{Ker}(Q))$, $f \in F$.

Thus, $P(x) = x = g_1 + P(f)$, so $g_1 = x - P(f) \in P(F)$. Since $P(g_1) = g_1$ and $\mathrm{Proj}(g_1, F) = 0$ we have $\mathrm{Proj}(g_1, P(F)) = 0$ according to Lemma 4.3.3. This implies that $g_1 = 0$. Similarly, we have $(I-P)(x) = 0 = g_2 + (I-P)(f)$, so $g_2 = -(I-P)(f) \in (I-P)(F)$. Since $(I-P)(g_2) = g_2$ and $\mathrm{Proj}(g_2, F) = 0$, we have $\mathrm{Proj}(g_2, (I-P)(F)) = 0$. This implies that $g_2 = 0$. Therefore, $x = f \in F$, hence $P(F) \subset F$ and $\phi(P)$ is well defined.
Now, let $x \in X$. Property 4.3.9 and Corollary 4.3.4 imply that

$$(x - \mathrm{Proj}(x, F)) = \mathrm{Rep}(\overline{x}) = \mathrm{Rep}(Q(\overline{x}) + (I-Q)(\overline{x})) = \mathrm{Rep}(Q(\overline{x})) + \mathrm{Rep}((I-Q)(\overline{x})).$$

Therefore $P(x - \mathrm{Proj}(x, F)) = \mathrm{Rep}(Q(\overline{x}))$. Since we have $P(\mathrm{Proj}(x, F)) \in P(F) \subset F$, we obtain

$$Q(\overline{x}) = \overline{\mathrm{Rep}(Q(\overline{x}))} = \overline{P(x) - P(\mathrm{Proj}(x, F))} = \overline{P(x)}.$$

As this is true for every $x \in X$, we get $\phi(P) = Q$, so the morphism ϕ is surjective.
- (ii) Since ϕ is surjective, we can use item (v) of Proposition 4.3.13 to see that ϕ is an isomorphism of Boolean algebras between $\{P \in \mathcal{P}_p(X) \colon PP_F = 0, P(F) \subset F\}$ and $\mathcal{P}_p(X/F)$.
- (iii) A non-trivial L^p-projection on X/F identifies with a non-trivial element

$$Q \in \{P \in \mathcal{P}_p(X) \colon PP_F = 0, P(F) \subset F\}.$$

By taking $P = Q + P_F - QP_F = Q + P_F$, we get $P_F < P < I$ and $P(F) \subset Q(F) + P_F(F) \subset F$ since Q is non-trivial and $Q(F) \subset F$. The converse is obtained by noticing that $Q = P(I - P_F)$. □

Example 4.3.17. Let $1 < p < +\infty$, $p \neq 2$. Let X be a Banach space. Let F be a closed subspace of X such that the metric projection on F is well-defined. Let $P \in \mathcal{P}_p(X)$ be such that $P(F) \neq P(X)$ and $(I-P)(F) \neq (I-P)(X)$. Take $x \in \mathrm{Ran}(P) \setminus P(F)$. We then have $x - \mathrm{Proj}(x, P(F)) \in \mathrm{Ran}(P)$ and $\mathrm{Proj}(x - \mathrm{Proj}(x, P(F)), P(F)) = 0$, so $\mathrm{Proj}(x - \mathrm{Proj}(x, P(F)), F) = 0$ according to Lemma 4.3.3. Thus, $\mathrm{Rep}(X/F) \cap \mathrm{Ran}(P)$ is not reduced to $\{0\}$. Similarly, $\mathrm{Rep}(X/F) \cap \mathrm{Ker}(P)$ is also not reduced to $\{0\}$.
Since for any $y \in \mathrm{Rep}(X/F) \cap \mathrm{Ran}(P)$ and any $z \in \mathrm{Rep}(X/F) \cap \mathrm{Ker}(P)$ we have $\overline{y} \perp_p \overline{z}$ according to Corollary 4.3.4, X/F possesses elements that have non-trivial p-orthogonal sets, which also means that certain subspaces of X/F admit non-trivial L^p-projections.

4.3.B L^p-projections on subspaces of quotients

As we went over L^p-projections on quotients of a Banach space X, we will broaden our field of view by looking at L^p-projections on subspaces of quotients of X. In the case of $L^p(\Omega)$, a quotient $L^p(\Omega)/F$ has an important structure since this quotient and F can recreate the Banach lattice $L^p(\Omega)$. When looking at subspaces of quotients of $L^p(\Omega)$, we gain way more variety since we are only looking at a piece of the whole structure. The following Proposition 4.3.18 gives a characterization of some subspaces F, G such that G/F possesses non-trivial L^p-projections, whereas Proposition 4.3.19 gives a converse result when X satisfies Properties 4.2.7 and 4.3.9 and when quotients of X have unique representatives of minimal norm.

Proposition 4.3.18. *Let $1 < p < +\infty$, $p \neq 2$. Let X be a Banach space. Let F be a closed subspace of X. Let $P \in \mathcal{P}_p(X)$. Suppose that every element of the quotients X/F,$X/P(F)$,$X/(I-P)(F)$ admits a unique representative of minimal norm. Let G_1, G_2 be closed subspaces of X, such that*

$$(F \cap P(F)) \subset G_1 \subset \mathit{Rep}(P(X)/P(F)) + (F \cap P(F)),$$
$$(F \cap (I-P)(F)) \subset G_2 \subset \mathit{Rep}((I-P)(X)/(I-P)(F)) + (F \cap (I-P)(F)).$$

Take $G = F + G_1 + G_2$ and let $Q \in \mathcal{P}_p(G/F)$. Then,

(i) G/F possesses an L^p-projection Q such that $P(Rep(Ran(Q))) = Ran(Q)$ and $P(Rep(Ker(Q))) = \{0\}$.

(ii) Q is non-trivial if and only if $G_1 \neq (F \cap P(F))$ and $G_2 \neq (F \cap (I-P)(F))$.

Proof. (i) We remark that $G_1 \subset \mathrm{Ran}(P)$ and $G_2 \subset \mathrm{Ker}(P)$. Let $g_1 \in G_1$. We have $g_1 = x + f_1$ with $x \in \mathrm{Rep}(P(X)/P(F))$ and $f_1 \in (F \cap P(F)) \subset G_1$. Thus $x = g_1 - f_1 \in G_1$, and $\mathrm{Proj}(x, P(F)) = 0$. Similarly, for $g_2 \in G_2$, we have $g_2 = y + f_2$ with $f_2 \in F \cap (I-P)(F)$, $y \in G_2$ and $\mathrm{Proj}(y, (I-P)(F)) = 0$. For $g = f + g_1 + g_2$ with $g_i \in G_i$ we have $\overline{g} = \overline{g_1} + \overline{g_2}$. With the previous notations we have $g_1 - x = f_1 \in F \cap P(F)$, so $\overline{g_1} = \overline{x}$, and $g_2 - y = f_2 \in F \cap (I-P)(F)$, so $\overline{g_2} = \overline{y}$. Since we have $P(x_1) = x_1$ and $P(x_2) = 0$, we can the apply Corollary 4.3.4 to obtain $\overline{g_1} = \overline{x_1} \perp_p \overline{x_2} = \overline{g_2}$. Therefore, if we denote $\pi : G \to G/F$ the canonical projection, then

$$G/F = \pi(G_1 + G_2) = \pi(G_1) \oplus_p \pi(G_2),$$

so G/F possesses an L^p-projection Q such that $Q(\overline{g_1 + g_2}) = \overline{g_1}$. We also get from 4.3.1 that $x = \mathrm{Rep}(\overline{g_1})$ and $y = \mathrm{Rep}(\overline{g_2})$, so $\mathrm{Rep}(\mathrm{Ran}(Q)) \subset G_1$ and $\mathrm{Rep}(\mathrm{Ker}(Q)) \subset G_2$, which gives the desired results.
- (ii) We have $\mathrm{Ran}(Q) \neq \{0\}$ and $\mathrm{Ker}(Q) \neq \{0\}$ if and only if $\mathrm{Rep}(\mathrm{Ran}(Q)), \mathrm{Rep}(\mathrm{Ker}(Q)) \neq \{0\}$. If this is true then the previous argument in the proof of item (ii) implies that $G_1 \neq (F \cap P(F))$ and $G_2 \neq (F \cap (I-P)(F))$.
Conversely, for $g_1 \in G_1$ we can see from the proof of item (i) that $\mathrm{Proj}(g_1, P(F)) \in F \cap P(F)$, so $\mathrm{Proj}(g_1, F) = \mathrm{Proj}(g_1, P(F))$. If $g_1 \notin (F \cap P(F))$ then $\mathrm{Proj}(g_1, F) \neq g_1$ and $\mathrm{Rep}(\overline{g_1}) \neq 0$. Similarly, for $g_2 \in G_2 \setminus (F \cap (I-P)(F))$ the same reasoning gives us $\mathrm{Rep}(\overline{g_2}) \neq 0$. Therefore $\mathrm{Rep}(\mathrm{Ran}(Q))$ and $\mathrm{Rep}(\mathrm{Ker}(Q))$ are non-trivial, which concludes the proof. □

Proposition 4.3.19 (Characterization of L^p-projections on subspaces of quotients)**.** *Let $1 < p < +\infty$, $p \neq 2$. Let X be a Banach space. Let $F \subset G$ be closed subspaces of X. Suppose that for every subspace Y, every element of X/Y admits a unique representative of minimal norm. Suppose that X satisfies Property 4.2.7 for p and Property 4.3.9 for F and p. Let $Q \in \mathcal{P}_p(G/F)$. We then have $Rep(G/F) = Rep(Ran(Q)) \oplus_p Rep(Ker(Q))$. Let $P \in \mathcal{P}_p(X)$ be an L^p-projection such that $Rep(Ran(Q)) \subset Ran(P)$ and $Rep(Ker(Q)) \subset Ker(P)$. Then*

(i) For all $g_1, g_2 \in P(Rep(G/F))$, $Proj(g_1 + g_2, F)$ lies in $(F \cap P(F))$. A similar result holds for $I - P$.

(ii) $G \cap P(G) = Rep(Ran(Q)) + (F \cap P(F))$,
$Rep(Ran(Q)) = Rep(P(G)/P(F)) = Rep((G \cap P(G))/(F \cap P(F)))$.
A similar result holds for $I - Q$ and $I - P$.

(iii) $G = F + (G \cap P(G)) + (G \cap (I-P)(G))$, with

$$\begin{aligned}(F \cap P(F)) \subset (G \cap P(G)) &\subset Rep(P(X)/P(F)) + (F \cap P(F))\\ (F \cap (I-P)(F)) \subset (G \cap (I-P)(G)) &\subset Rep((I-P)(X)/(I-P)(F))\\ &\quad + (F \cap (I-P)(F)).\end{aligned}$$

Thus G has the same relationship with F and P as in Proposition 4.3.18.

(iv) $G/F \simeq P(G)/P(F) \oplus_p (I-P)(G)/(I-P)(F)$,
$G/F \simeq (G \cap P(G))/(F \cap P(F)) \oplus_p (G \cap (I-P)(G))/(F \cap (I-P)(F))$.

Proof. (i) Let $g_1, g_2 \in P(\mathrm{Rep}(G/F))$. Since $P(\mathrm{Rep}(G/F)) = \mathrm{Rep}(\mathrm{Ran}(Q))$, we have $\overline{g_1+g_2} \in \mathrm{Ran}(Q)$, so

$$\mathrm{Rep}(\overline{g_1+g_2}) = (g_1+g_2) - \mathrm{Proj}(g_1+g_2, F) \in P(\mathrm{Rep}(G/F)).$$

As $g_1, g_2 \in \mathrm{Ran}(P)$ we have $\mathrm{Proj}(g_1+g_2, F) \in \mathrm{Ran}(P)$. Since $\mathrm{Proj}(g_1+g_2, F) \in F$, we end up with $\mathrm{Proj}(g_1+g_2, F) \in (F \cap P(F))$. $(I-Q)$ is an L^p-projection on G/F and as $(I-P)(\mathrm{Rep}(G/F)) = \mathrm{Rep}(\mathrm{Ran}(I-Q))$, we can now mimick the proof with $I-P$ instead of P to get a similar result for $I-P$.
- (ii) Let $g \in \mathrm{Rep}(\mathrm{Ran}(Q))$, $f \in F \cap P(F)$. Since $F \subset P(F)$, we have $f \in G \cap P(G)$. We have $g \in G$ and $g \in \mathrm{Ran}(P)$, so $g \in G \cap P(G)$. Therefore, $g + f \in G \cap P(G)$.
Conversely, let $x \in G \cap P(G)$. Since $G = \mathrm{Rep}(G/F) + F$, we have $x = (g_1+g_2)+f$, with $g_1 \in \mathrm{Rep}(\mathrm{Ran}(Q))$,$g_2 \in \mathrm{Rep}(\mathrm{Ker}(Q))$, $f \in F$. This implies that

$$0 = (I-P)(x) = g_2 + (I-P)(f).$$

As $\mathrm{Proj}(g_2, F) = 0$, Lemma 4.3.3 also gives $\mathrm{Proj}(g_2, (I-P)(F)) = 0$. Since $g_2 = -(I-P)(f) \in (I-P)(F)$ we must have $g_2 = 0 = (I-Q)(f)$. Therefore, $f = P(f)$ and $x = g_1 + f$, so $x \in \mathrm{Rep}(\mathrm{Ran}(Q)) + (F \cap P(F))$.
Let $g \in \mathrm{Rep}(\mathrm{Ran}(Q))$. As $F \cap P(F) \subset F$, the condition $\mathrm{Proj}(g, F) = 0$ implies $\mathrm{Proj}(g, F \cap P(F)) = 0$. Since $g \in G \cap P(G)$, we have $g \in \mathrm{Rep}((G \cap P(G))/(F \cap P(F)))$. As we also have $g \in P(G)$ and $\mathrm{Proj}(g, P(F)) = 0$, we have $g \in \mathrm{Rep}(P(G)/P(F))$.
Conversely, let $h \in \mathrm{Rep}((G \cap P(G))/(F \cap P(F)))$. Since $h \in G \cap P(G)$, we can write $h = g' + f'$, $g' \in \mathrm{Rep}(\mathrm{Ran}(Q))$, $f' \in (F \cap P(F))$. This implies that

$$0 = \mathrm{Proj}(h, F \cap P(F)) = \mathrm{Proj}(g'+f', F \cap P(F)) = \mathrm{Proj}(g', F \cap P(F)) + f' = f'.$$

Thus $h = g' \in \mathrm{Rep}(\mathrm{Ran}(Q))$. Let $k \in \mathrm{Rep}(P(G)/P(F))$. Since $G = \mathrm{Rep}(G/F) + F$, we get $P(G) = P(\mathrm{Rep}(G/F)) + P(F) = \mathrm{Rep}(\mathrm{Ran}(Q)) + P(F)$. Thus, $k = g" + f"$ with $g" \in \mathrm{Rep}(\mathrm{Ran}(Q))$, $f" \in P(F)$. This implies that

$$0 = \mathrm{Proj}(k, P(F)) = \mathrm{Proj}(g" + f", P(F)) = \mathrm{Proj}(g", P(F)) + f" = f".$$

Therefore $k = g" \in \mathrm{Rep}(\mathrm{Ran}(Q))$. We can then mimick the proof of this item with $I-Q$ and $I-P$ instead of Q and P to get similar results for $I-P$.
- (iii) By using item (ii) we have

$$\begin{aligned} G =& F + \mathrm{Rep}(G/F) = (F + (F \cap P(F)) + (F \cap (I-P)(F)) + (\mathrm{Rep}(\mathrm{Ran}(Q)) + \mathrm{Rep}(\mathrm{Ker}(Q))) \\ =& F + (G \cap P(G)) + (G \cap (I-P)(G)). \end{aligned}$$

Since this item also implies

$$\begin{aligned} \mathrm{Rep}(\mathrm{Ran}(Q)) &\subset \mathrm{Rep}(P(X)/P(F)), \\ \mathrm{Rep}(\mathrm{Ker}(Q)) =& \mathrm{Rep}(\mathrm{Ran}(I-Q)) \subset \mathrm{Rep}((I-P)(X)/(I-P)(F), \end{aligned}$$

we can obtain the desired inclusions for $(G \cap P(G))$ and $(G \cap (I-P)(G))$.
- (iv) Denote

$$G_1 = (P(G) \oplus_p (I-P)(G)),\ F_1 = (P(F) \oplus_p (I-P)(F)),$$
$$G_2 = ((G \cap P(G)) \oplus_p (G \cap (I-P)(G))),\ F_2 = ((F \cap P(F)) \oplus_p (F \cap (I-P)(F))).$$

Since $P(F_1) \subset F_1$ and $P(G_1) \subset G_1$, we can apply item (i) of Proposition 4.3.13 to get

$$\begin{aligned}\mathrm{Rep}(G_1/F_1) &= \mathrm{Rep}(P(G_1)/P(F_1)) + \mathrm{Rep}((I-P)(G_1)/(I-P)(F_1)),\\ \mathrm{Rep}(G_2/F_2) &= \mathrm{Rep}(P(G_2)/P(F_2)) + \mathrm{Rep}((I-P)(G_2)/(I-P)(F_2)).\end{aligned}$$

The results of item (ii) then imply that $\mathrm{Rep}(G/F) = \mathrm{Rep}(G_1/F_1) = \mathrm{Rep}(G_2/F_2)$. Denote $\pi_1 : X \to X/F_1$ and $\pi_2 : X \to X/F_2$ the quotient maps to X/F_1 and X/F_2. We define the maps $\psi_1 := \pi_1 \circ Rep_F$ and $\psi_2 := \pi_2 \circ Rep_F$. As $Rep_F : G/F \to \mathrm{Rep}(G/F)$, $\pi_1 : \mathrm{Rep}(G_1/F_1) \to G_1/F_1$ and $\pi_2 : \mathrm{Rep}(G_2/F_2) \to G_2/F_2$ are bijections, the previous result implies that the maps $\psi_1 : G/F \to G_1/F_1$ and $\psi_2 : G/F \to G_2/F_2$ are bijections. We can also see that for any $\overline{g} \in G/F$ we have

$$\|\overline{g}\| = \|\mathrm{Rep}_F(\overline{g})\| = \|\pi_1(\mathrm{Rep}_F(\overline{g}))\| = \|\pi_2(\mathrm{Rep}_F(\overline{g}))\|,$$

so ψ_1,ψ_2 are isometries. We will show that they are linear maps in order to obtain isometric isomorphisms between the spaces. As the maps Rep_F and π_1 are homogeneous, ψ_1 is homogeneous. Let $\overline{g_1}, \overline{g_2} \in G/F$. We saw in the proof item (i) that for $g_1, g_2 \in \mathrm{Rep}(\mathrm{Ran}(Q))$ we have

$$\begin{aligned}\mathrm{Rep}(\overline{g_1+g_2}) - (\mathrm{Rep}(\overline{g_1}) + \mathrm{Rep}(\overline{g_2})) &= \mathrm{Rep}(\overline{g_1+g_2}) - (g_1+g_2)\\ &= \mathrm{Proj}(g_1+g_2, F) \in F \cap P(F).\end{aligned}$$

Similarly, for $h_1, h_2 \in \mathrm{Rep}(\mathrm{Ker}(Q))$, we have

$$\begin{aligned}\mathrm{Rep}(\overline{h_1+h_2}) - (\mathrm{Rep}(\overline{h_1}) + \mathrm{Rep}(\overline{h_2})) &= \mathrm{Rep}(\overline{h_1+h_2}) - (h_1+h_2)\\ &= \mathrm{Proj}(h_1+h_2, F) \in F \cap (I-P)(F).\end{aligned}$$

Since $\mathrm{Rep}(G/F) = \mathrm{Rep}(\mathrm{Ran}(Q)) \oplus \mathrm{Rep}(\mathrm{Ker}(Q))$, for any $a, b \in \mathrm{Rep}(G/F)$ we then have

$$\mathrm{Rep}(\overline{a+b}) - (\mathrm{Rep}(\overline{a}) + \mathrm{Rep}(\overline{b})) = \mathrm{Rep}(\overline{a+b}) - (a+b) \in (F \cap P(F)) + (F \cap (I-P)(F)).$$

Therefore, $a + b = \mathrm{Rep}(\overline{a+b})\ mod(F_1)$ and $a + b = \mathrm{Rep}(\overline{a+b})\ mod(F_2)$, so

$$\psi_1(a) + \psi_1(b) = \psi_1(a+b)\ \psi_2(a) + \psi_2(b) = \psi_2(a+b).$$

This proves that ψ_1, ψ_2 are linear, bijective, and isometric, which concludes the proof. □

4.4 Generalizations for $L^p(\Omega, X)$ and L^q-projections in L^p-spaces

4.4.A Generalizations for $L^p(\Omega, X)$

In sections 4.2 and 4.3 we obtained several results for some classes of Banach spaces sharing common properties with L^p-spaces, mainly Properties 4.2.7 and 4.3.9. In this section we exhibit conditions on a Banach space X that allow us to generalize previous results to the spaces $L^p(\Omega, X)$.

Proposition 4.4.1. *Let $1 \le p < +\infty$, $p \neq 2$. Let X be a Banach space that satisfies the conditions*

(i) $\|f+g\|_p^p + \|f-g\|_p^p \le 2(\|f\|_p^p + \|g\|_p^p)$, $\forall f, g \in X$ if $1 \le p < 2$
and there is equality if and only if $f = 0$ or $g = 0$;

(ii) $\|f+g\|_p^p + \|f-g\|_p^p \geq 2(\|f\|_p^p + \|g\|_p^p)$, $\forall f, g \in X$ *if* $2 < p$
and there is equality if and only if $f = 0$ *or* $g = 0$.

Then all the results regarding L^p*-projections on subspaces or quotients of* $L^p(\Omega)$ *are true for* $L^p(\Omega, X)$.

Proof. The space X satisfies the same Clarkson inequality and the same equality case as the scalar field $\mathbb{C}$. We can thus prove a version of Lemma 4.1.12 for $L^p(X)$ by mimicking on with an analogue proof as the classical one [Roy88, Ch15-7,Lem 22,p.416] while changing $\mathbb{C}$ by X and $|.|$ by $\|\cdot\|_X$.
Then, all proofs regarding L^p-projections on subspaces or quotients of $L^p(\Omega)$ in this chapter or in [BDE+77], [Li79] can be mimicked for $L^p(\Omega, X)$ as it possesses a similar Banach lattice structure and the same Clarkson inequalities. ☐

Remark 4.4.2. Let $1 \leq p < +\infty$, $p \neq 2$. Let X be a Banach space that is non-zero.

(i) Recall that X is said to be a *strictly convex* Banach space if it satisfies the property

$$\|x+y\| = \|x\| + \|y\| \;\Rightarrow\; x = \alpha y \text{ or } y = \alpha x \text{ for some } \alpha \geq 0.$$

If $p = 1$ and X is strictly convex, then the condition of Proposition 4.4.1 is satisfied, as a short computation shows.

(ii) If X is a subspace, quotient, or subspace of quotient of $L^p(\Omega)$, then X satisfies Clarkson inequalities from either Lemma 4.1.12 or Proposition 4.3.10. Both lemmas also say that

$$x \perp_p y \Leftrightarrow \|x+y\|^p + \|x-y\|^p = 2(\|x\|^p + \|y\|^p).$$

Thus, X satisfies the equality condition of Proposition 4.4.1 if and only if the p-orthogonality relationship is trivial on X, which is in turn equivalent to $X = \{0\}$ or $\alpha(X) = 2$, according to Corollary 4.2.35.

(iii) It was proved in [KT97] that a Banach space satisfies Clarkson's inequalities if and only if its "type or cotype constant" is 1. We refer to [KT97] for more information.

4.4.B L^q-projections in L^p-spaces

When a Banach space X possesses non-trivial L^p-projections, it cannot have non-trivial L^q-projections. However, when X is a subspace or quotient of an L^p-space that has trivial L^p-projections, it is not known if X can possess non-trivial L^q-projections. The following lemma gives a partial answer to this question.

Lemma 4.4.3. *Let* $1 \leq p < +\infty$. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. Let* X *be a subspace, quotient, or subspace of a quotient of* $L^p(\Omega)$. *Suppose that* X *possesses a non-trivial* L^q*-projection* P, *for some* $1 \leq q < +\infty$.

(i) If $p = 2$ *then* $q = 2$;

(ii) If all Hermitian projections on X *have the form* M_{χ_A}, *then* $q = p$;

(iii) If $1 \leq p < 2$, *then* $p \leq q$;

(iv) If $2 < p$, then $1 < q \leq p$.

Proof.
- (i) Suppose that $p = 2$. Since a L^q-projection is of norm 1, P is a norm 1 projection on a Hilbert space, so P is an orthogonal projection and $q = 2$.
- (ii) Since a L^q-projection is a Hermitian projection, if X is a subspace of $L^p(\Omega)$ where all Hermitian projections have the form M_{χ_A}, then $P = M_{\chi_B}$ for some $B \subset \Omega$ and thus $q = p$.
- (iii) If $p \neq 1$, we cannot have $q = 1$ as the strict convexity of the L^p norm on X implies that

$$\|x + y\| = \|x\| + \|y\| \ \Rightarrow\ x = \alpha y \text{ or } y = \alpha x, \text{ for some } \alpha \geq 0$$

This property implies that a L^1-projection on X must be trivial.
- (iv) Since P is non-trivial, let $x \in \mathrm{Ran}(P)$, $y \in \mathrm{Ker}(P)$, with $x, y \neq 0$. We have $\|x \pm y\|^q = \|x\|^q + \|y\|^q$. Denote

$$a = (\frac{\|y\|}{\|x\|})^p > 0.$$

Thus,

$$\|x \pm y\|^p = (\|x\|^q + \|y\|^q)^{p/q} = \|x\|^p(1 + (\frac{\|y\|}{\|x\|})^q)^{p/q} = \|x\|^p(1 + a^{q/p})^{p/q},$$

and $\|x\|^p + \|y\|^p = \|x\|^p(1 + a)$.
If $1 \leq p < 2$, we will have

$$(1 + a^{q/p})^{p/q} \leq (1 + a) \ \Rightarrow\ 1 + a^{q/p} \leq (1 + a)^{q/p}.$$

If $2 < p$, we will have

$$(1 + a^{q/p})^{p/q} \geq (1 + a) \ \Rightarrow\ 1 + a^{q/p} \geq (1 + a)^{q/p}.$$

Let us study the map $h : r \in \mathbb{R}_+ \mapsto (1 + a)^r - (1 + a^r)$. We can see that $h(0) = -1$, $h(1) = 0$, h is continuous, $\lim_{r \to +\infty}(h(r)) = +\infty$, and for $r > 0$,

$$h'(r) = \log(1 + a).(1 + a)^r - \log(a)a^r.$$

Since we have $\log(a) < \log(1 + a)$ and $0 < a^r < (1 + a)^r$, then $h'(r) > 0$ for all $r > 0$, so h is strictly increasing on $\mathbb{R}_+$. Thus, $h(r) \leq 0$ on $[0, 1]$ and $h(r) \geq 0$ on $[1, +\infty[$. Hence, if $1 \leq p < 2$, we must have $h(q/p) \geq 0$, so $p \leq q$, and if $2 < p$ we must have $h(q/p) \leq 0$, so $q \leq p$. $\square$

We do not have for now a better result on the values of q, as we would think that $q = p$ is the only possible choice. The argument in the proof also tells us that for all $x \in \mathrm{Ran}(P)$,$y \in \mathrm{Ker}(P)$ with $x, y \neq 0$, we have

$$\|x + y\|^p < \|x\|^p + \|y\|^p \text{ if } 1 \leq p < 2,$$

and that

$$\|x + y\|^p > \|x\|^p + \|y\|^p \text{ if } p > 2.$$

This also means that we are always in the strict case of the Clarkson inequalities for $x \in \mathrm{Ran}(P)$, $y \in \mathrm{Ker}(P)$ unless $x = 0$ or $y = 0$. This fact could be exploited to bring out an example or to rule out the cases $q \neq p$. This thesis ends here. I sincerely thank you, the reader, for taking

time to read this work.

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